Parameterized Complexity of 1-Planarity

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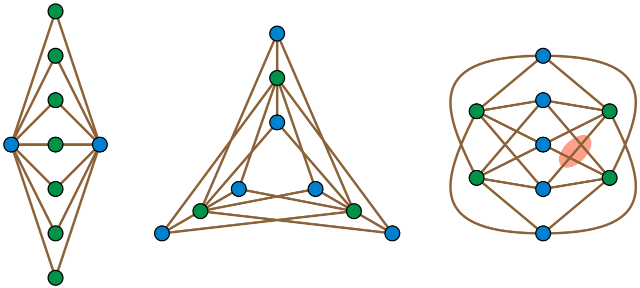
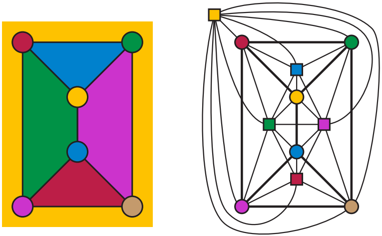
What is 1-planarity?

A graph is 1-planar if it can be drawn in the plane (vertices as points, edges as curves disjoint from non-incident vertices) so that each edge is crossed at most once (in one point, by one edge)



E.g. $K_{2,7}$ is planar, $K_{3,6}$ is 1-planar, and $K_{4,5}$ is not 1-planar
[Czap and Hudák 2012]

History and properties



Original application of 1-planarity: simultaneously coloring vertices and faces of planar maps [Ringel 1965]

1-planar graphs have:

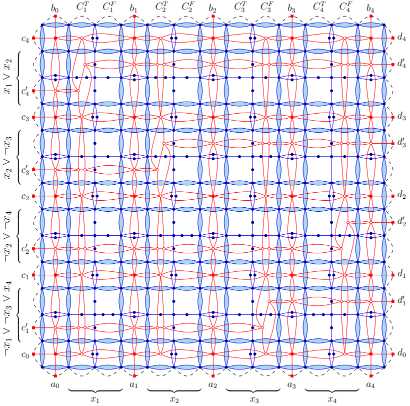
- At most $4n - 8$ edges [Schumacher 1986]
- At most $n - 2$ crossings [Czap and Hudák 2013]
- Chromatic number ≤ 6 [Borodin 1984]
- Sparse *shallow minors* [Nešetřil and Ossona de Mendez 2012]

Computational complexity of 1-planarity

NP-complete ...
[Grigoriev and Bodlaender 2007; Korzhik and Mohar 2013]

even for planar + one edge
[Cabello and Mohar 2012]

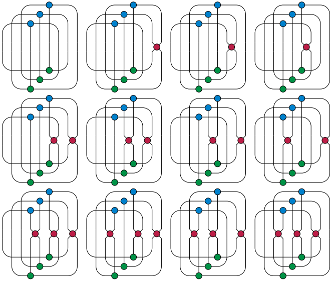
But that shouldn't stop us from seeking exponential or parameterized algorithms for instances of moderate size



Reduction from Cabello and Mohar [2012]

A naive exponential-time algorithm

1. Check that #edges $\leq 4n - 8$
2. For each pairing of edges
   - Replace each pair by $K_{1,4}$
   - Check if result is planar
   - If so, return success
3. If loop terminated normally, return failure



Time dominated by #pairings (*telephone numbers*)

$\approx m^{m/2 - o(m)}$ [Chowla et al. 1951]

E.g. the 9 edges of $K_{3,3}$ have 2620 pairings

Graphs with 18 edges have approximately a billion pairings

Parameterized complexity

NP-hard $\Rightarrow$ we expect time to be (at least) exponential

But exponential in what?

Maybe something smaller than instance size

Goals:

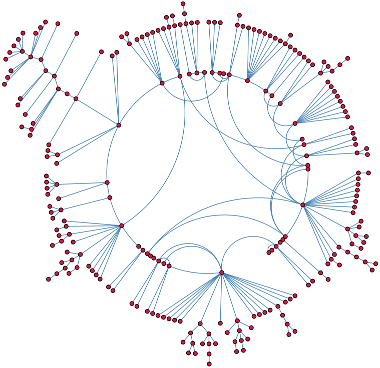
- Find a parameter p defined from inputs that is often small
- Find an algorithm with time $O(f(p)n^c)$
- f must be *computable* and c must be independent of p

If possible, then the problem is **fixed-parameter tractable**

Cyclomatic number

Remove a spanning tree, count remaining edges $\Rightarrow m - n + 1$

Often $\ll n$ for social networks (if closing cycles is rare) and utility networks (redundant links are expensive)



HIV transmission network

[Potterat et al. 2002]

$n = 243$ cyclomatic$\# = 15$

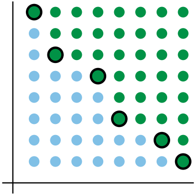
[Bannister et al. 2013]

A hint of fixed-parameter tractability

For any fixed bound k on cyclomatic number,
all properties preserved when degree ≤ 2 vertices are suppressed
(e.g. non-1-planarity) can be tested in linear time

Proof idea:

- Delete degree-1 vertices
- Partition into paths of degree-2 vertices
- Find $O(k)$-tuple of path lengths
- Check vs $O(1)$ minimal forbidden tuples



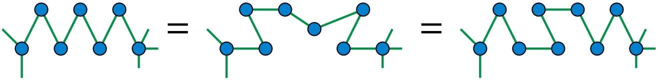
Every set of $O(1)$-tuples of positive integers has $O(1)$ minimal tuples [Dickson 1913]

But don't know how to find minimal tuples or construct drawing

Not FPT because dependence on k isn't explicit and computable

Kernelization

Suppose sufficiently long paths of degree-2 vertices – longer than some bound $\ell(k)$ – are indistinguishable with respect to 1-planarity



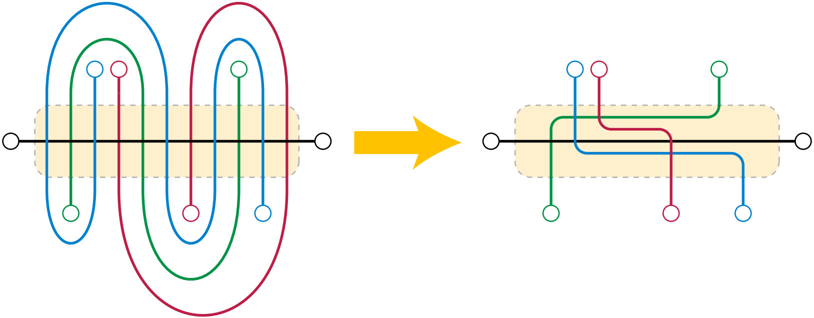
Leads to a simple algorithm:

- Delete degree-1 vertices
- Compress paths longer than $\ell(k)$ to length exactly $\ell(k)$, giving a kernel of size $O(k \cdot \ell(k))$
- Apply the naive algorithm to the resulting kernel
- Uncompress paths and restore deleted vertices, updating drawing to incorporate restored vertices

FPT: Running time $O(n + \text{naive}(\text{kernel size}))$

Rewiring

Suppose that path p is crossed by t other paths, each $\geq t$ times



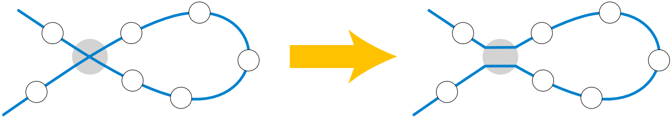
Then can reconnect near p, remove parts of paths elsewhere so:

- Each other path crosses p at most once
- Crossings on other paths do not increase

How long is a long path?

In a crossing-minimal 1-planar drawing, with q degree-two paths:

- No path crosses itself



- No path has $2(q-1)!$ or more crossings
  ...else we have a rewirable sequence of crossings

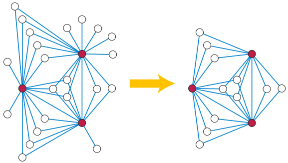
Path length longer than #crossings does not change 1-planarity

$q \leq 3k - 3 \quad \Rightarrow \quad \ell(k) \leq 2(3k-4)! - 1 \quad \Rightarrow \quad$ FPT

FPT algorithms for other parameters

- *k*-almost-tree number: max cyclomatic number of biconnected components
- Vertex cover number: min size of a vertex set that touches all edges

  "the *Drosophila* of fixed-parameter algorithmics" [Guo et al. 2005]
- Tree-depth: min depth of a tree such that every edge connects ancestor-descendant

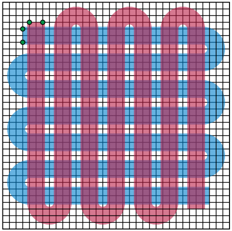


Kernelization for vertex cover

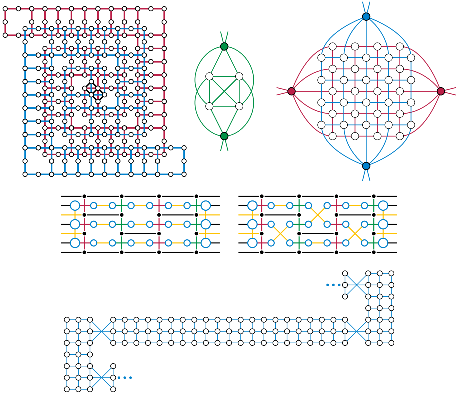
For vertex cover and tree-depth, *existence* of a finite set of forbidden subgraphs follows from known results [Nešetřil and Ossona de Mendez 2012]; difficulty is making dependence *explicit*

Negative results

NP-hard for graphs of bounded treewidth, pathwidth, or bandwidth



Reduction from satisfiability with three parts: substrate (black), variables (blue), and clauses (red)



Some of the gadgets

Conclusions

Results:

- First algorithmic investigation of 1-planarity
- Semi-practical exact exponential algorithm (18-20 edges)
- Impractical but explicit FPT algorithms
- Hardness results for other natural parameters

For future research:

- Make usable by reducing dependence on parameter
- Parameterize by feedback vertex set number?
  Would unify vertex cover and cyclomatic number
- Use similar kernelization for cyclomatic number / almost-trees
  in other graph drawing problems [Bannister et al. 2013]

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