A Brief History of Curves in Graph Drawing

David Eppstein
Computer Science Dept., University of California, Irvine

Workshop on Drawing Graphs and Maps with Curves
Dagstuhl, March 2012

Why curves?

"It is not the right angle that attracts me, nor the straight line, hard and inflexible, created by man. What attracts me is the free and sensual curve—the curve that I find in the mountains of my country, in the sinuous course of its rivers, in the body of the beloved woman."

— Oscar Niemeyer [2000]

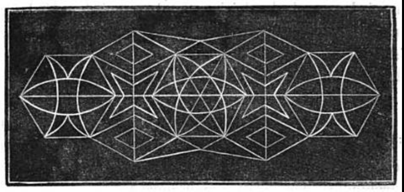
Amazon River near Fonte Boa, Amazonas, Brazil

NASA WORDL WIND 4.1

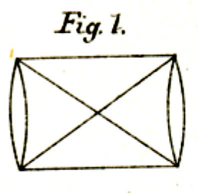
Central_Amazon_River.jpg on Wikimedia commons

Hand-generated graph drawings...

...have long used curves, independently of graph drawing research



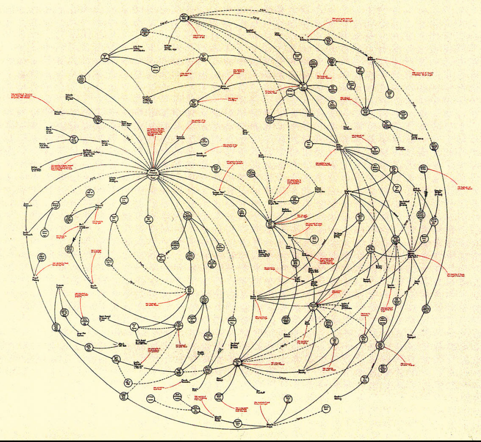
Example of a planar graph with an Euler path [Listing, 1848]



Multigraph for the polynomial
$(x_1 - x_2)^2(x_3 - x_4)^2(x_1 - x_3)$
$(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)$
[Petersen, 1891]

Plimmer et al. [2009] study automated rearrangement of hand-drawn graphs, preserving features such as edge curves

Curved graph drawings as art



Mark Lombardi
(1951–2000)

World Finance Corporation and Associates, ca 1970–84: Miami, Ajman, and Bogota–Caracas (Brigada 2506: Cuban Anti-Castro Bay of Pigs Veteran), 7th version, 1999

Graphite on paper, 69 $^{1}/_{8} \times$ 84 inches

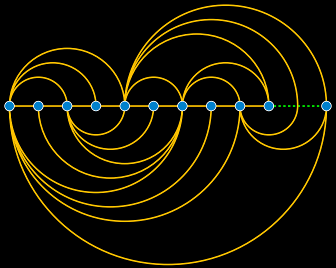
Hobbs [2003], Cat. no. 15, p. 71

Arc diagrams

Vertices placed on a line; edges drawn on one or more semicircles

Used by Saaty [1964] and Nicholson [1968] to count crossings

Minimizing crossings is NP-hard [Masuda et al., 1990];
see Djidjev and Vrt'o [2002], Cimikowski [2002] for heuristics



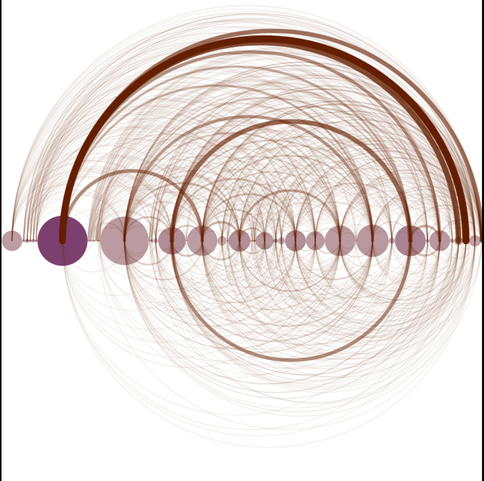
For *st*-planar graphs (or undirected planar graphs) the edges can be oriented left to right, with at most two semicircles per edge [Giordano et al., 2007, Bekos et al., 2013]

Using curvature to indicate directionality

Orient directed edges clockwise from source to destination [Fekete et al., 2003]

A good fit for arc diagrams [Pretorius and van Wijk, 2007]

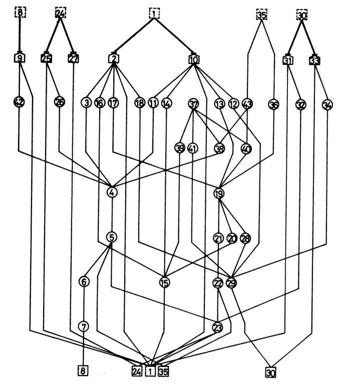
For other layouts, can be confusing to readers [Holten and van Wijk, 2009b]



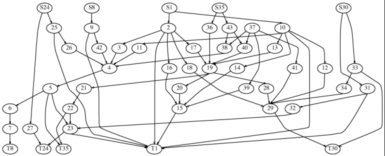
Visualization of internet chat connections, Martin Dittus, 2006, http://datavis.dekstop.de/irc_arcs/

Curving around obstacles

Layered drawings of Sugiyama et al. [1981] subdivide edges by adding dummy vertices drawn as sharp bends (preventing edge-vertex overlaps)



Instead, use dummy vertices to guide splines for smooth edges [Gansner et al., 1988]



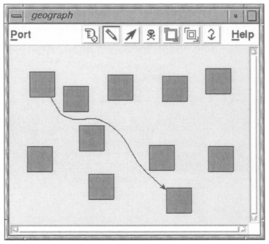
(Both are redrawings of a graph from Forrester [1971].)

Sander [1995] simplifies spline calculation, won 1994 GD contest

Curving around obstacles, II

Dobkin et al. [1997] route edges after vertices have been placed:

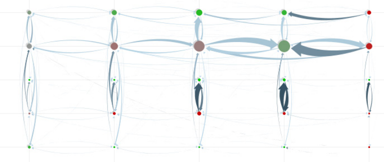
- Find a shortest obstacle-avoiding polyline
- Replace the polyline with a spline curve
- Adjust locally to eliminate intersections with obstacles



Much related work in motion planning on finding smooth curves for a fixed obstacle-avoiding route e.g. Lutterkort and Peters [1999]

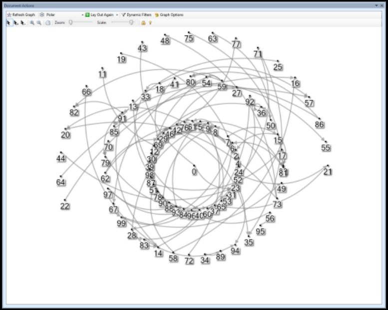
Curving around obstacles, III

PivotGraph [Wattenberg, 2006] places vertices on a grid to indicate two dimensions of multivariate vertex data



Curved edges (clockwise by directionality) avoid intermediate grid points

NodeXL [Smith et al., 2009], "polar" layout places vertices on concentric circles

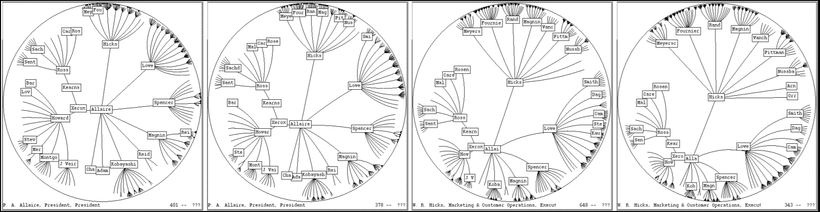


Curved edges avoid inner circles [Cf. Bachmaier et al., 2010]

Focus + context

Sarkar and Brown [1992] suggested interactive *fisheye views* of graphs to zoom in on a point of interest while showing its context

Poincaré model of hyperbolic geometry (with edges drawn as circular arcs) automatically has this effect [Lamping and Rao, 1994]

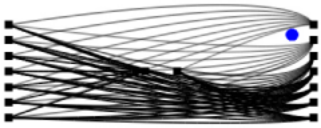


Can morph from one focus to another,
"maintaining the mental map"

Later work on focus + context

Mohar [1999] proves a version of Fáry's theorem (existence of drawings in which each edge follows a geodesic path) for graphs in the hyperbolic plane or on surfaces of negative curvature

Wong et al. [2003] bend edges locally away from a point of interest without distorting vertex placements

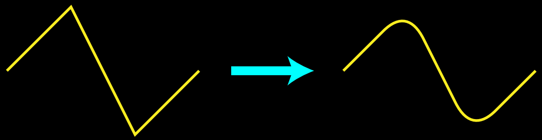


"Edge plucking" allows interactive user control of local bending of bundles of edges [Wong and Carpendale, 2005]

Edge complexity

Much research in graph drawing has focused on drawing styles with angular bends but low *curve complexity* (bends per edge)

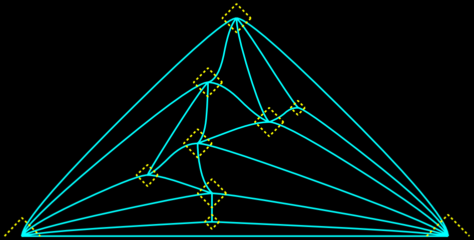
But graph drawing researchers have long known that bends can be replaced by smoothed curves [Eades and Tamassia, 1989]



Bekos et al. [2013] formalize the *edge complexity* of graphs drawn with piecewise-circular-arc smooth curves, as the maximum number of arcs and straight segments per edge. They observe that edge complexity is within a constant factor of bend complexity.

Planar drawings with bounded edge complexity

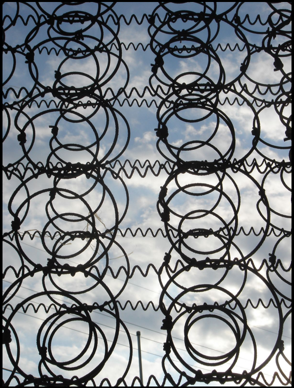
Goodrich and Wagner [2000] modify the (straight line) planar drawing algorithm of Fraysseix et al. [1988] by surrounding each vertex with a protected region of radius proportional to its degree, and placing equally spaced "ports" on the boundary of this region.



Splines through ports have constant edge complexity, near-optimal angular resolution, and do not cross

Similar ideas used by Cheng et al. [2001] and Duncan et al. [2012a]

Force-directed graph drawing (spring systems)



Long a mainstay of practical graph drawing

Use forces (springs) to attract adjacent pairs of vertices and repel other pairs

Somewhat slow but very flexible

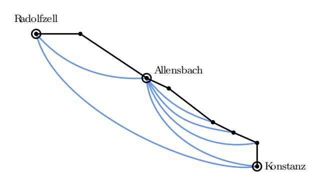
Tutte [1963] shows that springs can generate planar (straight line) drawings of planar graphs; other early research (also using straight edges) by Eades [1984], Kamada and Kawai [1989], and Fruchterman and Reingold [1991]

Force-directed train track bending

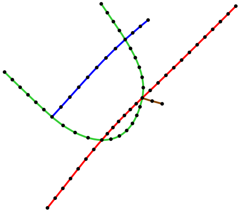
Brandes and Wagner [2000] draw graphs for train systems:

- Vertices are train stations
- Edges connect consecutive stations on the same line

Problem: Express train connections overlap local train stations
Solution: Bend them outwards using forces on spline control points



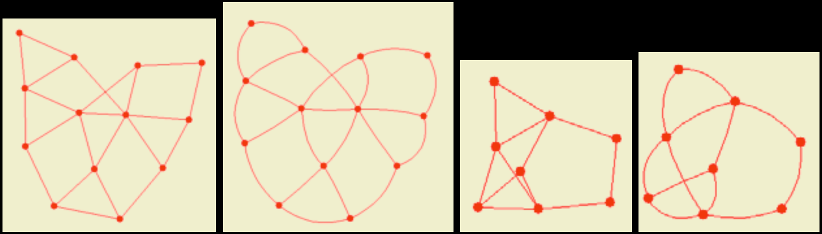
Fink et al. [2013] use force-directed drawing to schematize train system maps (replacing paths of degree-two vertices by splines)



Bending outwards can also be used in 3d to separate edges of geographic graphs from the Earth's surface [Munzner et al., 1996]

Forces and curves for arbitrary graphs

Finkel and Tamassia [2005] place a new vertex in the middle of each edge of a given graph, apply force-directed layout, and then use the new vertices as spline control points

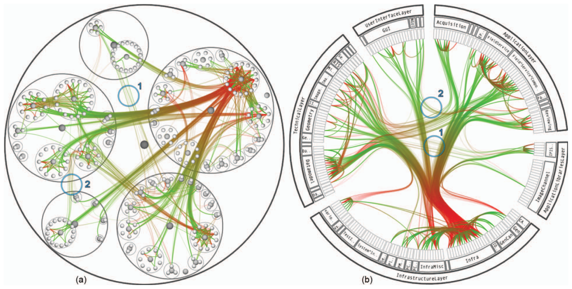


They report that this gives significant improvements in angular resolution and modest improvements in crossings

Similar ideas used for circular arcs by Chernobelskiy et al. [2012]

Edge bundling

In a hierarchically clustered graph, group edges that connect the same two clusters (at some level of the hierarchy) into "bundles" drawn as nearly-parallel curves

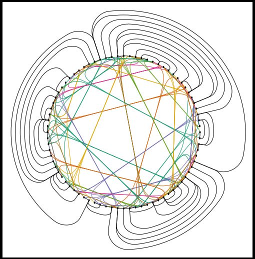


Introduced by Holten [2006] based on a physical analogy to electrical wiring bundles; closely related to flow maps for numerical geographic data [Phan et al., 2005]; hundreds of successor papers

Refinements and variations of edge bundling

Non-hierarchical bundling by modeling edges as springs that attract each other [Holten and van Wijk, 2009a]

Circular layout, unbundled edges outside (chosen to minimize crossings), ink-minimizing bundles inside [Gansner and Koren, 2007]



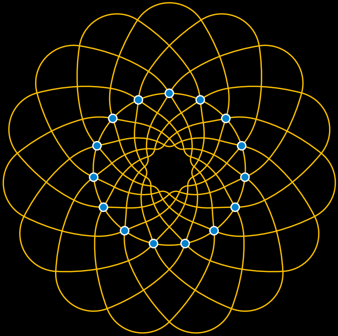
Edge bundling in Sugiyama-style layered drawing [Pupyrev et al., 2011]

Forbidding crossings inside bundles, routing edges on parallel tracks resembling metro maps, so edges are easier to follow [Bereg et al., 2012]

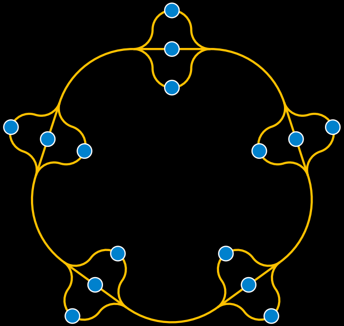
For a taxonomy of bundling-related curved edge techniques see Riche et al. [2012]

Confluent drawing

Represent a graph using *train tracks* (smooth one-dimensional cell complexes) rather than individual edges; vertices are adjacent if connected by a smooth curve along a track [Dickerson et al., 2004]



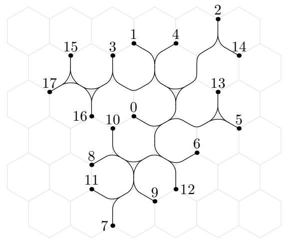
GD 2011 contest winner



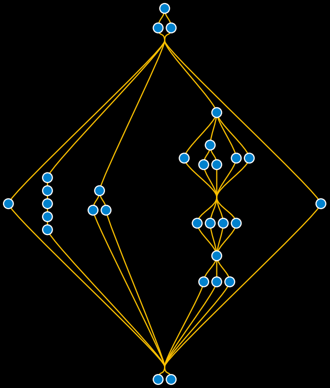
Same graph drawn confluently

Graphs with crossing-free confluent drawings

Interval graphs [Dickerson et al., 2004] and distance-hereditary graphs [Eppstein et al., 2006, Hui et al., 2007] have non-crossing confluent drawings

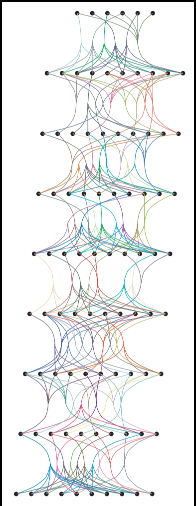


A partial order has an upward-planar confluent *Hasse diagram* if and only if its order dimension is at most two [Eppstein and Simons, 2012]



Combinations of confluence with other styles

Confluent edge routing in Sugiyama-style layered drawing [Eppstein et al., 2007]

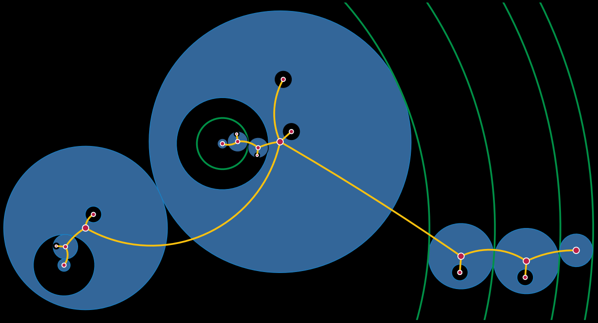


GD 2003 contest winner

Use confluence to draw high-degree planar graphs with axis-parallel edges [Quercini and Ancona, 2011]

Lombardi drawing of trees

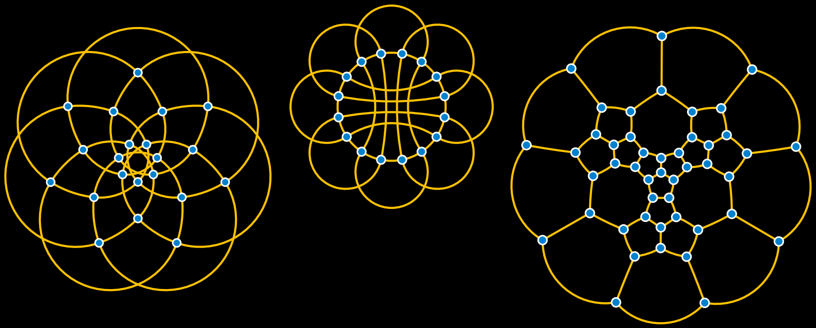
A very strict layout style: edges must be circular arcs, meeting at equal angles at each vertex



Allows plane trees to be drawn in balloon style (subtrees drawn recursively in disks surrounding root node) with polynomial area, not true for straight line drawing styles [Duncan et al., 2013]

Lombardi drawing of graphs

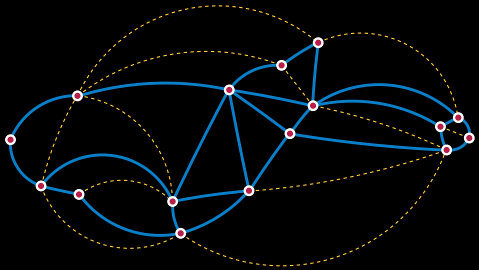
Works for regular graphs and symmetric graphs [Duncan et al., 2012b], planar graphs with max degree three [Eppstein, 2013], some other special cases [Löffler and Nöllenburg, 2013]



... but not for other graphs, causing researchers in this area to resort to force-based approximations [Chernobelskiy et al., 2012] or multi-arc relaxations [Duncan et al., 2012a]

Additional alternatives

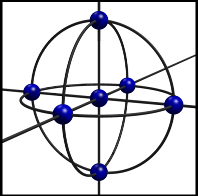
- Choosing one of two arcs per edge, with fixed vertex locations, to avoid crossings [Efrat et al., 2007]
- Angle-optimized arc-triangulations [Aichholzer et al., 2012]



- Circle-arrangement representations of 3-connected 4-regular planar graphs [Bekos and Raftopoulou, 2013]

Some open problems

- Planar drawings can always be made with straight edges (Fáry's theorem) but some other types of drawing cannot. What can we say about edge complexity of crossing-minimal drawings, 1-planar drawings, or thickness-1 drawings?
- With few exceptions [e.g. Munzner et al., 1996], most work on curved drawings has been two-dimensional. What can we say about 3d curved graph drawing?
- What is the complexity of recognizing graphs with non-crossing confluent drawings?



3d Lombardi drawing of K_7

- Does every outerplanar graph have an outerplanar Lombardi drawing? Is it NP-hard to recognize graphs with Lombardi drawings? (Known for circular layouts but not in general.)

Do curves work?

User studies on curves vs line segments in force-directed drawings found mixed results on aesthetics, little or no improved usability [Xu, 2012, Purchase et al., 2013]

Public domain image Flatout_100,_S-Bends_in_Ireland.jpg by Osioni on Wikimedia commons

Biggest future challenge: designing methods for curved drawing that are general, usable, and more consistently beautiful

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