Hyperconvexity and Metric Embedding

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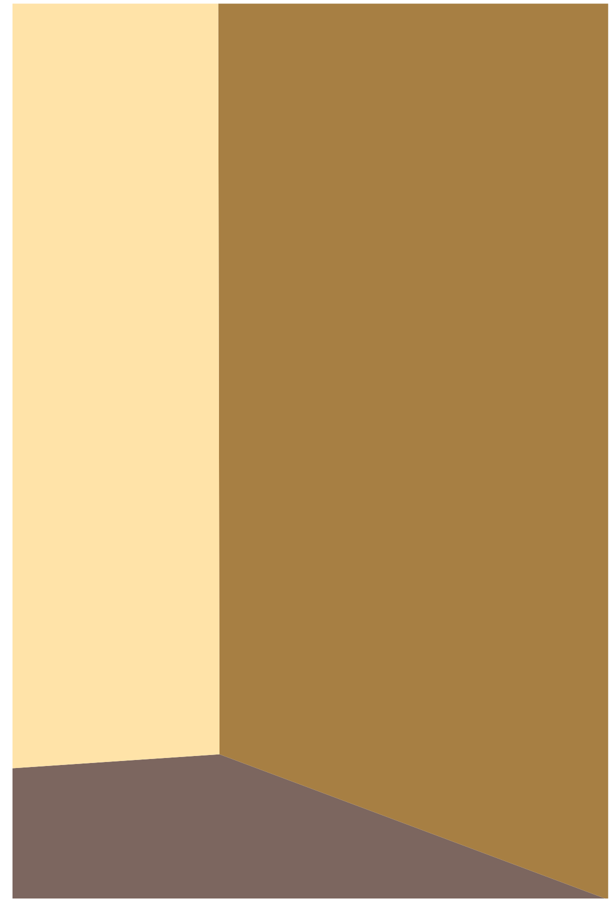
Metric embedding problems

Input: complicated metric space

Quadratic # degrees of freedom
e.g. distance matrix
shortest paths in weighted graph
n-dimensional L-infinity space



Output: simple metric space

As close as possible to
the input metric space

Linear # degrees of freedom

Hyperconvex metric spaces

Defined by Helly property on metric balls
Include trees, L_∞ metrics, other interesting spaces

Tight span

Embed any metric space into a hyperconvex space
"Convex hull" for metric spaces

Algorithms

Diameter in hyperconvex spaces
Planar tight spans and Manhattan embedding
Minimum dilation stars

Outline

Hyperconvexity

Tight spans

Planar tight span construction

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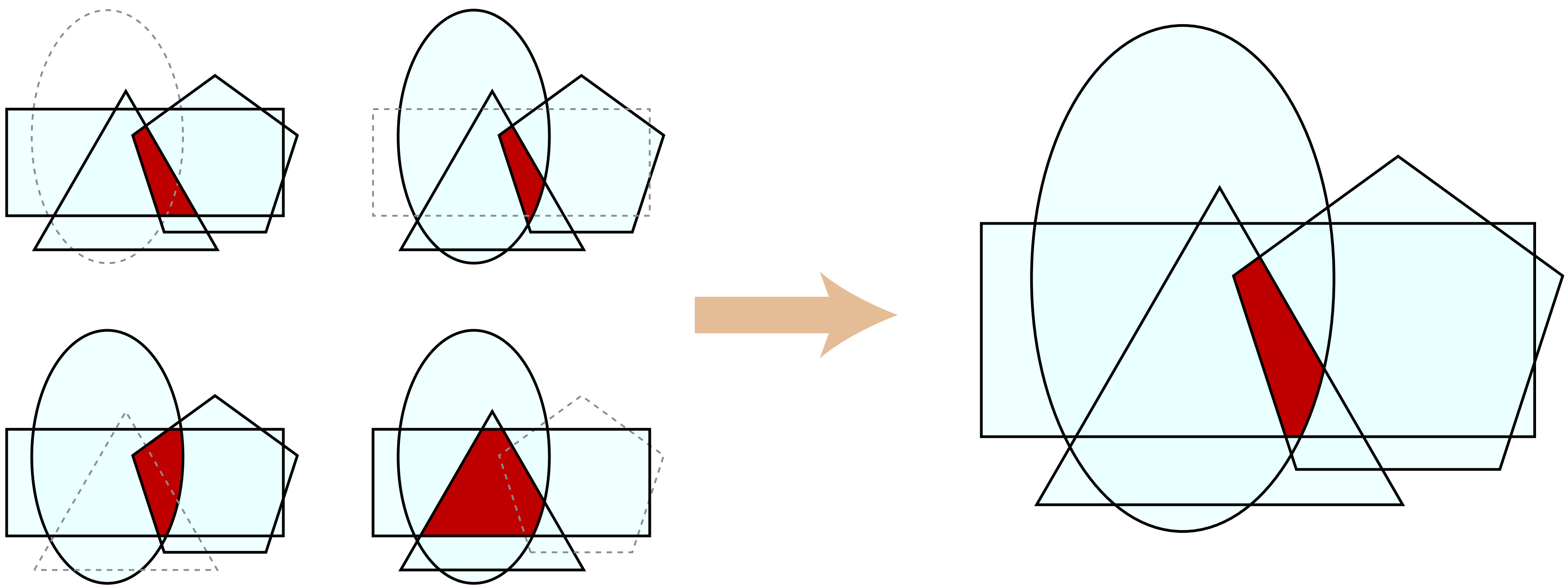
Helly's theorem [Helly 1923]

Given a family of convex objects in d-dimensional Euclidean space:

If each (d+1)-tuple has a common intersection, so does the whole family



k-Helly family

Any family of sets such that, for any subfamily,
if all k-tuples in the subfamily intersect,
then the whole subfamily has a common intersection

(Like convex sets in (k–1)-dimensional Euclidean space)

Helly family

Special case of a 2-Helly family

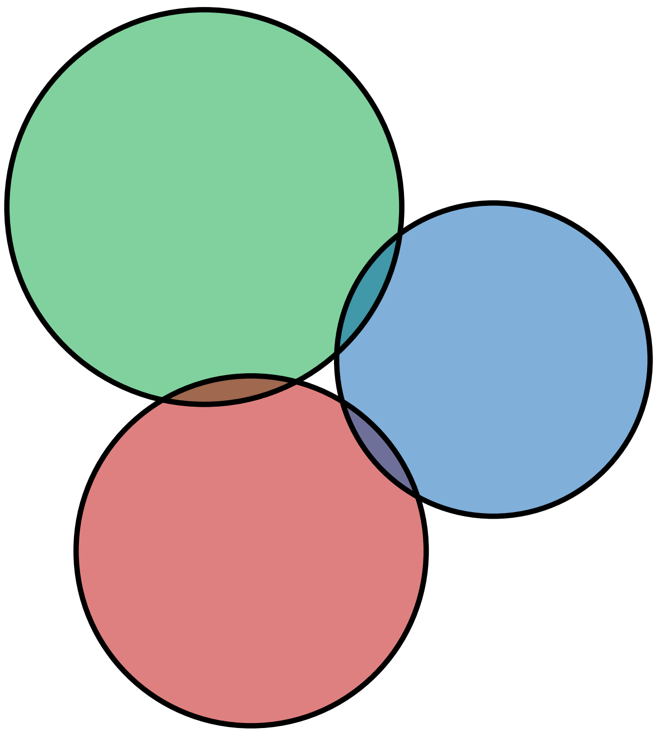
(the simplest nontrivial case of a k-Helly family,
like intervals of the real line)

Hyperconvex (aka injective) metric spaces:

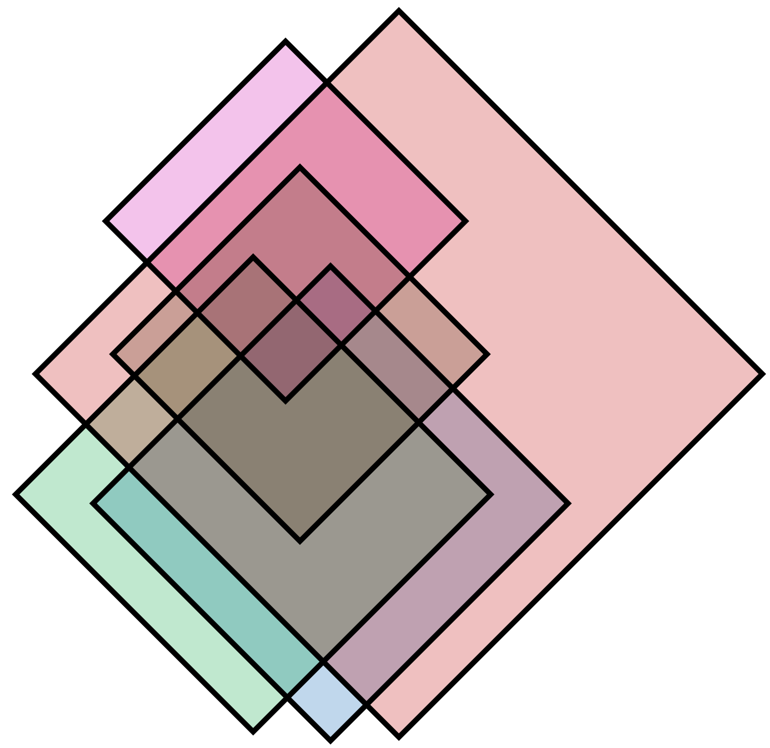
[Aronszajn and Panitchpakdi 1956; Isbell 1964]

Balls form a Helly family, and

If two balls have radii adding to at least their center separation, they intersect
(equivalently, any two points have a geodesic connecting them)



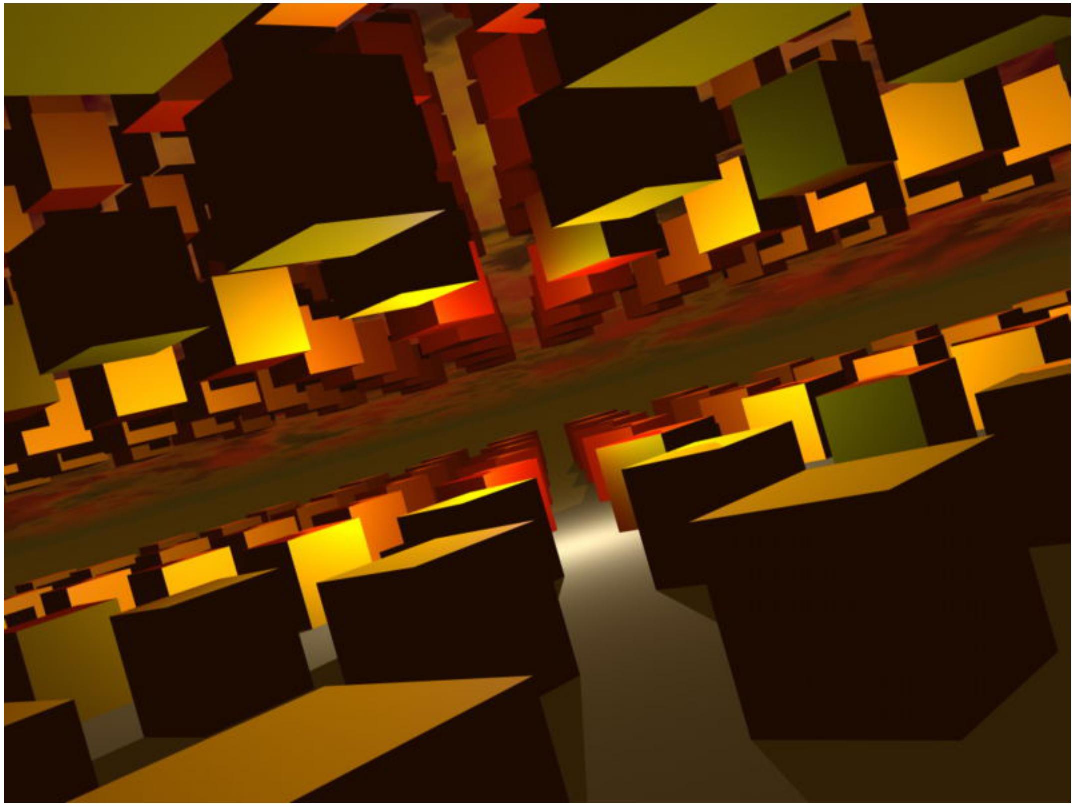
Euclidean plane: not hyperconvex



L_1 plane: hyperconvex

Example: Chebyshev distance

Points = d-dimensional vectors of real numbers
Distance = maximum coordinatewise difference (L_∞ norm)



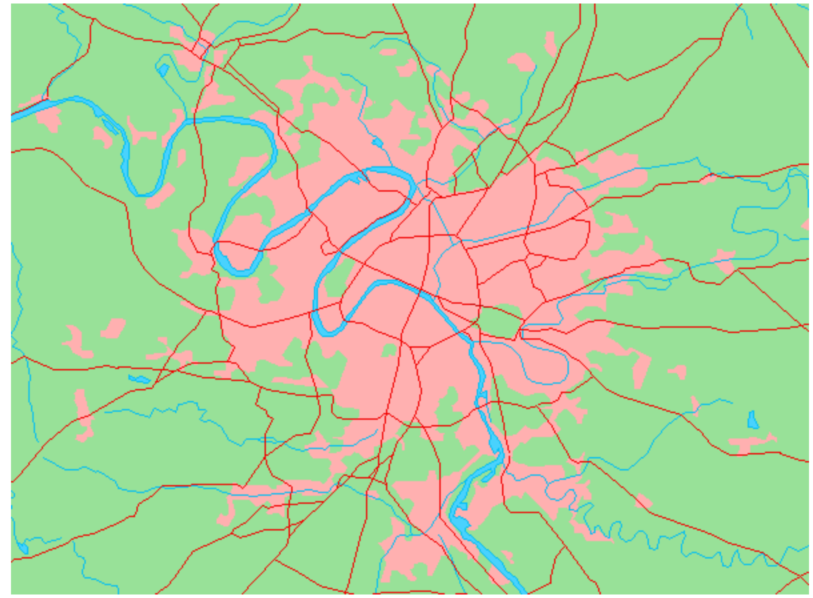
Metric balls = axis-aligned cubes
Family of balls has common intersection iff they intersect in each coordinate

Example: Paris metric (hedgehog space)

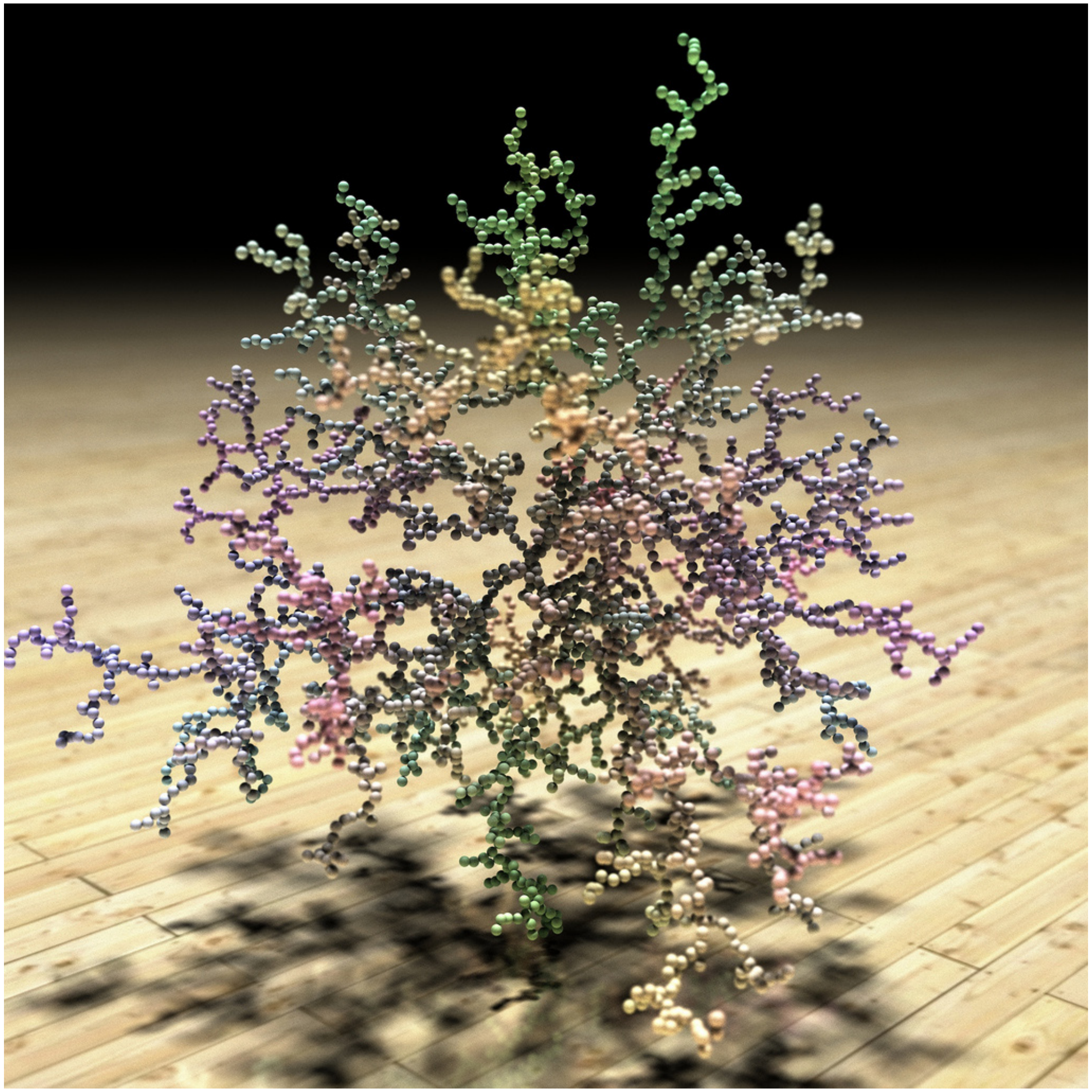
Points = plane with polar coordinates

All geodesics follow rays through origin

$\text{Distance}((\rho_1,\theta_1), (\rho_2,\theta_2))$
$= |\rho_1 - \rho_2|$ if $\theta_1 = \theta_2$,
$= |\rho_1| + |\rho_2|$ otherwise



Example: Real trees



Metric spaces in which

- Any two points are endpoints of a unique simple curve

- Curve length = distance

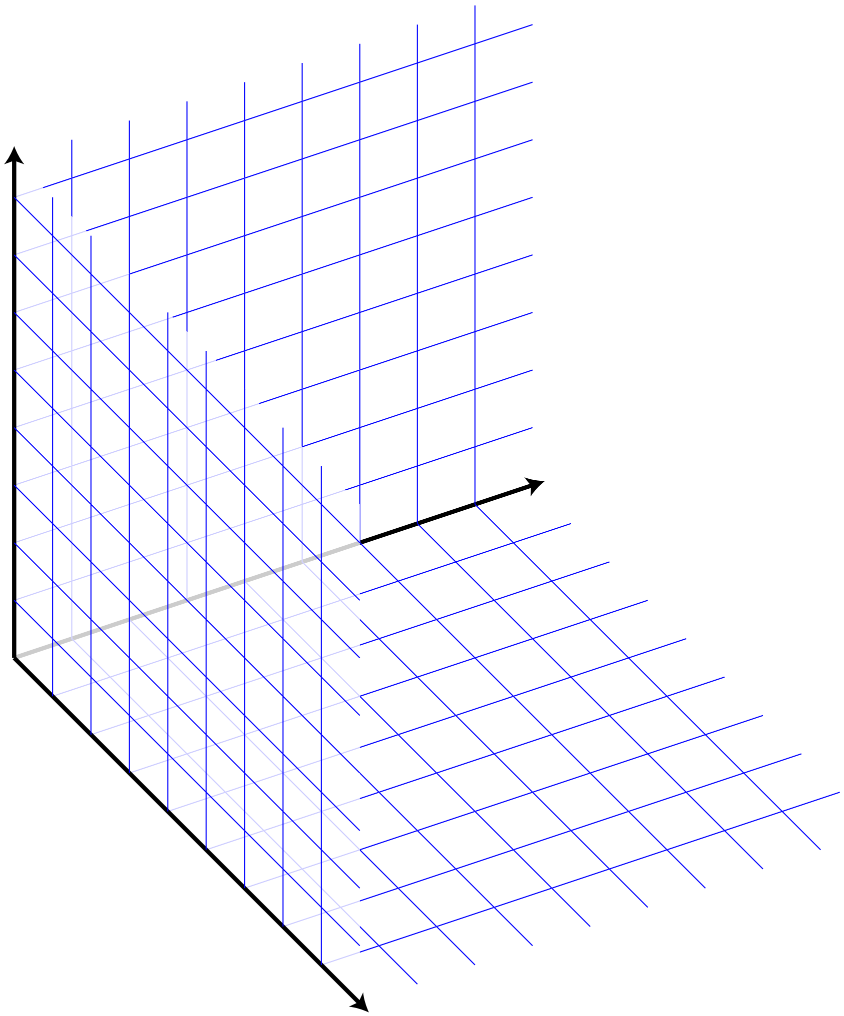
Include:

- Paris metric

- Acyclic connected undirected graphs, with edges replaced by line segments

- Diffusion-limited aggregation

Example: Manhattan orbifolds [E., arXiv:math/0612109]



Space locally modeled on the Manhattan (L_1 or L_∞) plane

Topologically, must be a simply-connected 2-manifold

Geometrically, most points must have neighborhood isometric to L_1 neighborhood of the origin

Discrete "cone points" allowed (e.g. cone of order 3 on boundary of L_1 orthant, depicted)

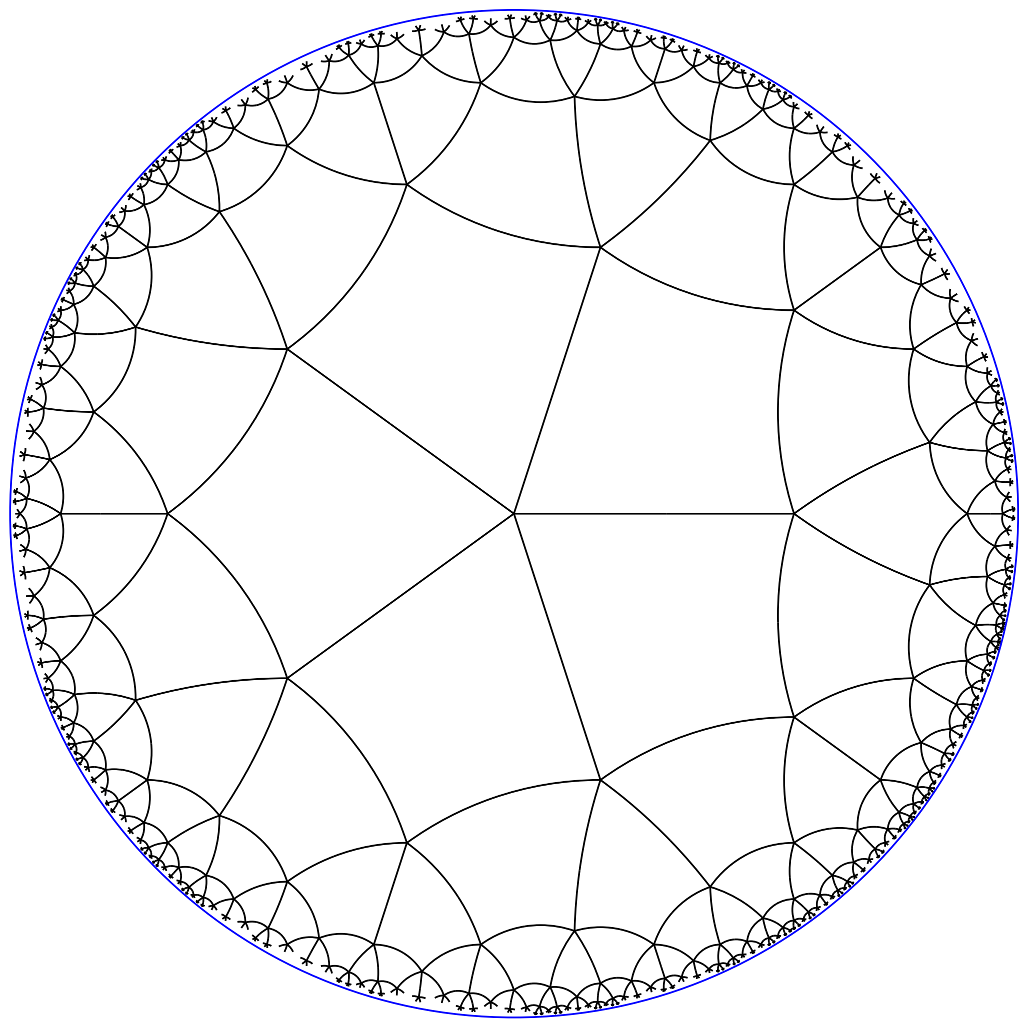
All cone points must have order > 4

Manhattan orbifold approximation to hyperbolic plane

(analogous to Manhattan plane as an approximation to Euclidean)

Form tesselation by quadrilaterals meeting five to a vertex

Fill each quadrilateral by an L_1 metric square



An algorithmic application

1-median: optimal location for facility minimizing sum of distances from given sites (e.g. minimize travel time from house to office, shopping, and beach)

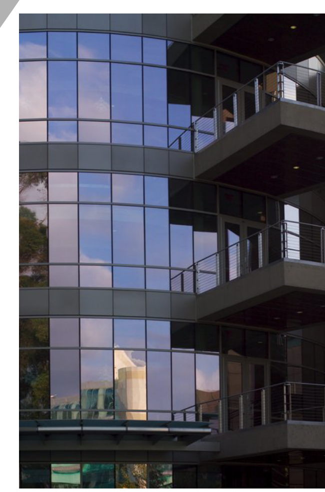
In Euclidean spaces:

Solution is root of high degree polynomial
Must be approximated numerically

In hyperconvex spaces:

Linear program
Variable = distance to site
Two vars per inequality

Strongly polynomial
[Aspvall & Shiloach 1979;
Lueker, Megiddo, & Ramachandran 1986]



Outline

Hyperconvexity

Tight spans

Planar tight span construction

Minimum dilation stars

The big picture:

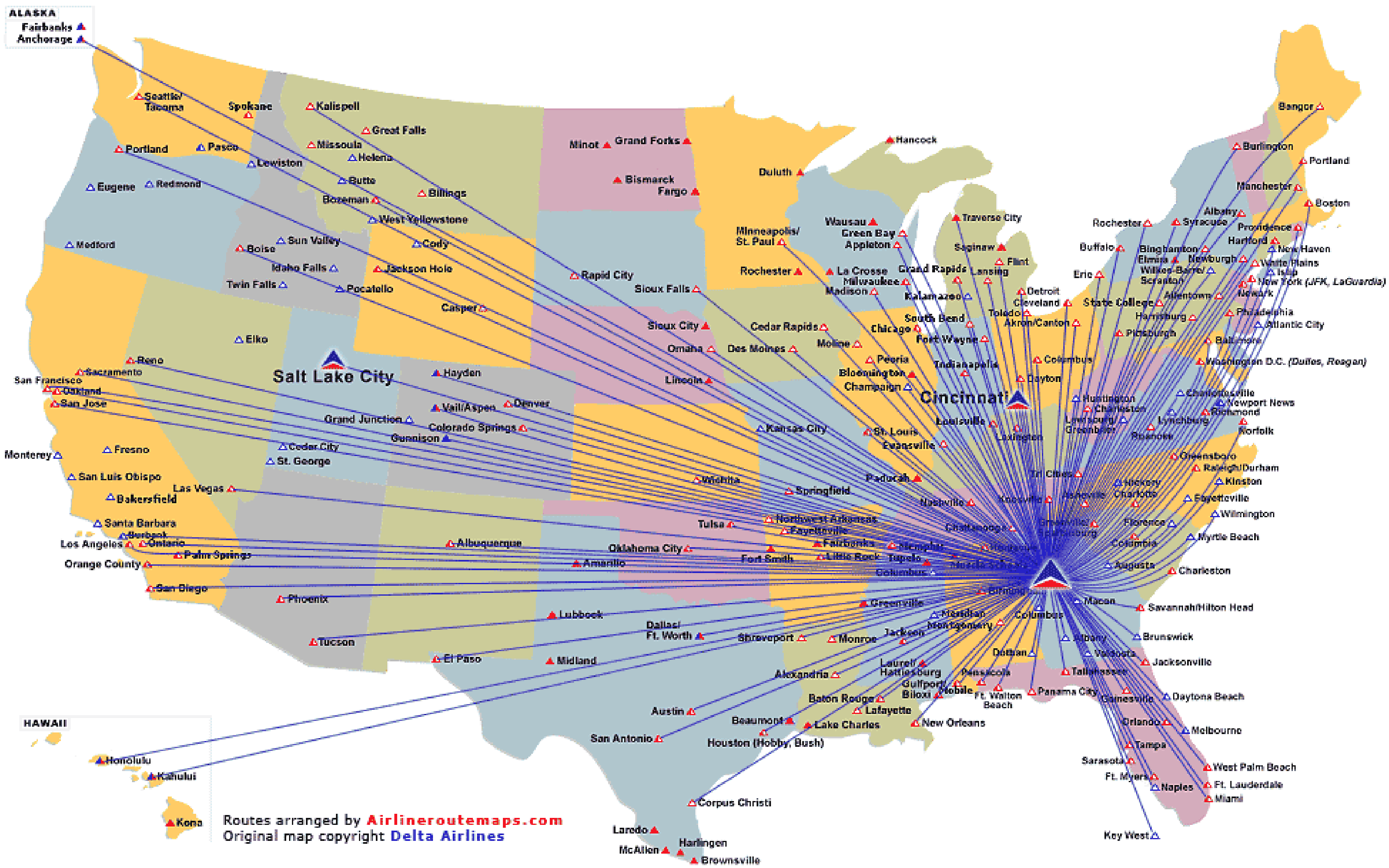
Any metric space (X,d) can be embedded without distortion into a minimal hyperconvex space, its tight span

Each point of the tight span represents embedding of (X,d) into a star metric (allowing distortion)

The 200-inch telescope at Mt. Palomar, DE, 08/01

Star metrics

Defining property: there exists a hub h that is between every other pair of points



For all s and t, $d(s,t) = d(s,h) + d(h,t)$

Once distances from the hub are known, all other distances are determined

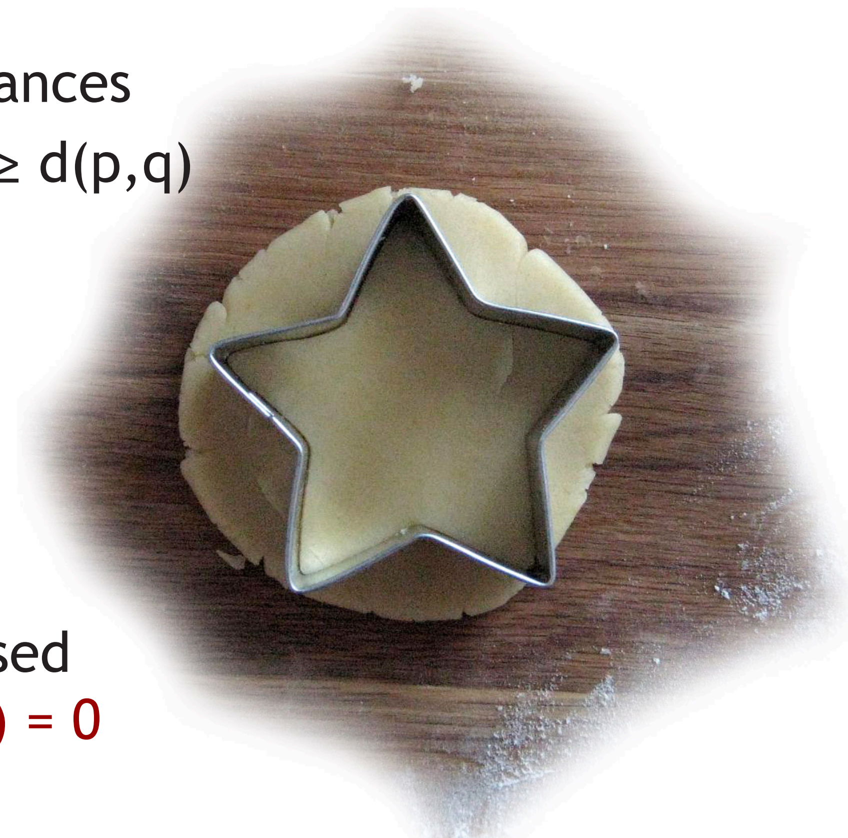
Star embedding

A map from a metric space (X,d) to a star
may be represented as a function D from X to $\mathbf{R}^+$
(D(x) = distance from image of x to hub)

Embedding: a map that
may distort but does not decrease distances

Equivalently: for all p and q, $D(p) + D(q) \geq d(p,q)$

Technical condition for infinite X:
$|D(x) - d(x,0)|$ must be bounded
for an arbitrary base point 0



The embedding is minimal if
no hub distance D(x) may be decreased
i.e., for all p, $\inf_q D(p) + D(q) - d(p,q) = 0$

Interstellar distances

Let D(x) and E(x)
be two star embeddings
(distances from x to hub centers)

Define Distance(D,E) = $\sup_x |D(x) - E(x)|$

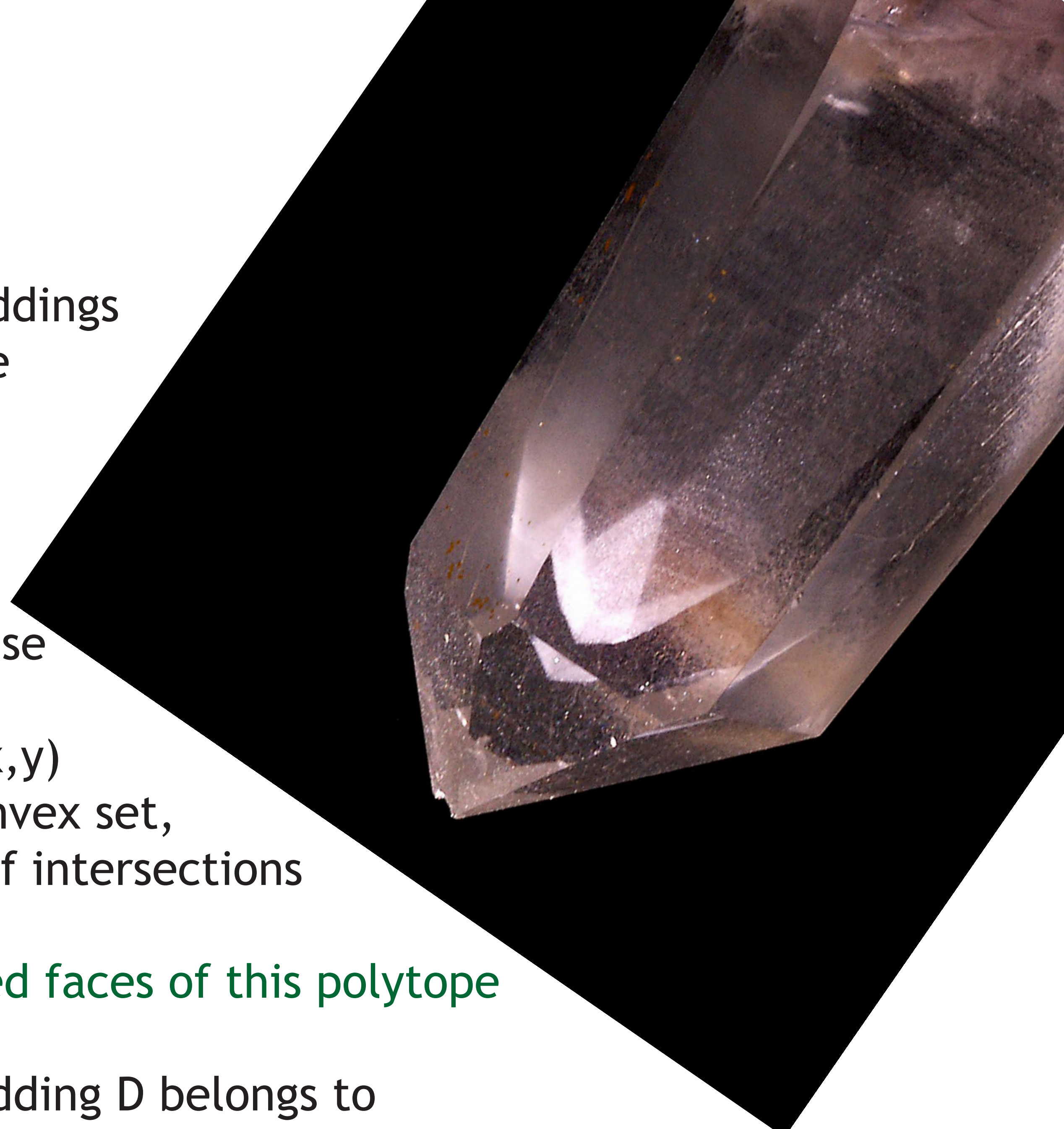
The original metric space (X,d)
embeds without distortion
into the space of minimal stars:

Map point x to function $D_x(y) = d(x,y)$

$\text{Distance}(D_x,D_y) = \sup_z |d(x,z) - d(y,z)|$
$= d(x,y)$
by triangle inequality for d

http://xkcd.com/482/

Hyperconvexity of minimal stars



The distance on star embeddings is just the L_∞ metric on the functions from X to **R**

Common intersection of pairwise intersecting balls may be found coordinatewise

Constraint $D(x) + D(y) \geq d(x,y)$ defines an (unbounded) convex set, doesn't change existence of intersections

Minimal stars lie on bounded faces of this polytope

If some non-minimal embedding D belongs to a set of balls centered at minimal stars, decrease distances in D to find a minimal embedding in the intersection

The tight span of a metric space (X,d)

[Isbell 1964, Dress 1984, Chrobak & Larmore 1994, Develin & Sturmfels 2004]

Equivalently (up to isometry), may be defined as:

1. The hyperconvex metric space of minimal star embeddings of (X,d)

2. The points on the bounded faces of the polytope $D(x) + D(y) \geq d(x,y)$ (with the L_∞ metric)

3. The tropical convex hull of the distance vectors

4. The smallest hyperconvex superspace (if (X,d) is a subspace of any hyperconvex space, so is the tight span)

5. The injective hull in the category of metric spaces and distance-decreasing functions

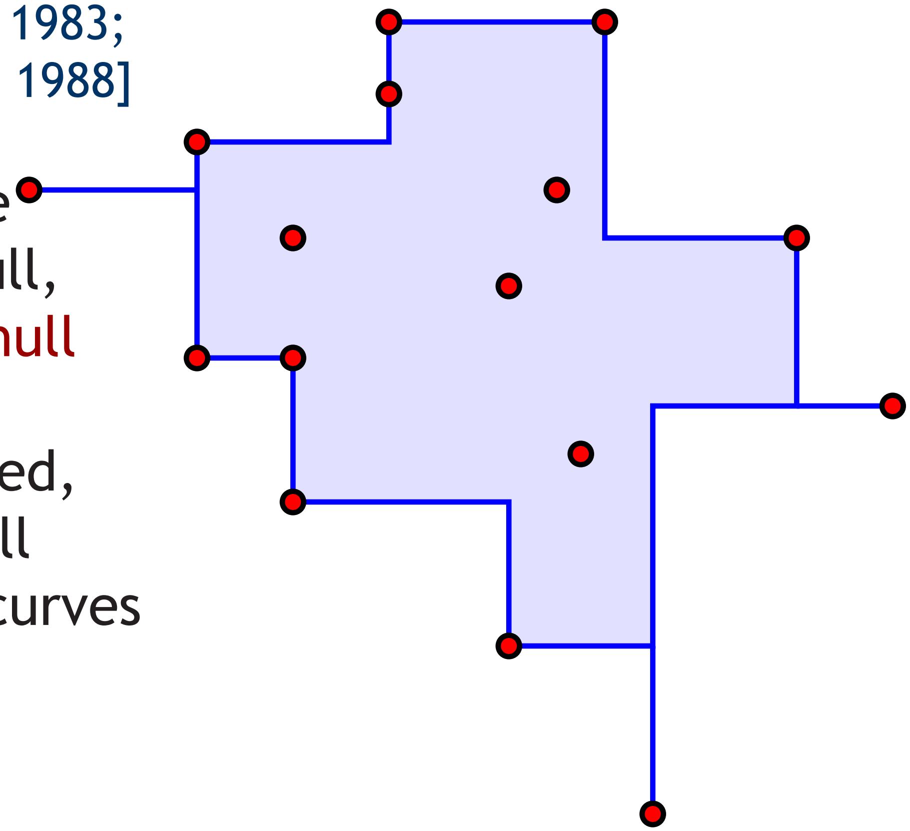
Tight spans in the Manhattan plane

Orthogonal convex set:
intersects any axis-parallel line in an interval

Orthogonal convex hull:
Intersection of all orthogonally convex connected supersets
[Montuno & Fournier 1982; Nicholl et al 1983;
Ottman et al 1984; Karlsson & Overmars 1988]

For any subset of the L_1 plane
with a connected orthogonal hull,
tight span = orthogonal convex hull

If orthogonal hull is not connected,
tight span = components of hull
connected together by monotone curves



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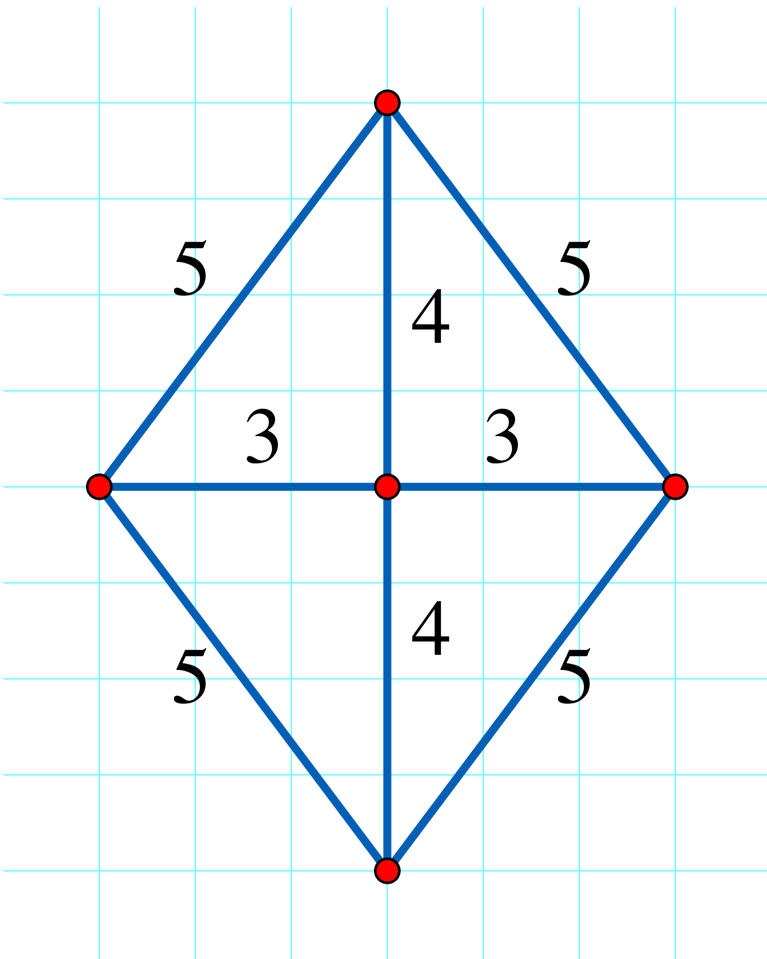


Easy warm-up problem:

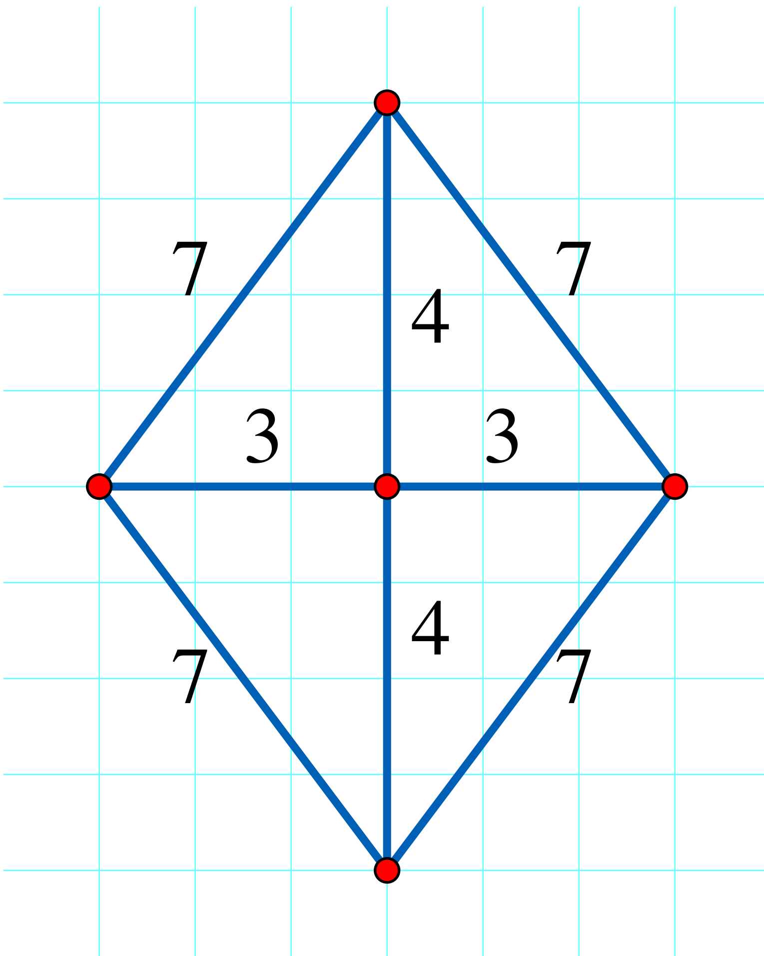
Given a finite metric space
can it be represented as
distances in Euclidean plane?

Find 3 non-collinear points
Place them in a triangle
Use distances from these points
to uniquely place each other point

Then verify all distances correct

Total time $O(n^2)$
linear in size of input distance matrix

Much additional research exists
when distance matrix is partial
or corrupted...



Same problem for Manhattan plane
(L_1 or L_∞ distance):

Given a finite metric space
can it be represented
as a set of points in the L_1 plane?

More difficult because of ambiguities:
three points do not uniquely
determine the location of the rest

Edmonds [Disc. Comput. Geom. 2008]:
algorithm with $O(n^2 \log^2 n)$ time

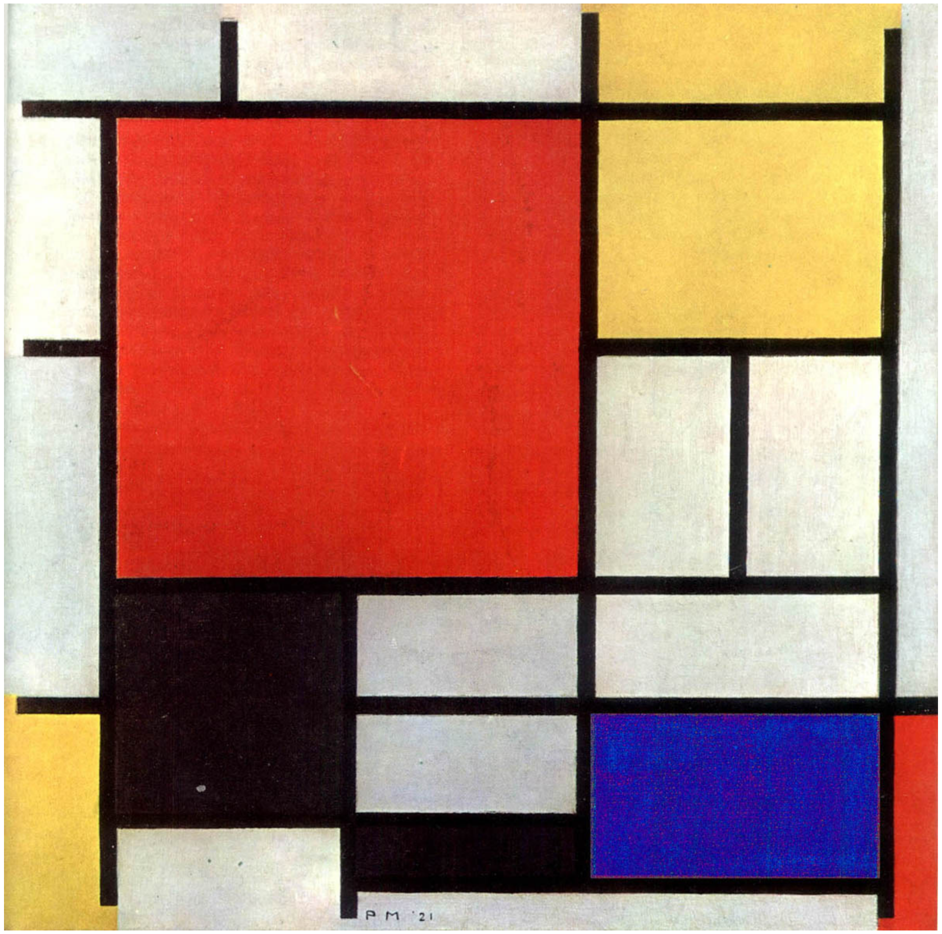
New result:
$O(n^2)$, linear in input size
based on finding planar tight span
then testing whether
span has Manhattan embedding

Main ideas of algorithm:

Build up metric space
by adding points one by one

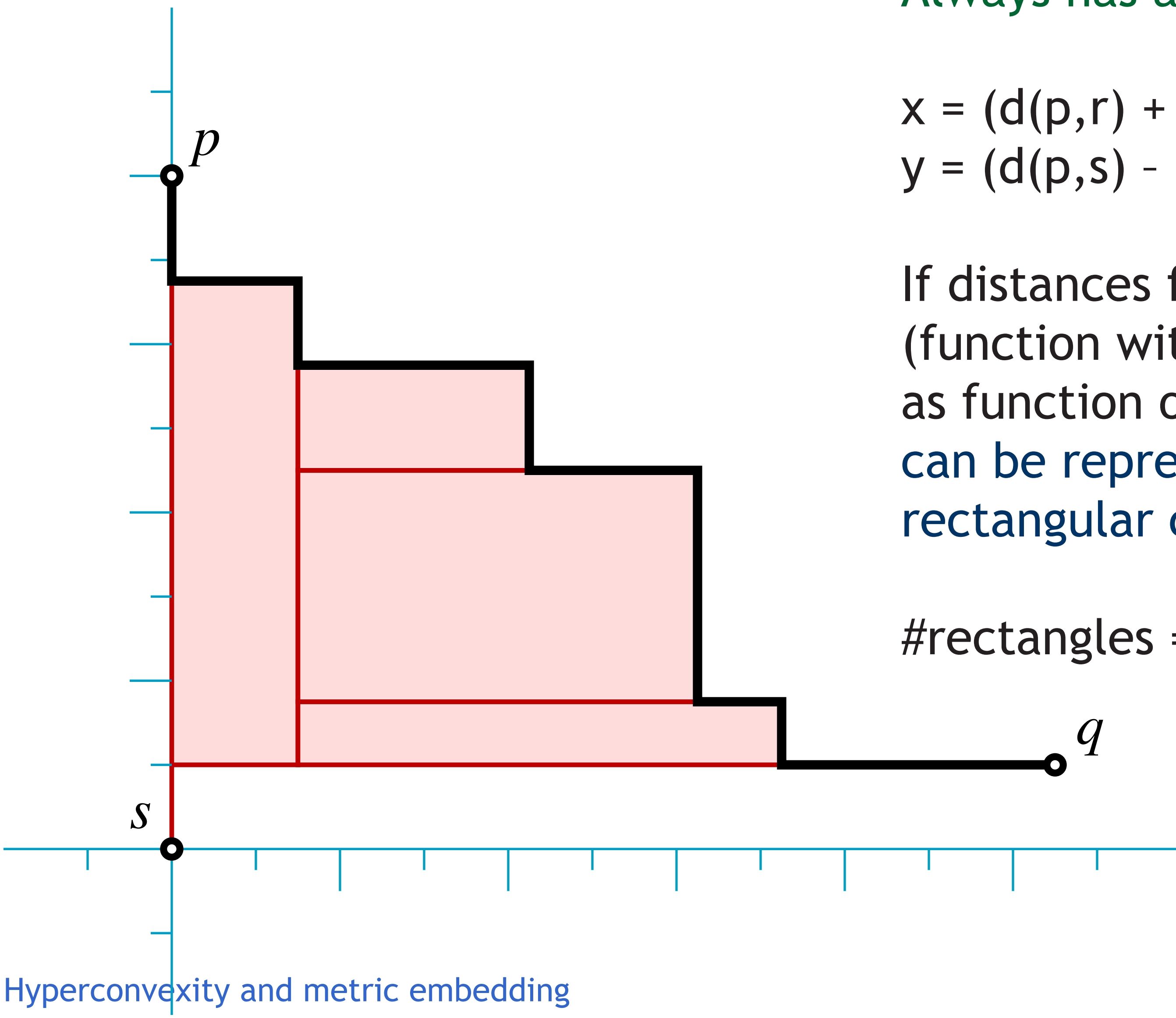
Maintain tight span of added points
as a complex of rectangles
with L_1 geometry
(not necessarily edge-to-edge)

Characterize the
rectangle complexes
that have Manhattan embeddings



Piet Mondrian, 1921

Important special case:
tight span of a geodesic path pq and one added point s



Always has an L_1 plane embedding:

$x = (d(p,r) + d(r,s) - d(p,s))/2$
$y = (d(p,s) - d(p,r) + d(r,s))/2$

If distances from s form a sawtooth (function with derivative +/– 1) as function of path position, then can be represented as a rectangular complex

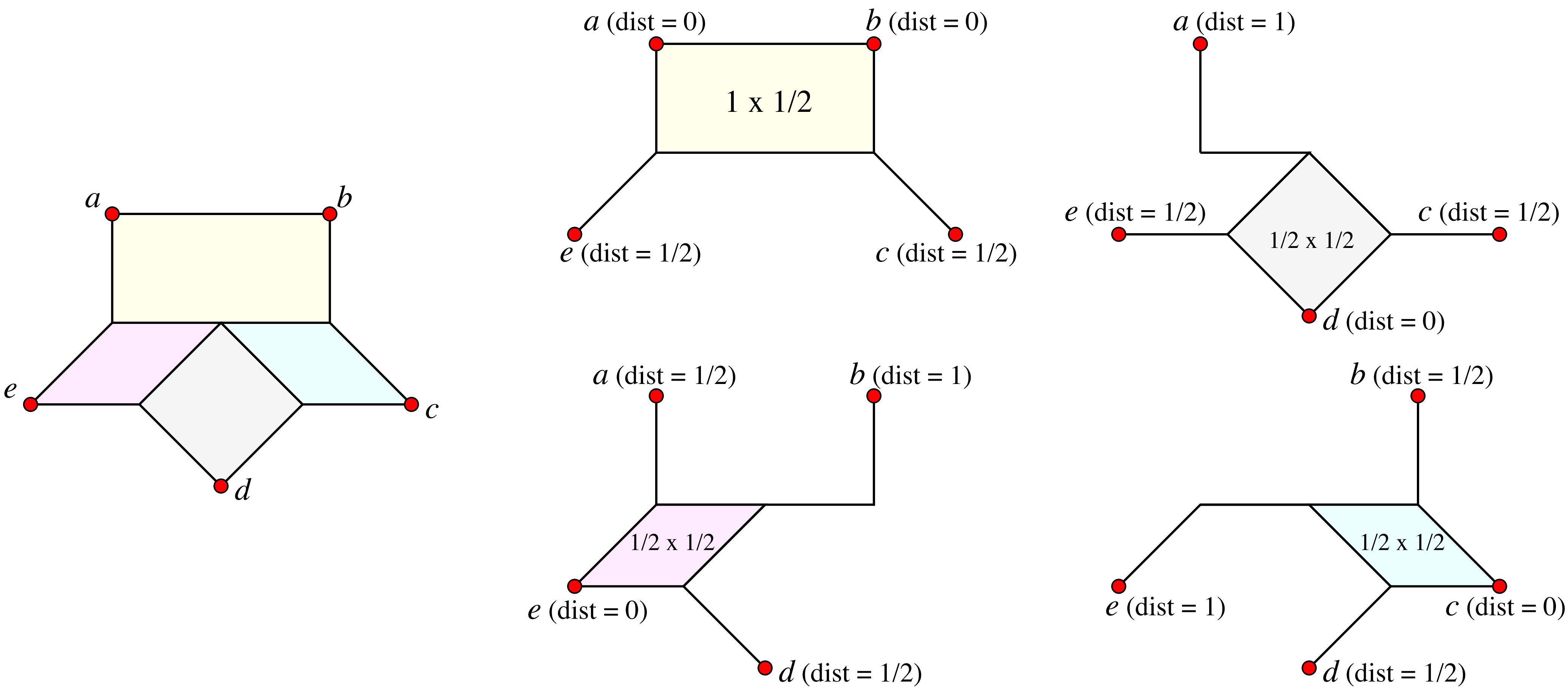
#rectangles = #sawtooth breakpoints – 1

Using landmarks to measure distances within rectangles

Each rectangle corner c_i stores a site s_i having c as closest point in the rectangle

If r is in rectangle and p is anywhere, **$dist(p,r) = max_i\ d(p,s_i) - d(r,c_i) - d(c,s_i)$**

Same idea works with two landmarks for edges of complex



Planar tight span algorithm

maintain a rectangular complex with landmarks
(initially empty)

for each point p of the metric space
(in an arbitrary order):

> compute distances to p within each cell of the complex
>
> if p is at zero distance to some point of complex:
>> add it there, verify distances to other sites
>
> else if local minimum of distance is interior to a rectangle:
>> tight span is not planar
>
> else:
>> use distance computations to determine attachment path
>> compute tight span of path+point and add to complex

Which planar tight spans have a Manhattan embedding?

Several easily-checked necessary and sufficient conditions, amounting to:

- Each point of the rectangle complex must be locally Manhattan

- Each biconnected component of the complex must have exactly four extreme boundary edges (both endpoints convex)

- Articulation points must lie on boundary edges

- Any biconnected component must have at most one neighboring component in each of its four directions (biconnected neighbors take up two directions each)

Planar tight span summary

The number of new rectangles created by each point insertion
is balanced by the number of vertices removed from the boundary

Therefore, total size of the rectangular complex = $O(n)$

Inserting each point takes time proportional to size of updated complex

Therefore, total time to create tight span = $O(n^2)$, linear in input size

Testing whether the span has a Manhattan embedding also takes $O(n)$

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How to measure the quality of a metric embedding?

Dilation: how much farther you would have to travel in the new metric

Scale the new metric so that all distances are at least as large as they were in the old metric

Dilation = maximum ratio
new distance / old distance
among all pairs of input points

(factor by which some distances grow)

Minimum dilation star problem

Given an n by n matrix D[s,t] of distances in a metric space

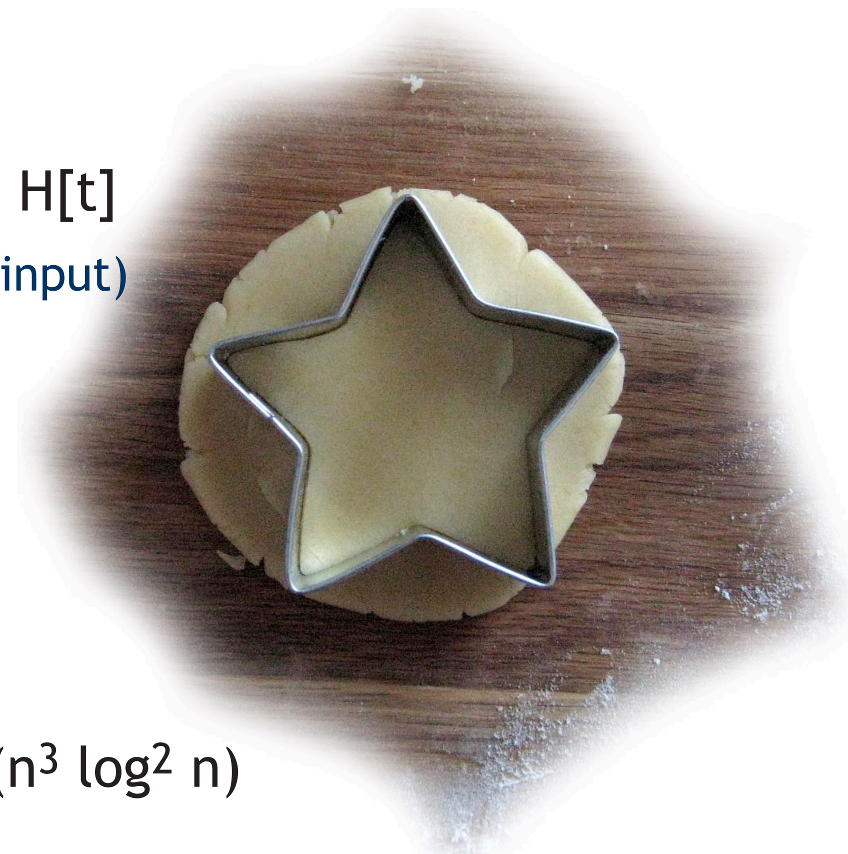
Produce a vector H[s] of distances
from each input point to a new hub

Satisfying, for all s and t, $D[s,t] \le H[s] + H[t]$

(scale output distances to be at least as big as input)

Minimize $\max_{s,t} (H[s] + H[t])/D[s,t]$

(find the star with optimal dilation)



We solve this in **strongly polynomial time** $O(n^3 \log^2 n)$

Related Work

Metric embedding into L1, L2, etc. with guaranteed dilation bounds
[Large literature]

Does not find optimal embedding

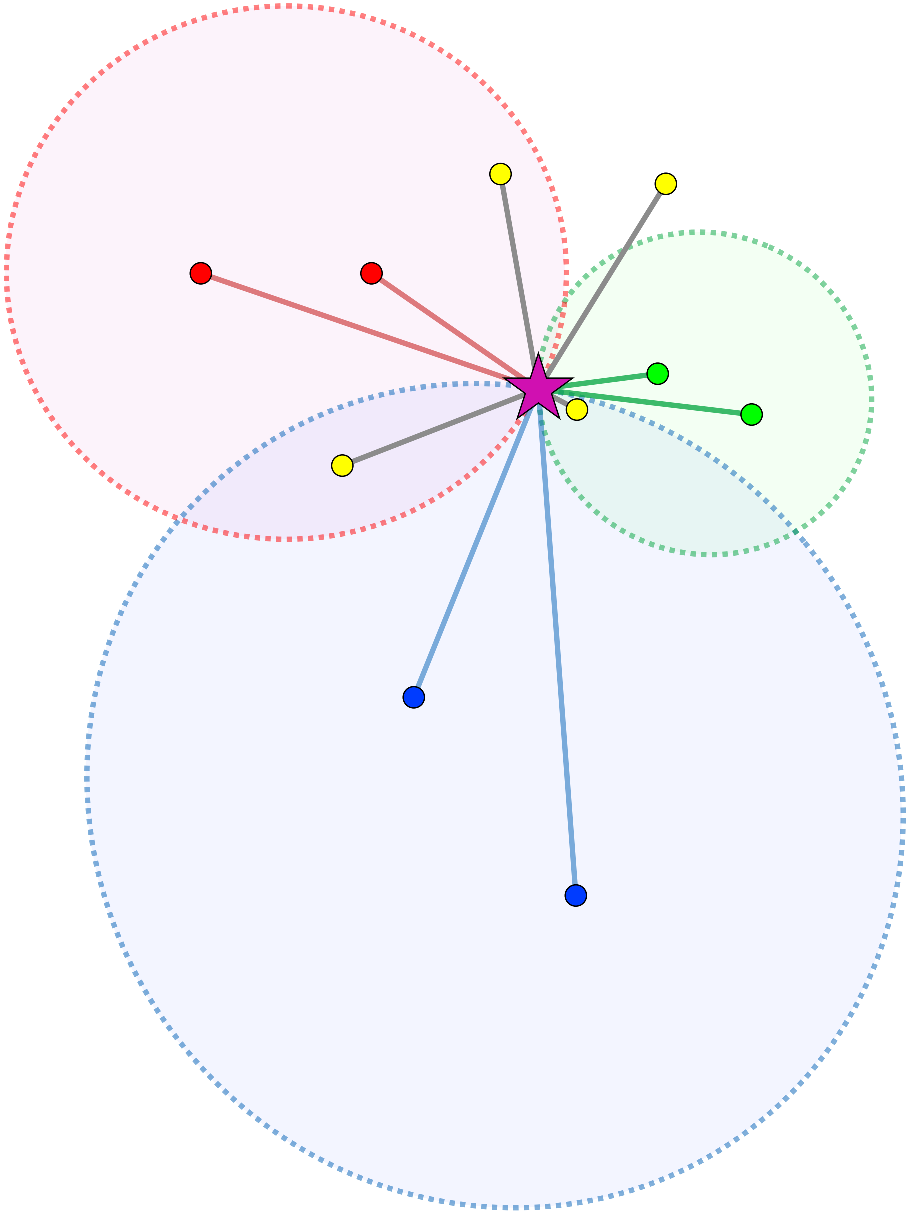
Optimal embedding into ultrametric
minimizing maximum difference (new distance - input distance)
[Farach, Kannan, Warnow, Algorithmica 1995]

Insensitive to distortion of small distances

Optimal embedding of unweighted graph to line
[Fomin, Lokshtanov, Saurabh, WG 2009]

Singly exponential time (at least not factorial)

Related work: Euclidean min dilation star [E & W, SoCG'05]



Given n points in Euclidean plane or higher dimensional space

Find a hub within that space minimizing dilation of star network with Euclidean distances as lengths

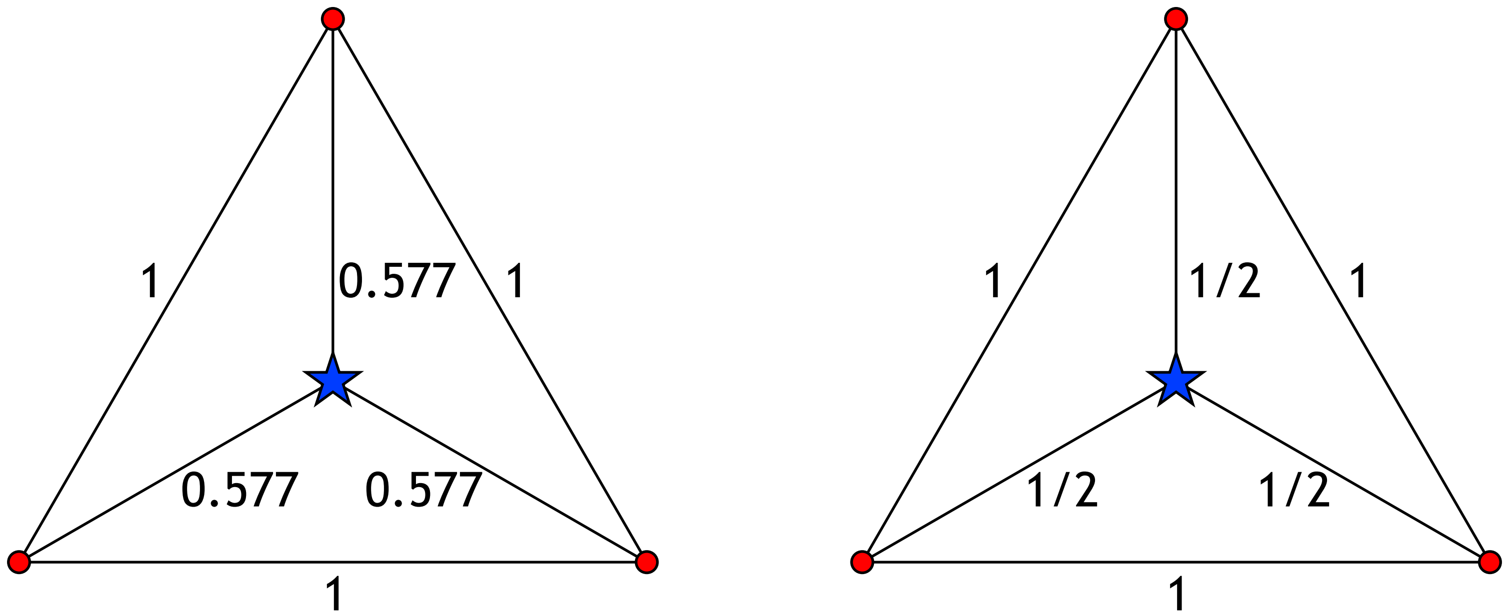
Equivalently, maximize min eccentricity of ellipses passing through the hub having pairs of input points as foci

$O(n \log n)$ in any fixed dimension if hub can be any point of the space

$O(n\, 2^{\alpha(n)} \log^2 n)$ if hub must be one of the input points (2d only)

The difference between geometric and metric stars

Example: input is an equilateral triangle
(three points with equal distances)



Euclidean min dilation star
dilation = 1.155

Metric min dilation star
dilation = 1

Solution ideas, I:
Express MDS as a linear program

Find H[x] and Δ

Satisfying

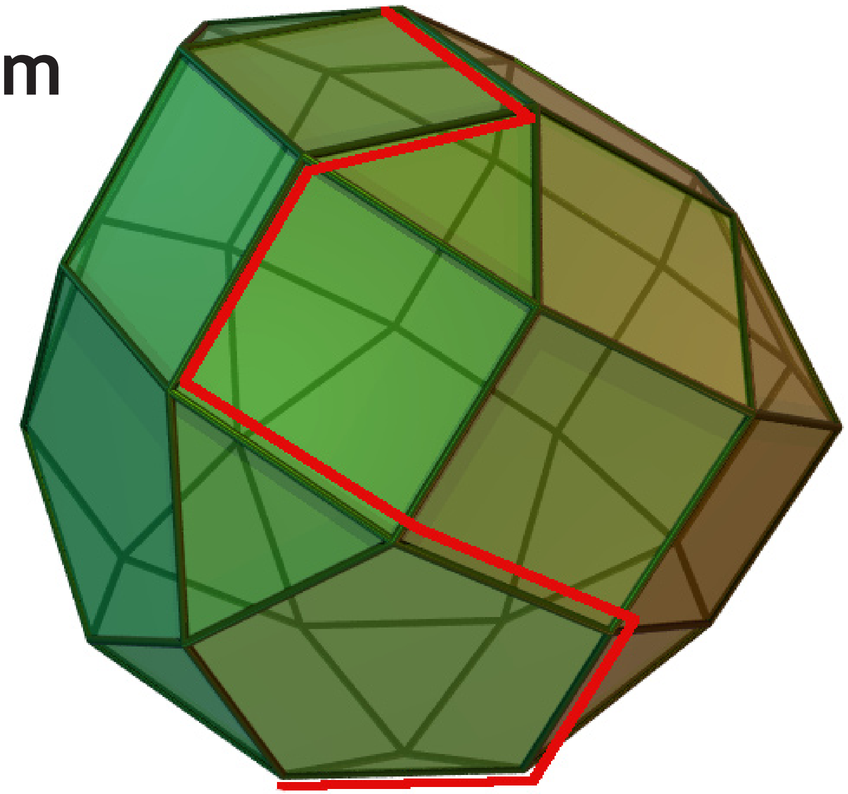
$H[x] \geq 0$

$H[x] + H[y] \geq \text{distance}(x,y)$

$H[x] + H[y] \leq \Delta\ \text{distance}(x,y)$

for all x and y

Minimizing Δ



Not in form for known strongly-polynomial LP algorithms

- more than O(1) variables total
- more than two variables in some inequalities

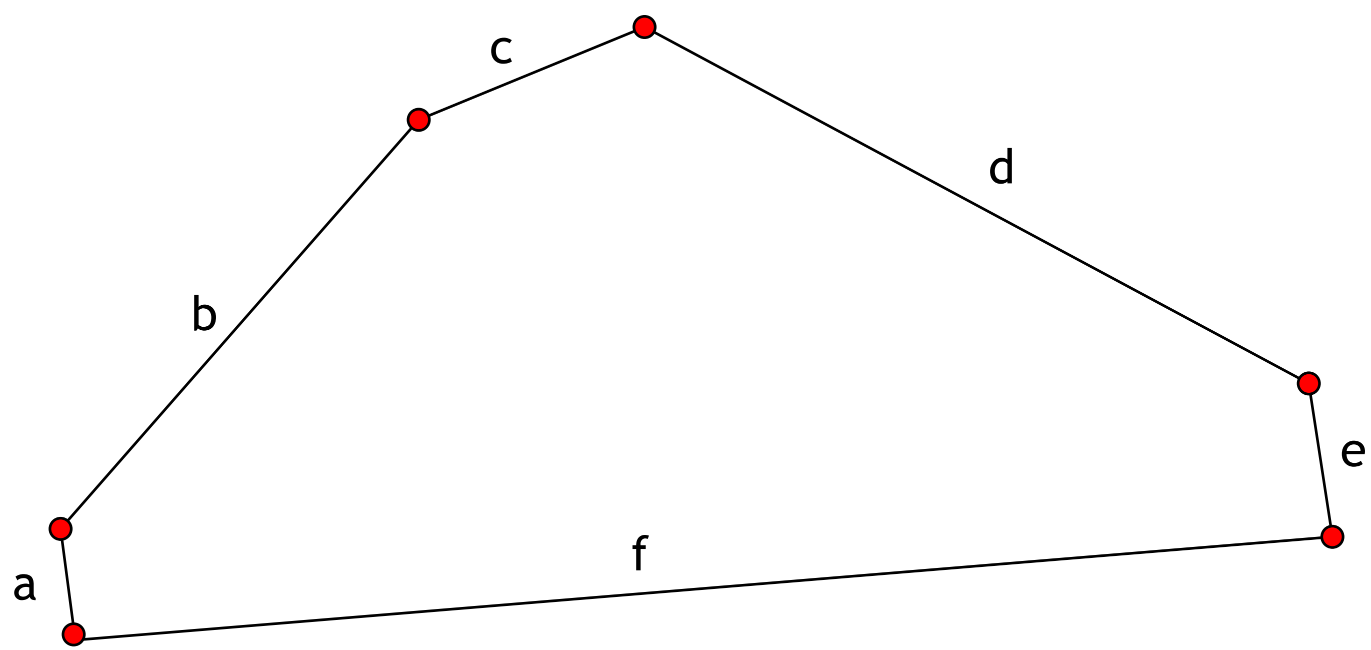
Solution ideas, II: Characterize LP basis

Any even cycle in original metric space lower-bounds the dilation:

dilation ≥ (sum of even edge lengths) / (sum of odd edge lengths)

(because in star, both sums are forced to equal each other)

Optimal dilation turns out to equal worst cycle of this type

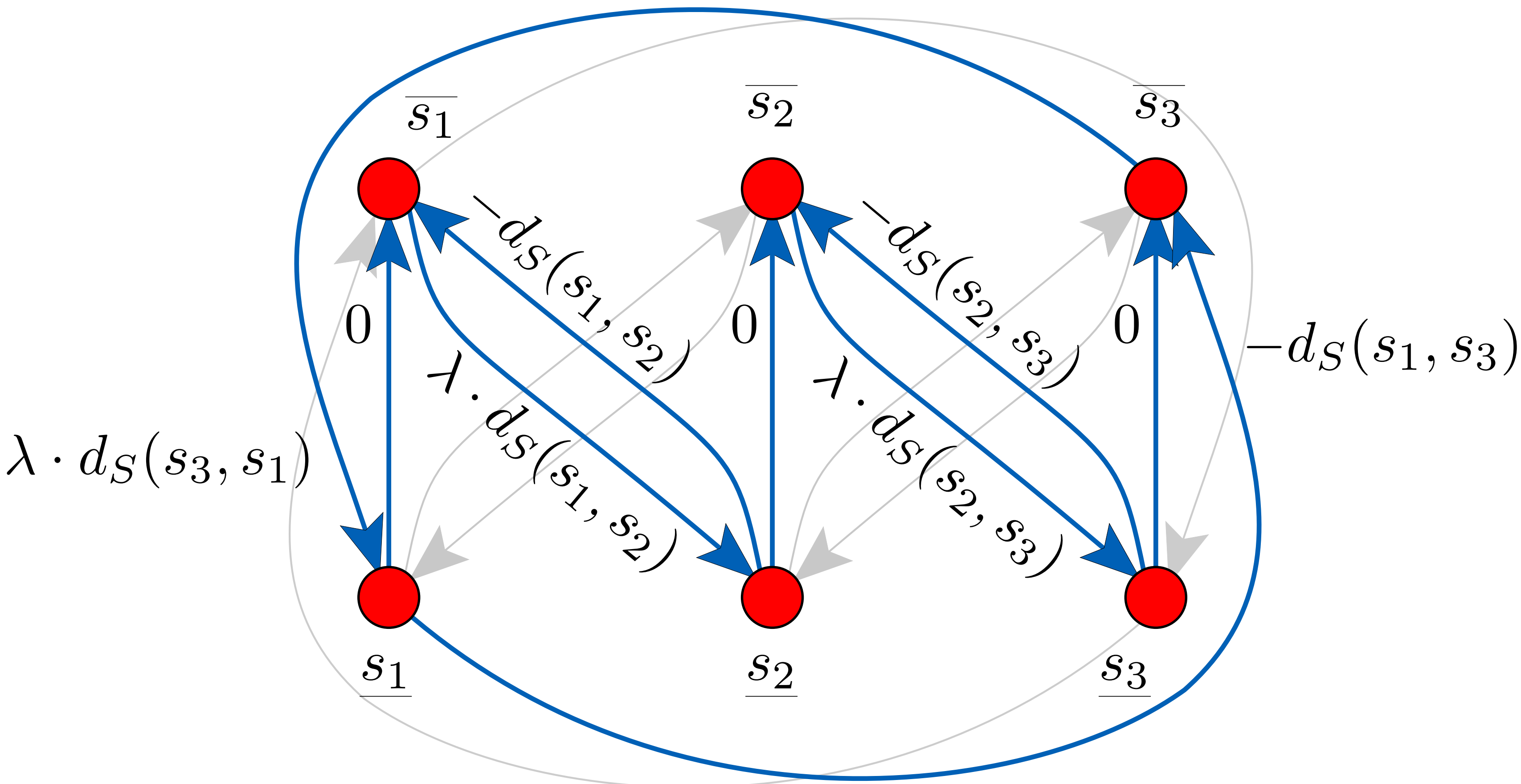


$$\text{dilation} \geq (b + d + f) \,/\, (a + c + e)$$

Solution ideas, III: transform to a graph problem

Parametric negative cycle detection:
digraph where edge weights are linear functions of a parameter λ,
find smallest value of λ such that all cycle lengths are non-negative

Optimal λ = optimal dilation Δ of original metric embedding problem



Solution ideas, IV: strongly polynomial solution for parametric negative cycle detection problem

Megiddo's parametric search [Megiddo, JACM 1983]:

Simulate parallel algorithm for optimization problem
as if it were given the optimal parameter value as input
using a decision algorithm to help simulate each branch step

Simulated algorithm [Savage, Ph.D. thesis 1977]:

Compute all pairs shortest paths by repeated matrix squaring

Decision algorithm for comparing parameter value to optimum:

Bellman–Ford, detect negative cycles in non-parametric graph

Solution ideas, V: cycle detection details

Store a matrix of piecewise linear parametric functions

Represents lengths of paths with at most 2^i hops

Initially: $i = 0$, matrix stores the edge length functions

Repeat log n times, until $2^i > n$:

Square in (min,+) matrix arithmetic to increment i

$O(n^3 \log n)$ time to combine piecewise linear functions

Binary search for the optimal parameter value
among the breakpoints of the path-length functions
simplifying matrix entries back to non-piecewise functions

$O(\log n)$ calls to Bellman-Ford

Solution ideas VI: finding the actual embedding

Now that we know the correct dilation...

Plug it into the same parametric graph

Add an extra "source" vertex to the graph

Compute distances to all other vertices (Bellman-Ford)

H[x] = (half of) difference between two distances
from source to two vertices representing x

Some algebra + triangle inequality shows this is a valid embedding

Lower bound shows this is the optimal embedding

Conclusions

Hyperconvexity is...

...as central to metric spaces as convexity is to Euclidean spaces

...important for the development of approximation algorithms

already used as such in k-server online algorithms

...a unifying point of view for finding metric embeddings

exact embeddings into Manhattan plane

optimal-dilation embeddings into stars

...an interesting subject for more algorithmic research