Bipartite and Series-Parallel Graphs Without Planar Lombardi Drawings

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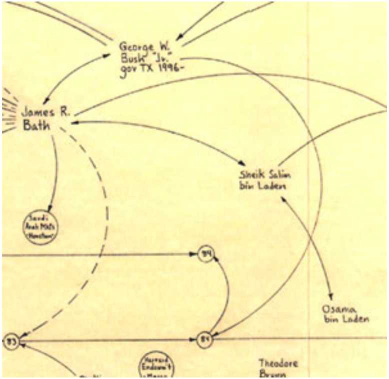
Mark Lombardi

Artist whose "narrative structures" showed links in international financial and criminal conspiracies



George W. Bush, Harken Energy, and Jackson Stephens (1999, detail)

Connected Bush to Bin Laden two years before 9/11/2001

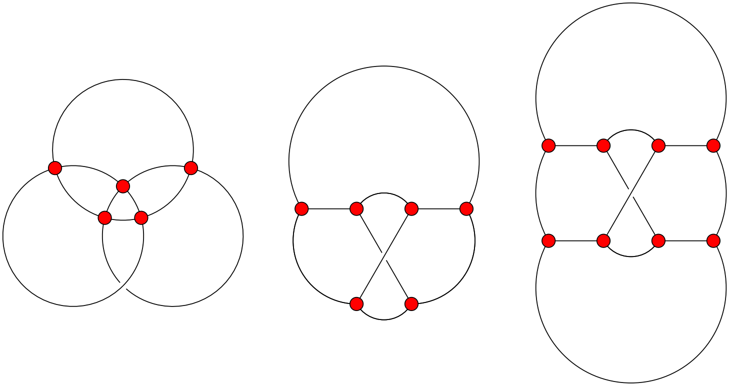
Died under mysterious circumstances (apparent suicide) in 2000

Lombardi drawing

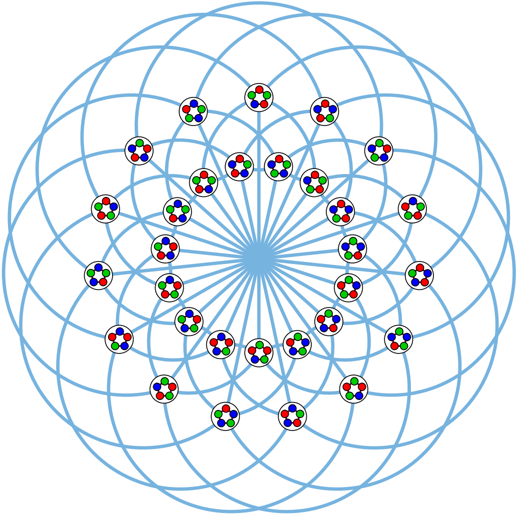
Style of drawing named in honor of Lombardi by Duncan et al. [2012, 2013]

Edges must be drawn as circular arcs

At each vertex, incident edges form equal angles



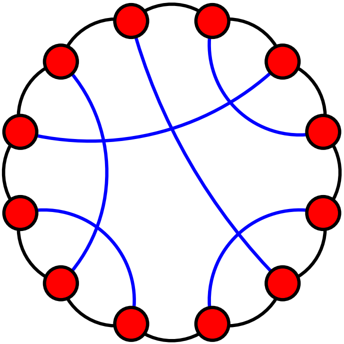
This style works well for symmetric graphs

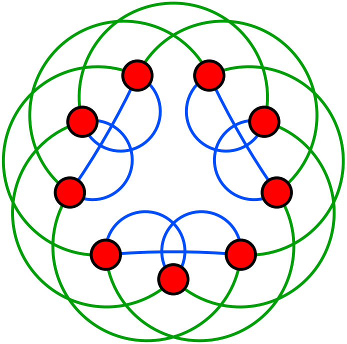


Wang–Swendsen–Kotecký dynamics on 3-colorings of a 5-cycle

Random walks rapidly mix ⇒ random colorings

Most regular graphs have circular layouts

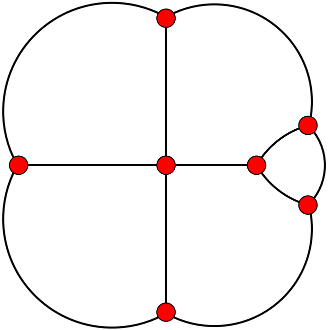


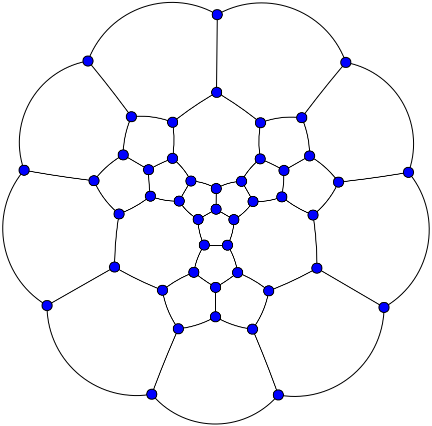


(With some exceptions when degree is not divisible by 4)

Which graphs have planar Lombardi drawings?

Known: Halin graphs and 3-regular planar graphs





3-regular planar Lombardi drawing ⇒ characterization of the graphs of two-dimensional soap bubble foams [Eppstein 2014]

What about bipartite planar graphs?

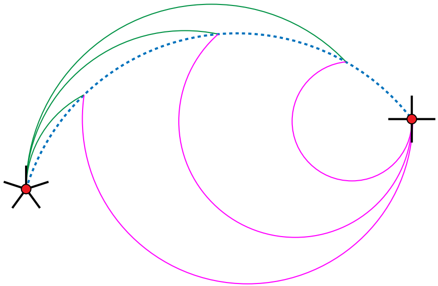
They usually (always?) have Lombardi drawings

Remove a vertex of degree ≤ 3

Draw remaining graph with correct angles recursively

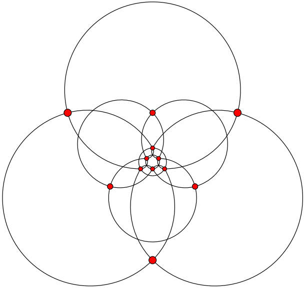
Add back the removed vertex

It will always be possible to get arc angles to match

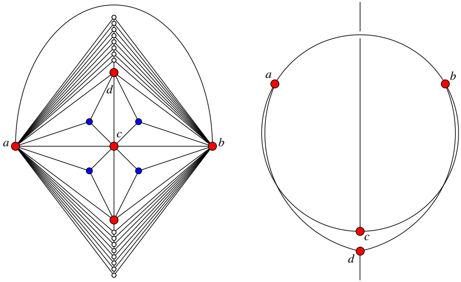


But do they always have planar Lombardi drawings?

Planar + Lombardi ≠ planar Lombardi



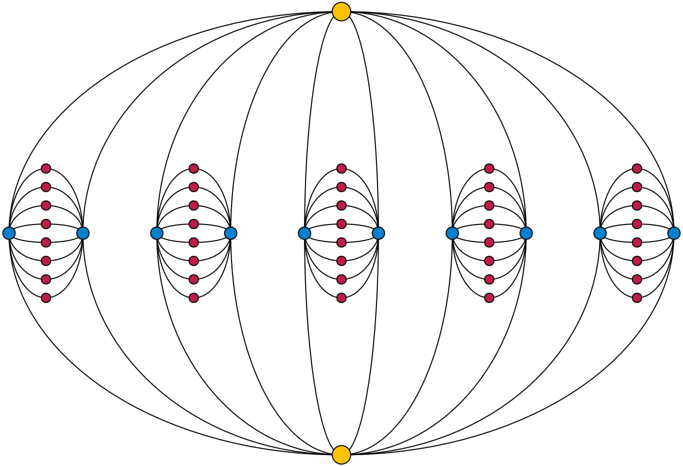
4-nested triangles



A planar 3-tree

Our main result

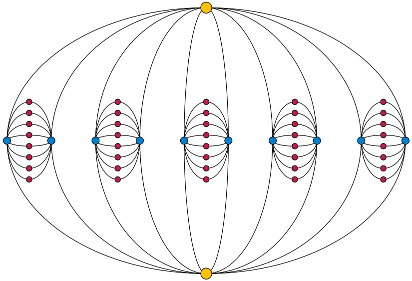
Bipartite planar graphs like this one
(but with higher degree vertices)
do not have planar Lombardi drawings



What the faces must look like

The yellow–blue–yellow–blue quadrilateral faces
must be quadrilaterals with all four vertex angles very sharp
all sharing the same two yellow vertices

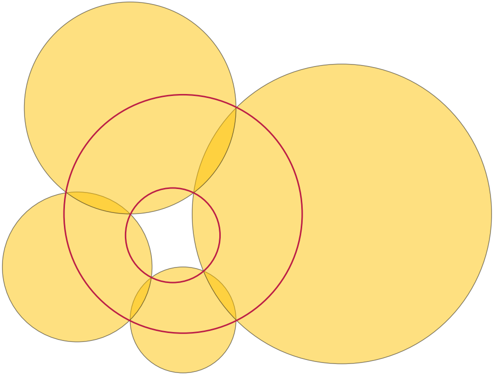
(The blue–red subgraphs are only there to cause the blue and yellow vertices to have equal degrees)



Plan: Show that this ring of sharp-angled arc-quads cannot exist

A property of arc-quads with equal angles

They exist for any interior angle $0 < \theta < 2\pi$, but…

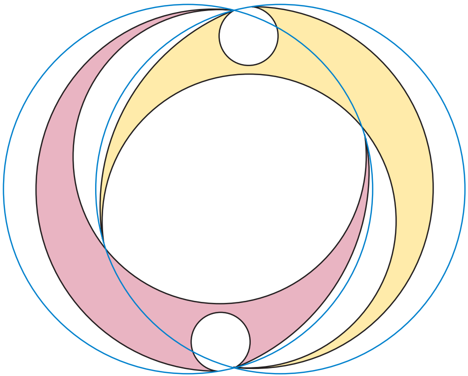


All four vertices must lie on a circle

(proof idea: Möbius transform to make more symmetric; after transforming, vertices form a rectangle)

Consecutive quadrilaterals in the circular sequence

Can form pockets allowing one quadrilateral to reach its circle without crossing the neighboring quadrilateral



But in each quad, one pocket is close to one of the two shared vertices and the other pocket is close to the other shared vertex

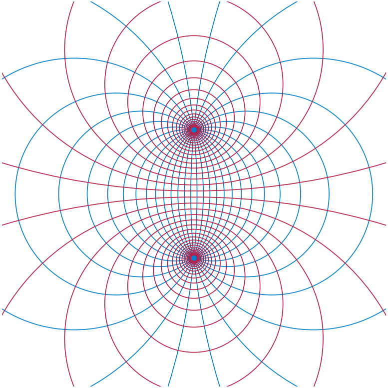
Overall proof strategy

As the quadrilaterals dance in a ring around their shared vertices

Image: Matisse [1909–1910]

. . . the pockets get closer and closer to one pole,
leading to a contradiction when they get back around to the start

A hint of the technical difficulties



How do we define closeness of a pocket to a shared vertex?

Bipolar coordinates!

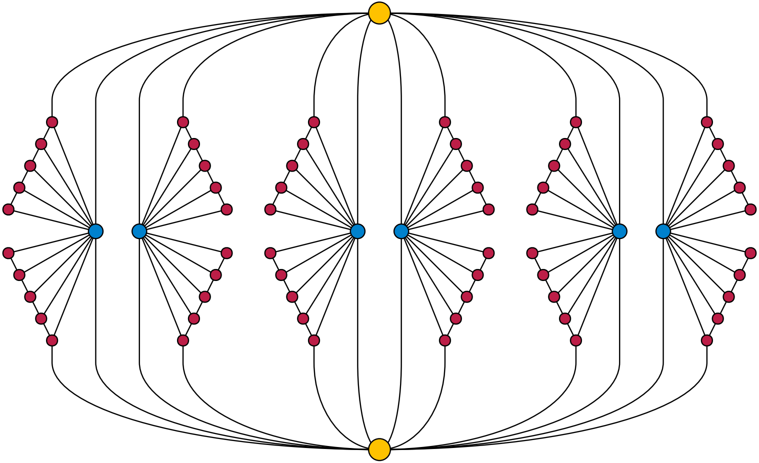
- Blue contours: angle to the two poles
- Red contours: log of ratio of distances to poles

Poles = yellow shared vertices

Acted on additively by Möbius transforms that fix the poles

Same method works for other graphs

The following series-parallel graph does not have a planar Lombardi drawing that is consistent with the embedding below (but it might have drawings with other embeddings)



Conclusions

Planar bipartite graphs might not have planar Lombardi drawings

Embedded series-parallel might not have planar Lombardi drawings

Open: Series-parallel without a fixed embedding?

Open: Outerplanar with the outerplanar embedding?

References and image credits

Christian A. Duncan, David Eppstein, Michael T. Goodrich, Stephen G. Kobourov, and Martin Nöllenburg. Lombardi drawings of graphs. *J. Graph Algorithms & Applications*, 16(1):85–108, 2012. doi: 10.7155/jgaa.00251.

Christian A. Duncan, David Eppstein, Michael T. Goodrich, Stephen G. Kobourov, and Martin Nöllenburg. Drawing trees with perfect angular resolution and polynomial area. *Discrete & Computational Geometry*, 49(2):157–182, 2013. doi: 10.1007/s00454-012-9472-y.

David Eppstein. A Möbius-invariant power diagram and its applications to soap bubbles and planar Lombardi drawing. *Discrete & Computational Geometry*, 52(3):515–550, 2014. doi: 10.1007/s00454-014-9627-0.

Henri Matisse. La danse (second version). Public domain image, 1909–1910.