Triangles and Squares

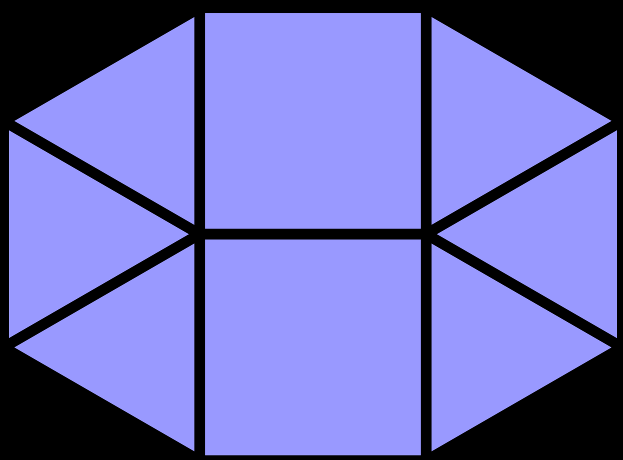
David Eppstein

COMB01, Barcelona, 2001

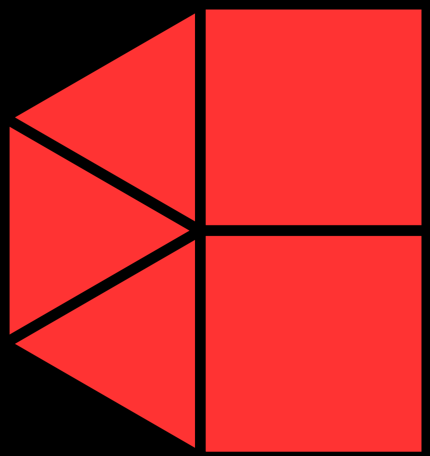
Joint work with Günter Ziegler
and Greg Kuperberg

Floor tile detail from Casa Milà
(La Pedrera), Antoni Gaudí, Barcelona

Which strictly convex polygons can be made
by gluing together unit squares and equilateral triangles?

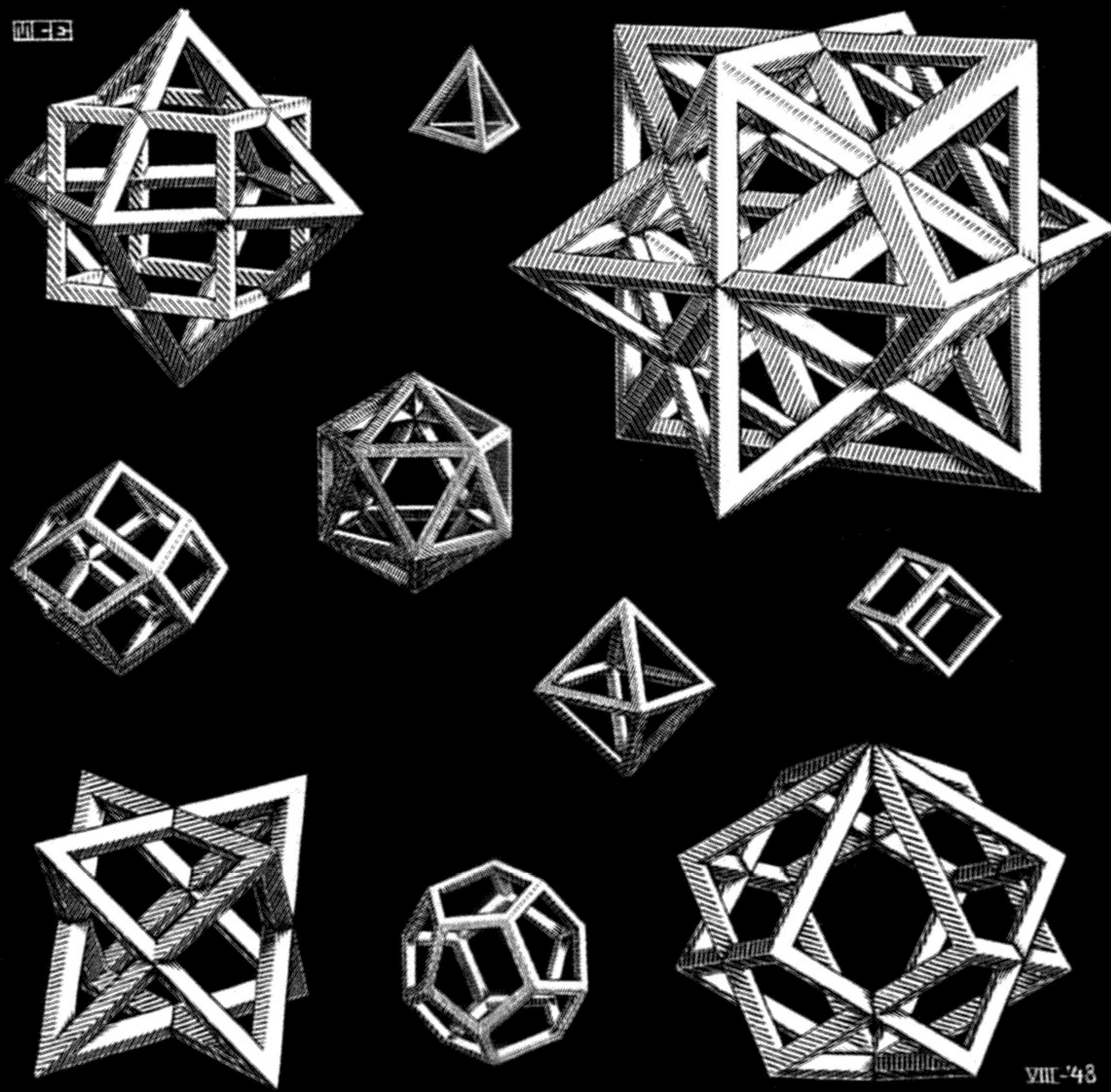


Strictly convex



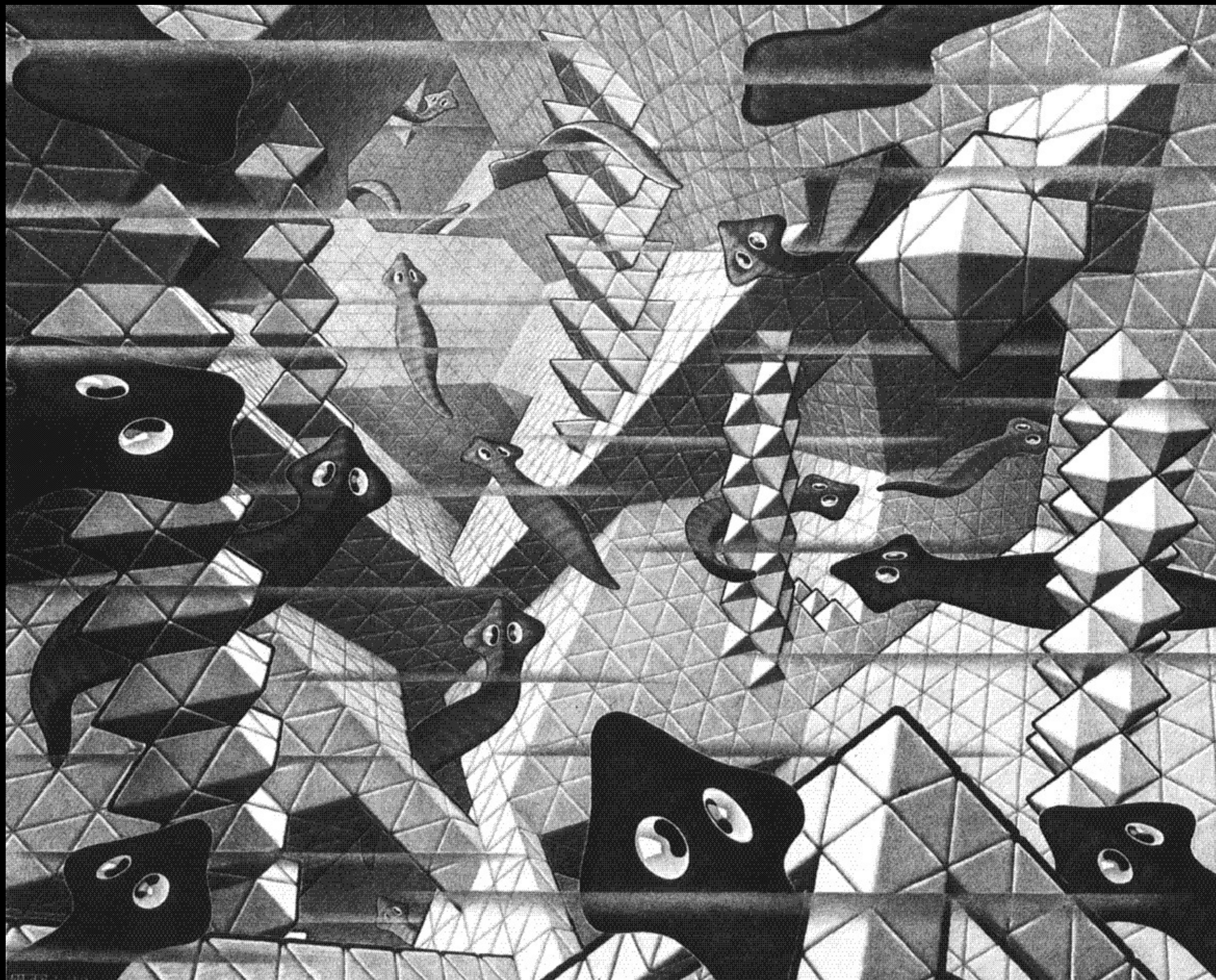
Not strictly convex

The Five Platonic Solids (and some friends)



M. C. Escher, *Study for Stars*, Woodcut, 1948

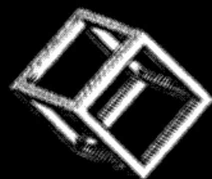
Octahedron and tetrahedron dihedrals add to 180!
So they pack together to fill space



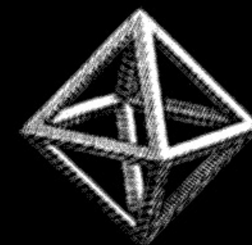
M. C. Escher, *Flatworms*, lithograph, 1959

The Six Regular Four-Polytopes

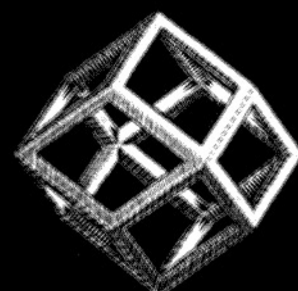
Simplex, 5 vertices, 5 tetrahedral facets, analog of tetrahedron



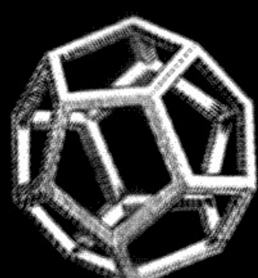
Hypercube, 16 vertices,
8 cubical facets, analog of cube



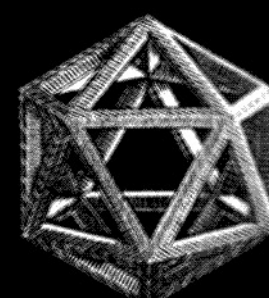
Cross polytope, 8 vertices,
16 tetrahedral facets, analog of octahedron



24-cell, 24 vertices, 24 octahedral facets, analog of rhombic dodecahedron



120-cell, 600 vertices,
120 dodecahedral facets, analog of dodecahedron



600-cell, 120 vertices,
600 tetrahedral facets, analog of icosahedron

Mysteries of four-dimensional polytopes...

What face counts are possible?

For three dimensions, $f_0 - f_1 + f_2 = 2$, $f_0 \leq 2\, f_2 - 4$, $f_2 \leq 2\, f_0 - 4$
describe all constraints on numbers of vertices, edges, faces

All counts are within a constant factor of each other

For four dimensions, some similar constraints exist
e.g. $f_0 + f_2 = f_1 + f_3$
but we don't have a complete set of constraints

Is "fatness" (f1 + f2)/(f0 + f3) bounded?

Known $O((f_0 + f_3)^{1/3})$ [Edelsbrunner & Sharir, 1991]

Further mysteries of four-dimensional polytopes...

How can we construct more examples like the 24-cell?

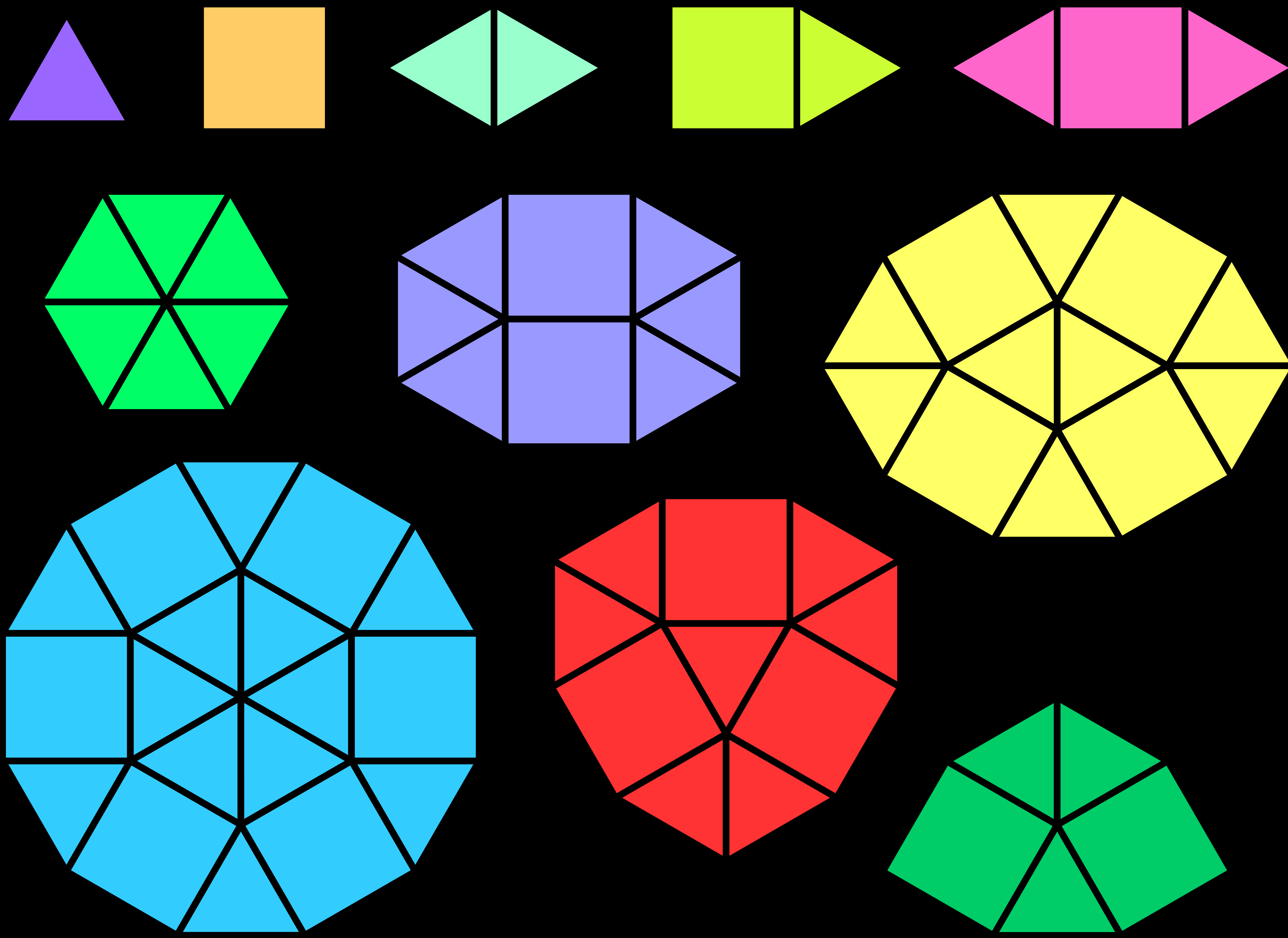
All 2-faces are triangles ("2-simplicial")

All edges touch three facets ("2-simple")

Only few 2-simple 2-simplicial examples were known:

simplex
hypersimplex and its dual
24-cell
Braden polytope and its dual

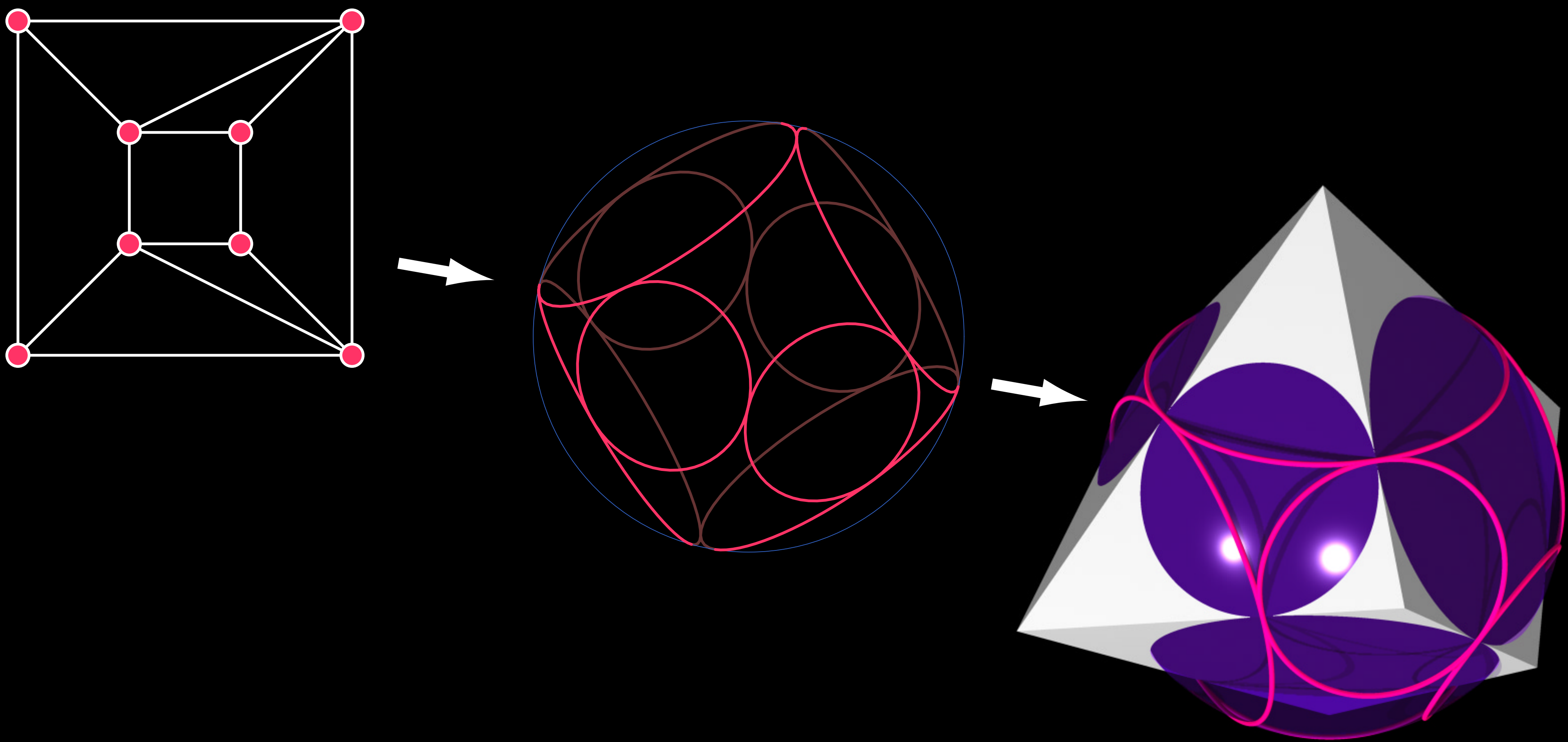
The Eleven Convex Square-Triangle Compounds



Polytopes and Spheres

M. C. Escher, *Order and Chaos*, lithograph, 1950

Theorem [Koebe, 1936]:



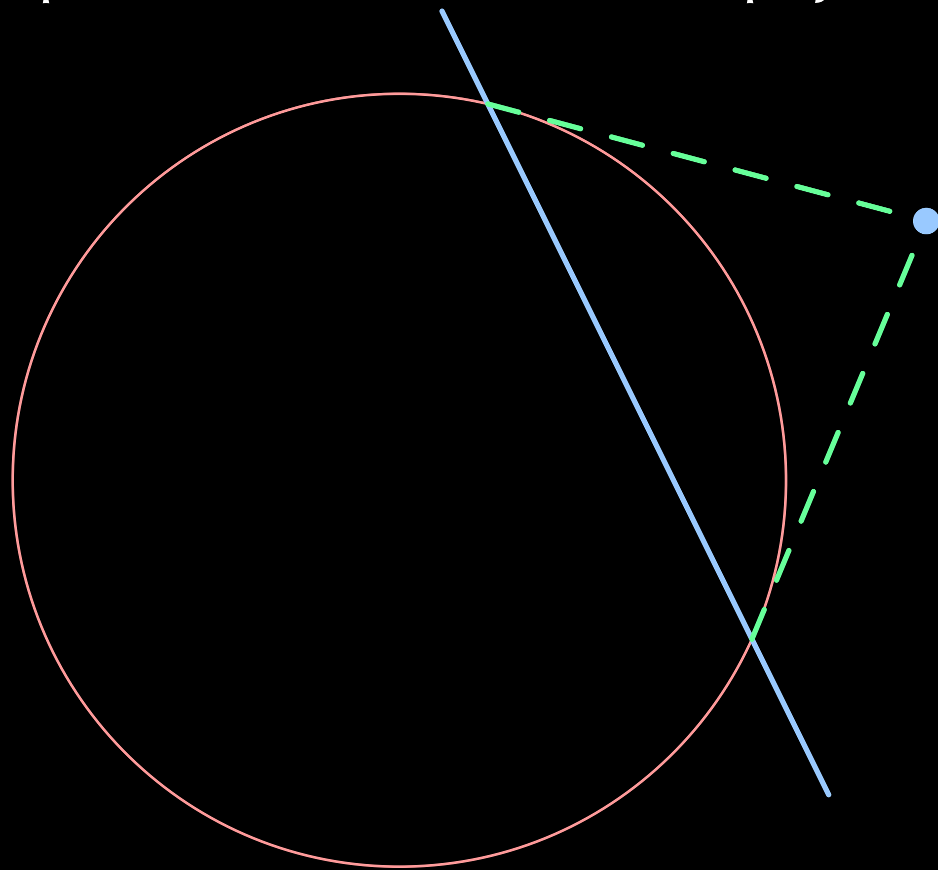
Any planar graph can be represented by circles on a sphere,
s.t. two vertices are adjacent iff the corresponding two circles touch

Replacing circles by apexes of tangent cones
forms polyhedron with all edges tangent to the sphere

Polarity

Correspond points to lines in same direction from circle center
distance from center to line = 1/(distance to point)

Line-circle crossings equal point-circle horizon
Preserves point-line incidences! (a form of projective duality)



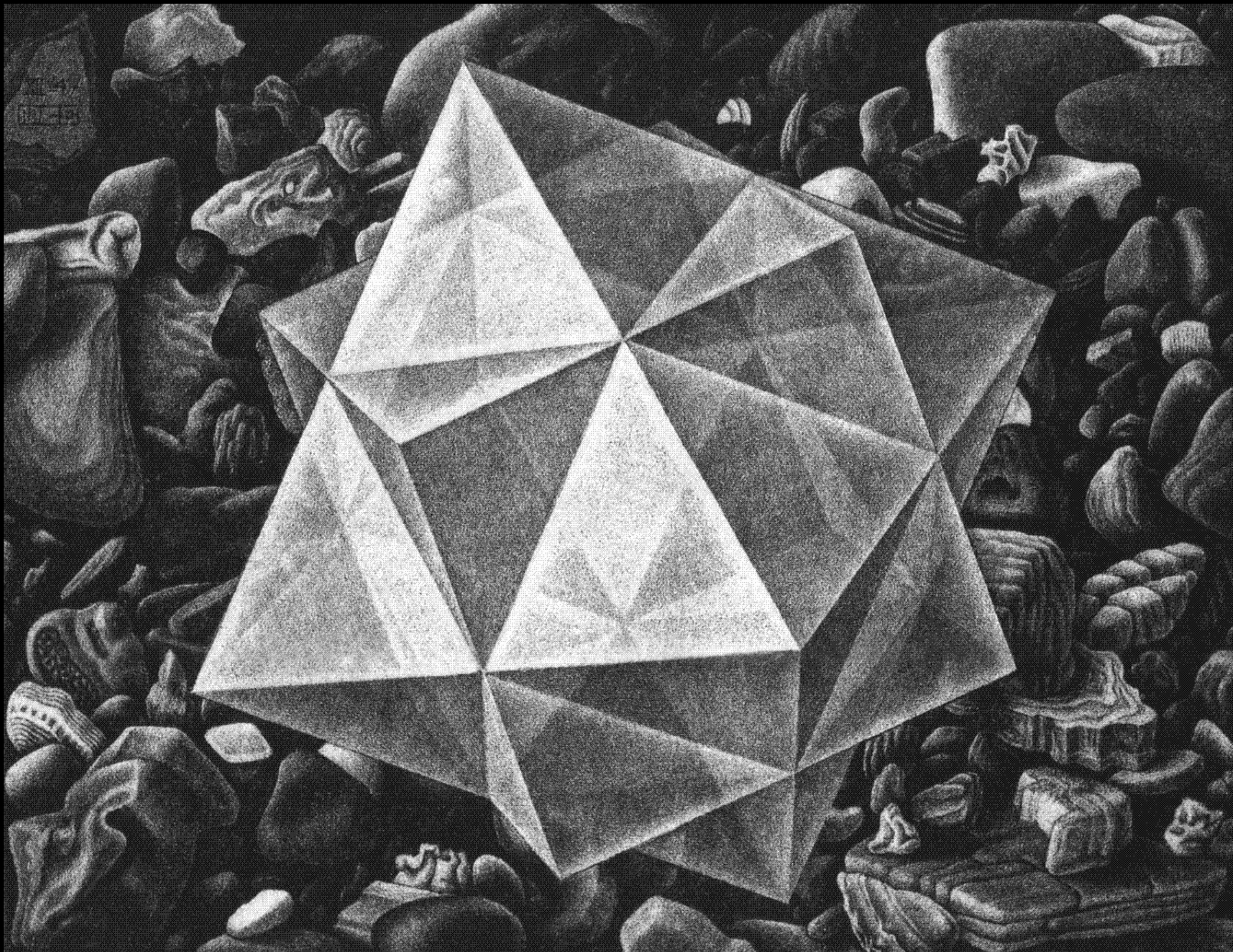
Similar dimension-reversing correspondence in any dimension

Converts polyhedron or polytope (containing center) into its dual

Preserves tangencies with unit sphere

Convex Hull of (P union polar), P edge-tangent

Edges cross at tangencies; hull facets are quadrilaterals



M. C. Escher, *Crystal*, mezzotint, 1947

Same Construction for Edge-Tangent 4-Polytopes?

Polar has 2-dimensional faces (not edges) tangent to sphere

Facets of hull are dipyramids over those 2-faces

All 2-faces of hull are triangles (2-simple)

Three facets per edge (2-simplicial) if and only if edge-tangent polytope is simplicial

This construction leads to all known 2-simple 2-simplicial polytopes

Simplex $\Rightarrow$ hypersimplex

Cross polytope $\Rightarrow$ 24-cell

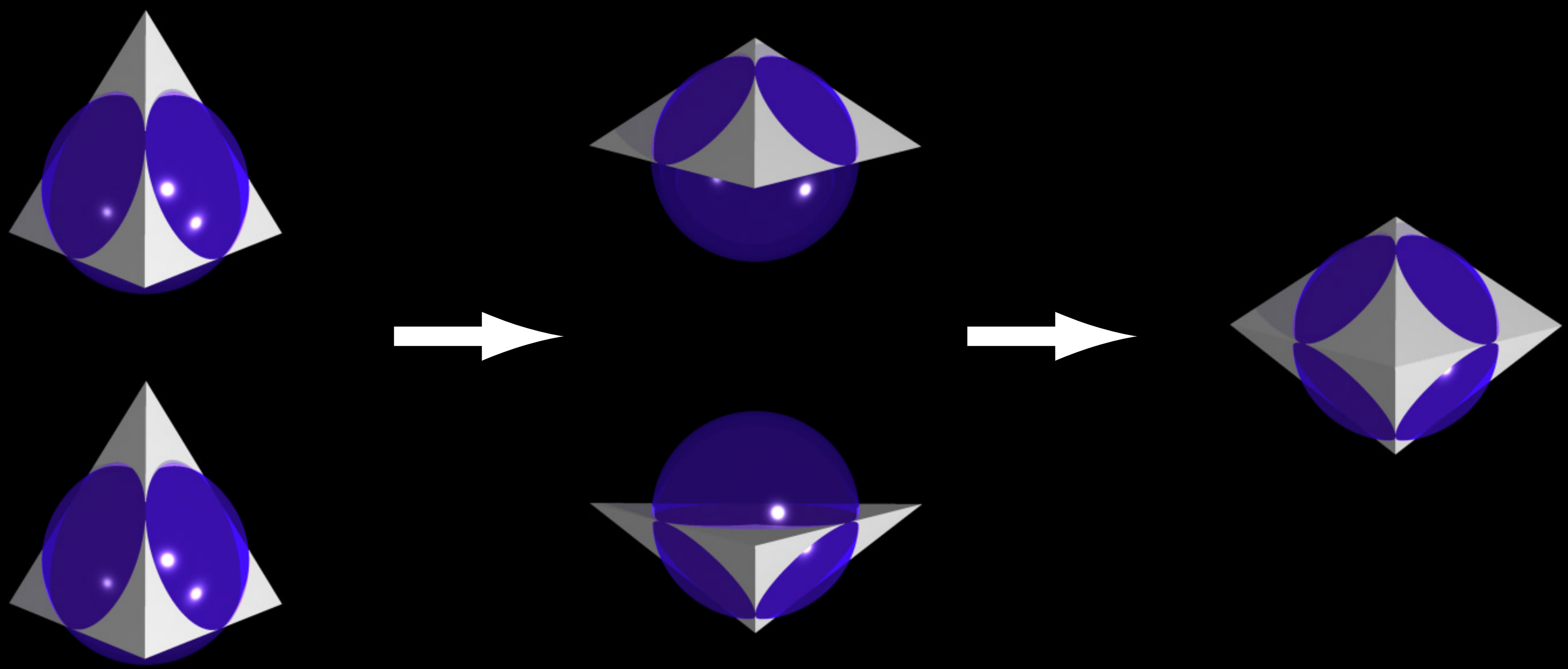
600-cell $\Rightarrow$ new 720-vertex polytope, fatness=5

So are there other simplicial edge-tangent polytopes?

How to make more edge-tangent simplicial 4-polytopes?

Glue together simple building blocks: regular polytopes

Warp (preserving tangencies) so facets match



But when will the result be convex?

Need a space where we can measure convexity
independent of warping (projective transformations)

Answer: hyperbolic geometry!

Hyperbolic Space (Poincaré model)

Interior of unit sphere; lines and planes are spherical patches perpendicular to unit sphere

M. C. Escher, *Circle Limit II*, woodcut, 1959

Size versus angle in hyperbolic space

Smaller shapes have larger angles

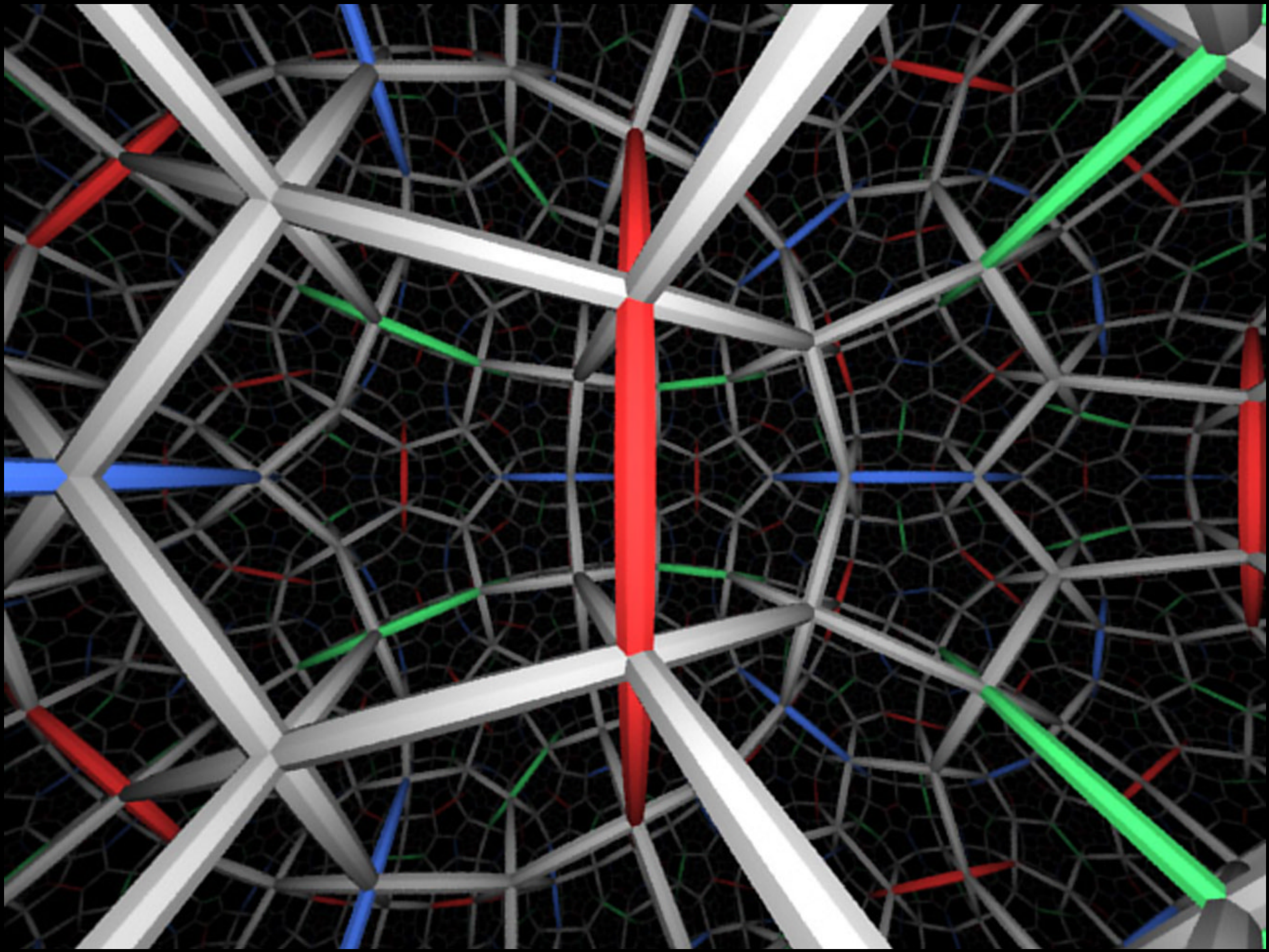
Larger shapes have smaller angles

What are the angles in Escher's triangle-square tiling?

3 triangle + 3 square = 360
2 triangle + 1 square = 180
square < 90, triangle < 60

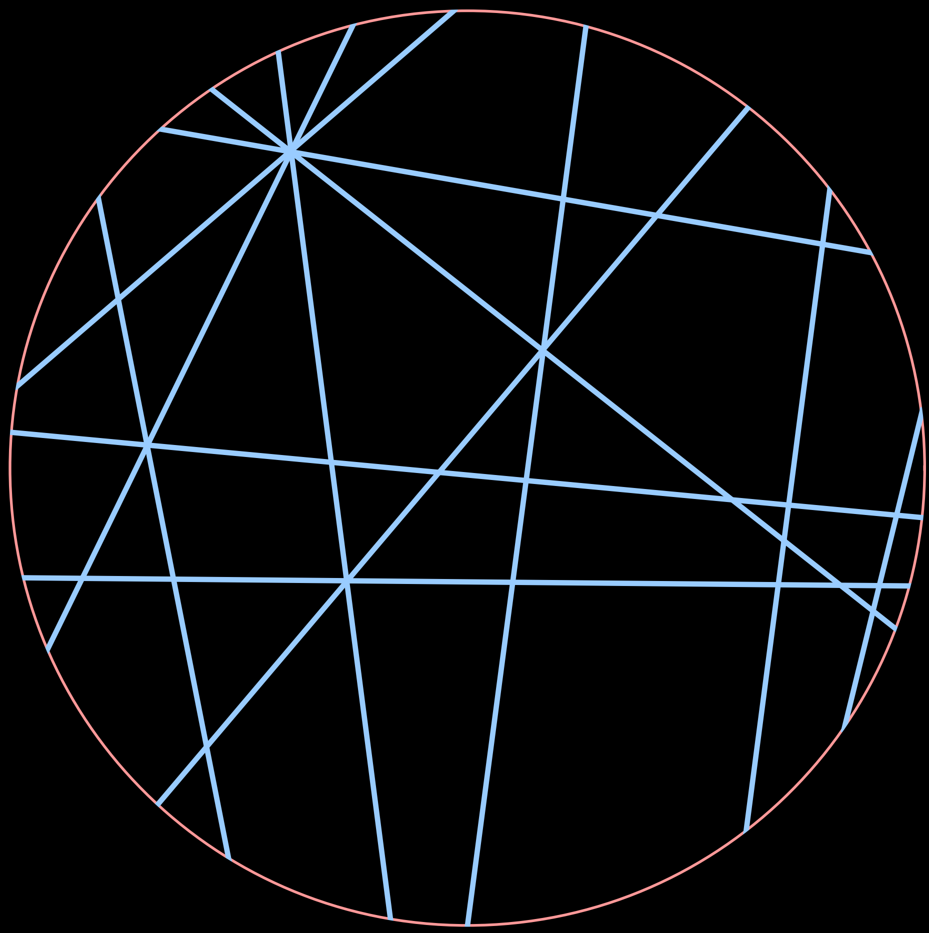
Another impossible figure!

Right-angled dodecahedra tile hyperbolic space



From *Not Knot*, Charlie Gunn, The Geometry Center, 1990

Two Models of Hyperbolic Space

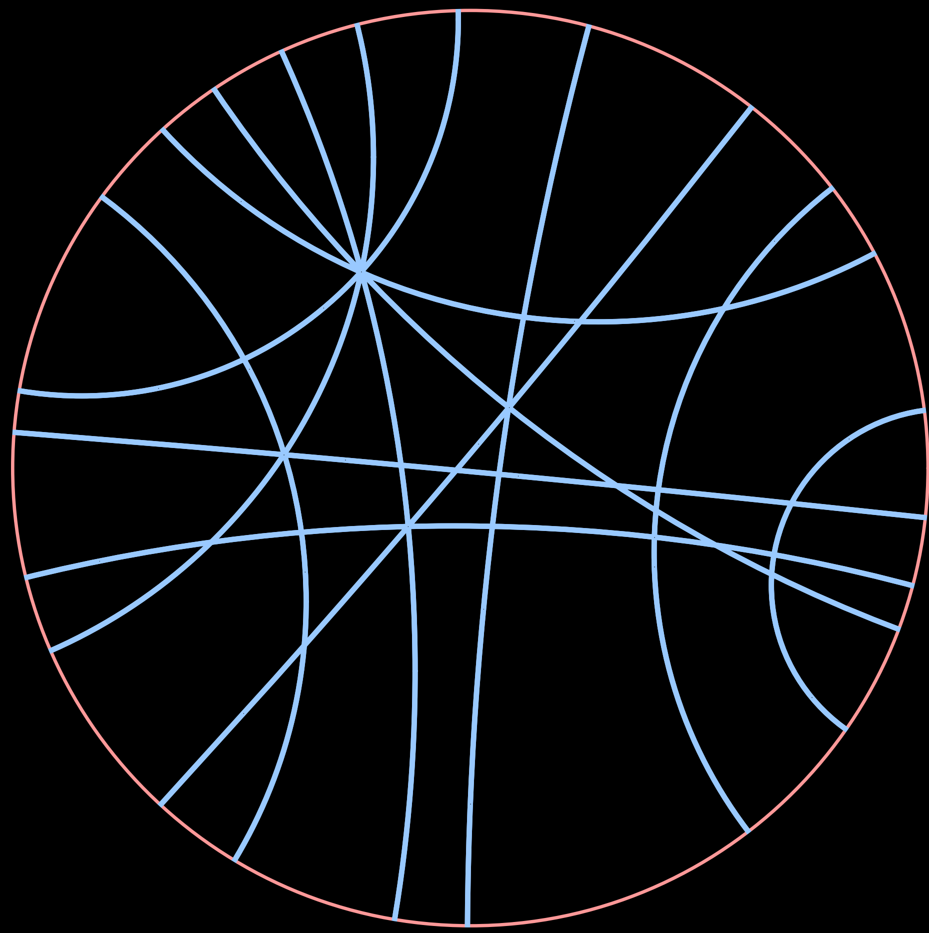


Klein Model

Hyperbolic objects are straight or convex
iff their model is straight or convex

Angles are severely distorted

Hyperbolic symmetries are modeled as
Euclidean projective transformations



Poincaré Model

Angles in hyperbolic space
equal Euclidean angles of their models

Straightness/convexity distorted

Hyperbolic symmetries are modeled as
Möbius transformations

Hyperbolic angles of edge-tangent 4-polytopes

Recall that we need a definition of angle that's invariant
under the projective transformations used to glue polytopes together

Given an edge-tangent 4-polytope, view portion inside sphere as Klein model of some (unbounded) hyperbolic polytope

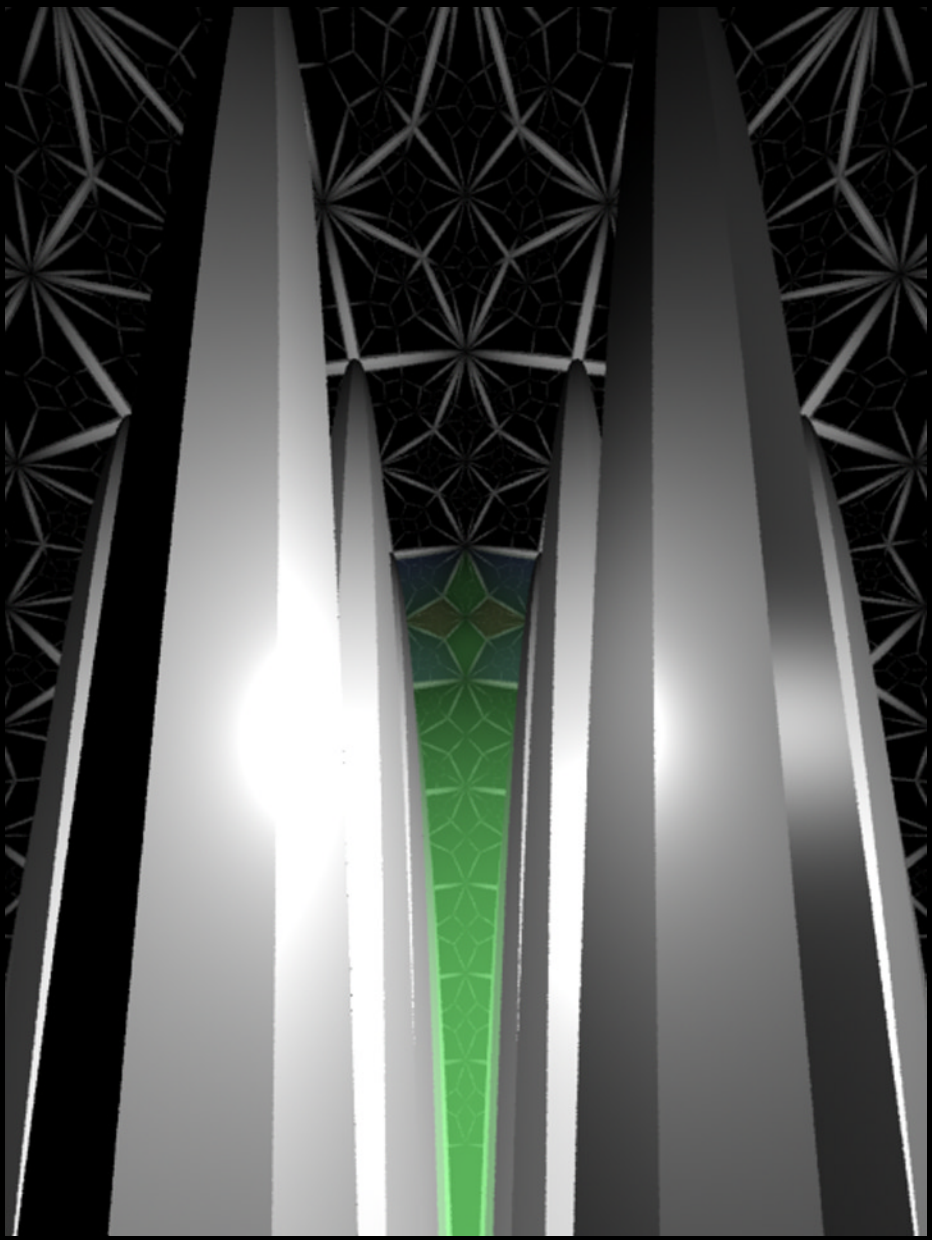
Use Poincaré model of the same polytope to measure its dihedral angles

Dihedrals are well-defined at finite hyperbolic points on incenter of each 2-face

Invariant since projective transformations correspond to hyperbolic isometries

Gluing is convex iff sum of dihedrals is less than 180 degrees
Can also glue cycle of polytopes with dihedrals adding to exactly 360 degrees

How to compute hyperbolic dihedrals



From *Not Knot*, Charlie Gunn, The Geometry Center, 1990

For ideal 3d polyhedron

Near ideal vertex, looks like a prism
dihedral = cross-section angle

e.g. here rhombic dodecahedron
has square cross-section
⇒ 90-degree dihedral angles

For edge-tangent 4-polytope

Slice by hyperplane through tangency
Apply 3d method to cross section

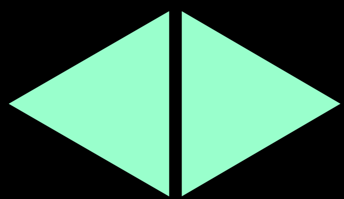
Regular polytopes have regular-polygon cross sections!

Dihedral(simplex) = 60 degrees
Dihedral(cross polytope) = 90 degrees
Dihedral(600-cell) = 108 degrees

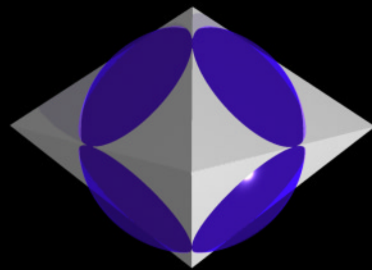
Simplex - Simplex Compounds

Only two possible

Two simplices (dual of tetrahedral prism)
leads to Braden's 2-simplicial 2-simple polytope

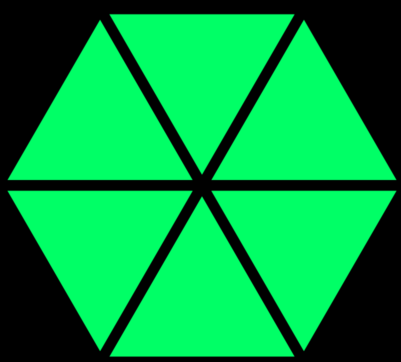


Two-dimensional analog



Three-dimensional analog

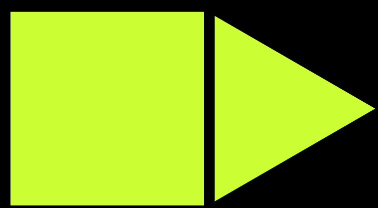
Six simplices sharing a common dihedral (dual of C_3 x C_6)
leads to new 2-simplicial 2-simple polytope

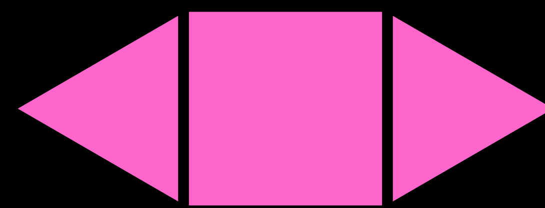


**All other stacked 4-polytopes (or other simplex compounds)
can not be edge-tangent**

Simplex / Cross-polytope Compounds I

Glue simplices onto independent facets of cross polytope





Dually, truncate independent vertices of a hypercube

How many different truncations possible?

i.e., how many independent subsets of a hypercube
modulo symmetries of the hypercube

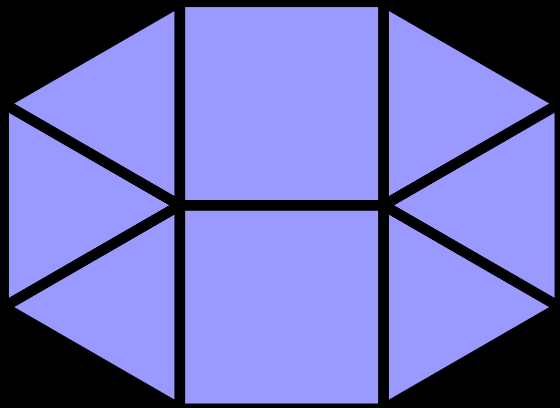
New entry in Sloan's encyclopedia, sequence A060631:
2, 2, 3, 6, 21, 288, ...
Starts at d=0; figures above are two of the three choices for d=2

So 20 (not counting original untruncated hypercube) for d=4

Simplex / Cross-polytope Compounds II

Glue two cross polytopes together at a facet

Dihedral = 90, so gluing forms four flat dihedrals
Make strictly convex by adding chains of three simplices



Can repeat, or glue additional simplexes onto the cross polytopes

As long as sets of glued faces are non-adjacent,
all dihedrals will be at most 90 + 60 = 150

E.g., glue *n* cross polytopes end-to-end, additional three-simplex chains don't interfere

Leads to infinitely many 2-simple 2-simplicial polytopes!

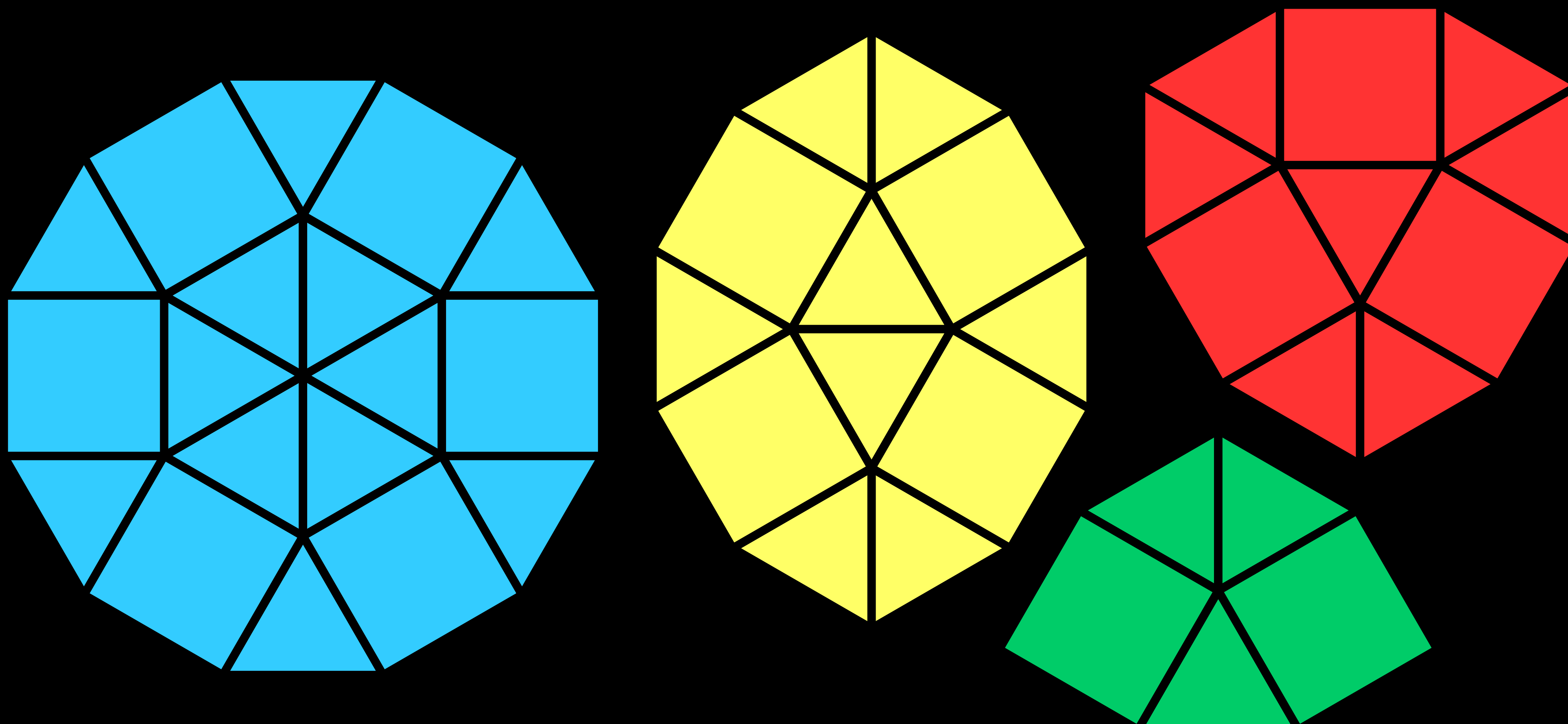
Simplex / Cross-polytope Compounds III

Start with a simplex, two simplices, or six simplices

Glue cross polytopes on some facets

At 120 degree dihedrals, both or neither facets must be glued

Fill remaining concavities with more simplices

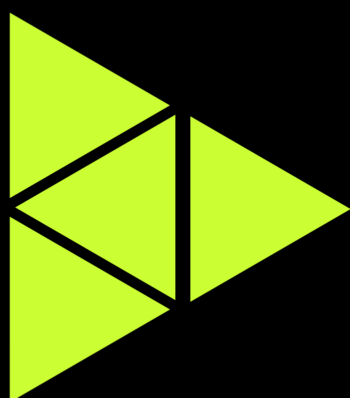


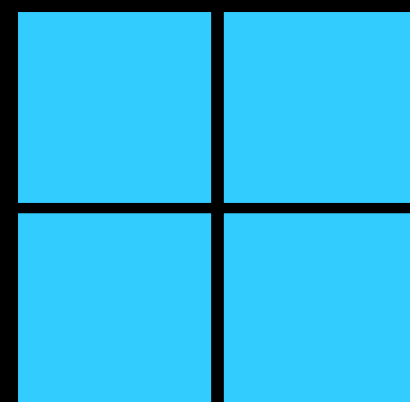
Many ways of combining these constructions...

Simplex / Cross-polytope Compounds IV

Same cross-section idea used in angle measurement can also lead to impossibility results, e.g.:

No edge-tangent simplex / cross polytope compound can include three simplices glued to a central simplex or four cross polytopes in a ring





Proof sketch:

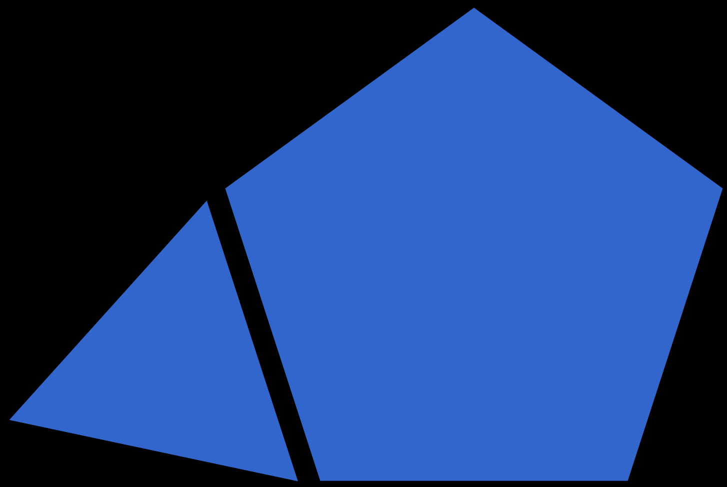
Some edge in the compound would have a link including one of the figures above
But these do not appear in the list of 11 convex triangle-square compounds

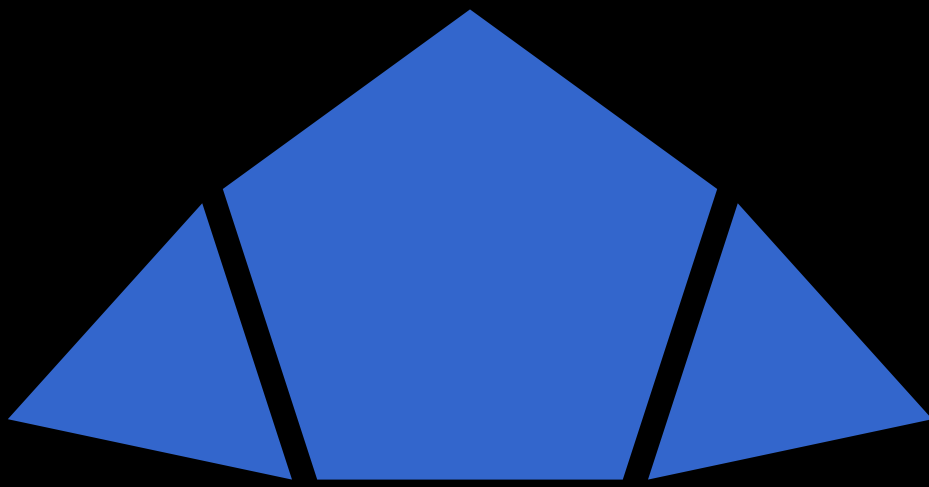
Further gluing can not cover over the bad edge link

Simplex / 600-cell compounds

Glue simplices onto independent faces of 600-cell

Dihedrals are 108 + 60 = 168 degrees



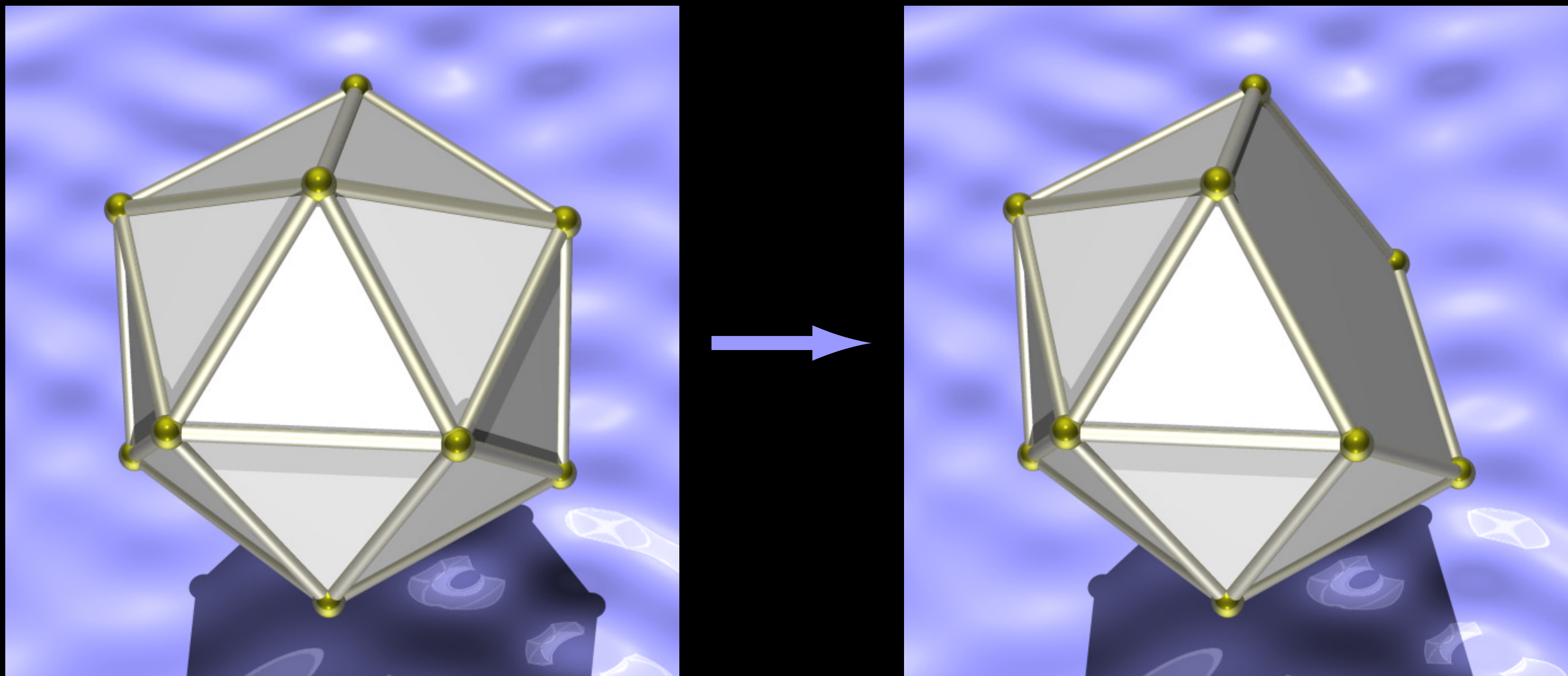


How many different choices are there?

Equivalently, how many independent sets of 120-cell vertices
up to 120-cell symmetries?

600-cell / 600-cell compounds

Slice one vertex from a 600-cell, form convex hull of remaining vertices



POV-ray rendering based on scene descriptions by Russell Towle

Sliced facet is an icosahedron bounded by 72-degree dihedrals
so can glue along slices forming 144-degree angles

End-to-end gluing is closely related to Kuperberg-Schramm construction of
3d sphere packings with high average kissing number

Even better: two adjacent slices form 36 degree dihedrals
wrap cycles of ten doubly-sliced 600-cells around a common triangle


M. C. Escher, *Dragon*, wood-engraving, 1952

Conclusions

2-simple 2-simplicial
4-polytope construction

New infinite families of
edge-tangent simplicial 4-polytopes
and 2*s*2*s* 4-polytopes

Slight improvement to
avg kissing # of spheres in 3d

Interesting combinatorics
of edge-tangent compounds

Fatter *f*-vectors but
boundedness still open

…topological-sphere cell complexes
do have unbounded fatness!
E, K, and Z, unpublished…