

Making Change in 2048

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9th International Conference on Fun With Algorithms
(FUN 2018)

La Maddalena, Italy, June 2018

2048 game



16 squares that can be filled by power-of-two tiles

Each turn: slide all squares in one of four directions

When two equal tiles slide into each other they merge

On each turn, one new small tile appears

Goal: Keep playing long enough to see big tiles

Finiteness of the 2048 game

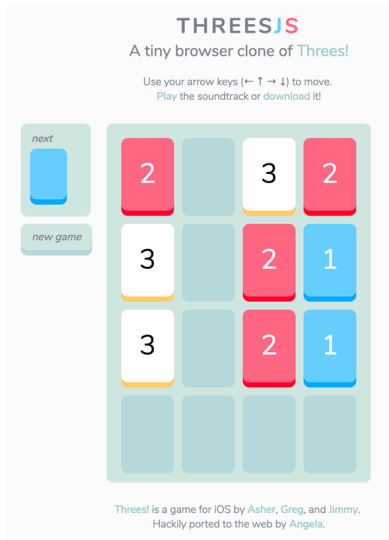


Best possible state at any step:

- ▶ No two tiles are equal
- ▶ They form the binary representation of the current total

Can't reach numbers with more than 16 ones in their binary representation

Related game: Threes



The original
(predecessor of 2048)

Tile values are 1, 2, and $3 \cdot 2^i$

Pairs of tiles can merge when
their sum is in this set
(except $1 + 1 = 2$)

Tiles only slide one step per
turn

Related game: 987

987

SCORE 6380 BEST 6380

Join the numbers and get to the 987 tile! **New Game**

610	89	34	21
		1	3
		1	

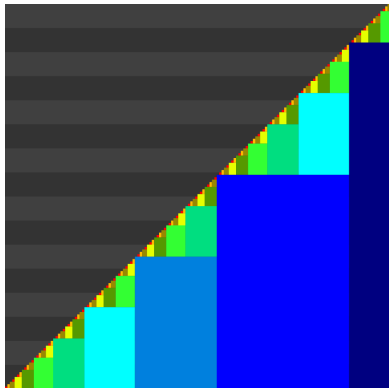
HOW TO PLAY: Use your **arrow keys** or **swipe** to move the tiles. When two neighbouring numbers from the **Fibonacci sequence** touch, they **merge into the next one!**

Tile values are all the Fibonacci numbers

Consecutive Fibonacci numbers can merge
(all pairs whose sum is a Fibonacci number)

Same game play as 2048

987 analysis



Zeckendorf representations.svg from
Wikimedia commons, released under
Free Art License by Parcly Taxel

Zeckendorf representation:

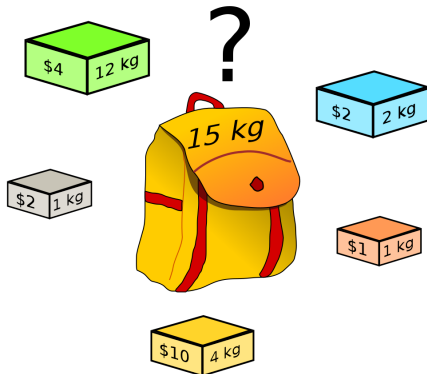
Every number is uniquely
representable as sum of
non-consecutive Fibonacci

E.g. $100 = 3 + 8 + 89$

When this exceeds 16 terms,
at $1 + 3 + 8 + \dots = 9227464$
you're stuck

Do we have to separately analyze each variant?

It's not easy to understand optimal representations as sums. . .



CC-BY-SA image Knapsack.svg by Dake from Wikimedia commons

. . . and what about when all representations have length $O(1)$?

Moser–de Bruijn sequence...

+	0	1	4	5	16	17	20	21
0	0	1	4	5	16	17	20	21
2	2	3	6	7	18	19	22	23
8	8	9	12	13	24	25	28	29
10	10	11	14	15	26	27	30	31
32	32	33	36	37	48	49	52	53
34	34	35	38	39	50	51	54	55
40	40	41	44	45	56	57	60	61
42	42	43	46	47	58	59	62	63

The sequence

$0, 1, 2, 4, 5, 16, \dots$

of numbers whose binary representations have zeros in all odd positions

Every number has a unique representation as $x + 2y$ for x, y in the sequence

...merged with its double

If we made coins either with values in the Moser–de Bruijn sequence or two times a value in the sequence,
1, 2, 4, 5, 8, 10, 16, 17, 20, ...



Composite of images from [Category:Coins by face value on Wikimedia commons](#)

... then cashiers could always make change using only two coins!

Does this game terminate?

What if we replace the powers of two in 2048 by this sequence?

Every total tile value can be represented by only two tiles!

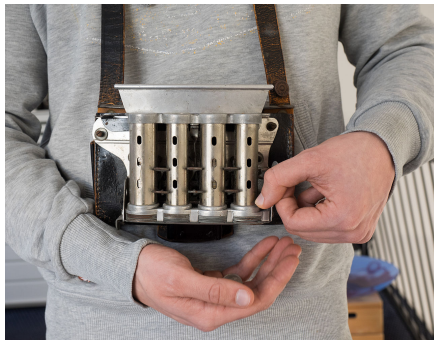
Does that mean we can keep playing forever?



Optimal vs greedy change-making

For usual coins, a greedy algorithm makes change optimally
(same for binary numbers and Fibonacci numbers)

But this is not true in general and optimal change-making is a
standard undergraduate dynamic programming exercise



These two problems are really the same!

Our main result:

For 2048-like games with n squares, where new tile value = 1 and you can merge any subset of tiles whose total value is another tile:

Max score = min # for which greedy change-making uses n coins



So when do these games terminate?

If tiles have arbitrarily large gaps, max score is finite
(and can be calculated quickly from stream of tile values)

In particular this is true for Moser–de Bruijn and its double



Micro chess by Erhan
Cubukcuoglu from
[http://kirill-kryukov.com/
chess/discussion-board/
viewtopic.php?f=6&t=344](http://kirill-kryukov.com/chess/discussion-board/viewtopic.php?f=6&t=344)

Application: choosing the right size for the game board

Max score for $n = 16$ is 954437167

Better to use $n = 9$ and max score 58249.

But some 2048 variants remain unanalyzed

2048

Numberwang

Join the numbers and get to the 12374 tile! Let's play Numberwang!

SCORE
448

BEST
270760

	1	0	5
6	11	63.6	19
		6	3
		9944	