The Topology of Bendless Orthogonal Three-Dimensional Graph Drawing

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What's in this talk?

Unexpected equivalence between a style of graph drawing
and a type of topological embedding

3d grid drawings in which each vertex has three perpendicular edges

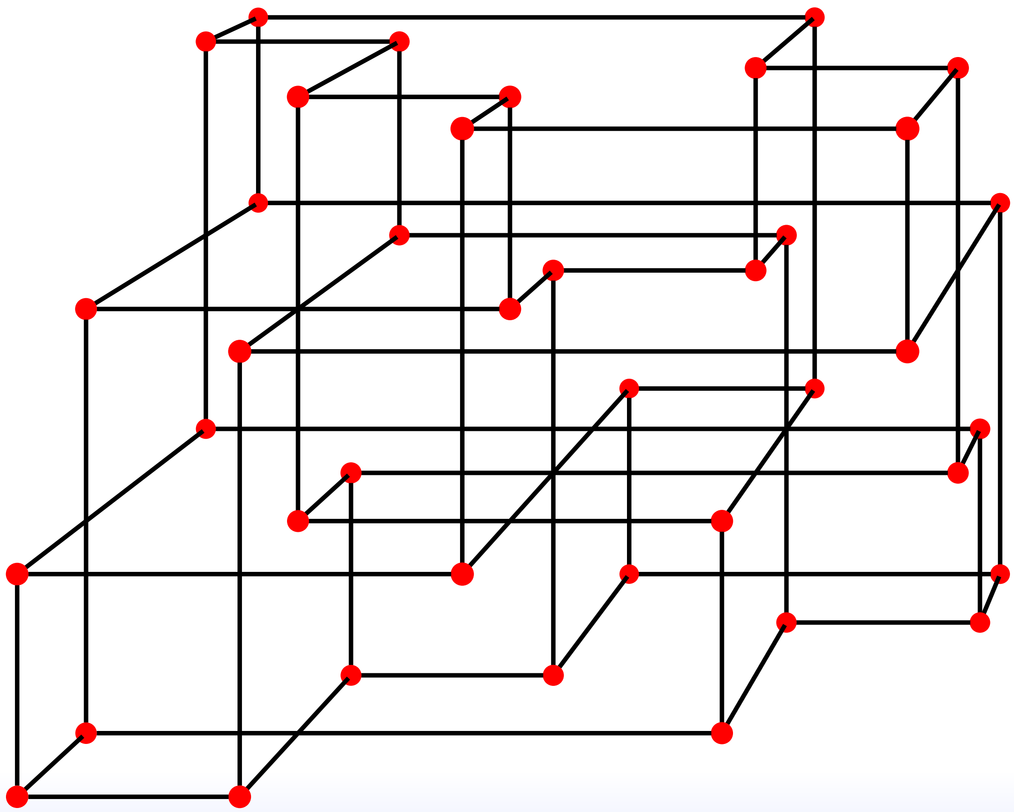
2d surface embeddings in which the faces meet nicely and may be 3-colored

...and its algorithmic consequences

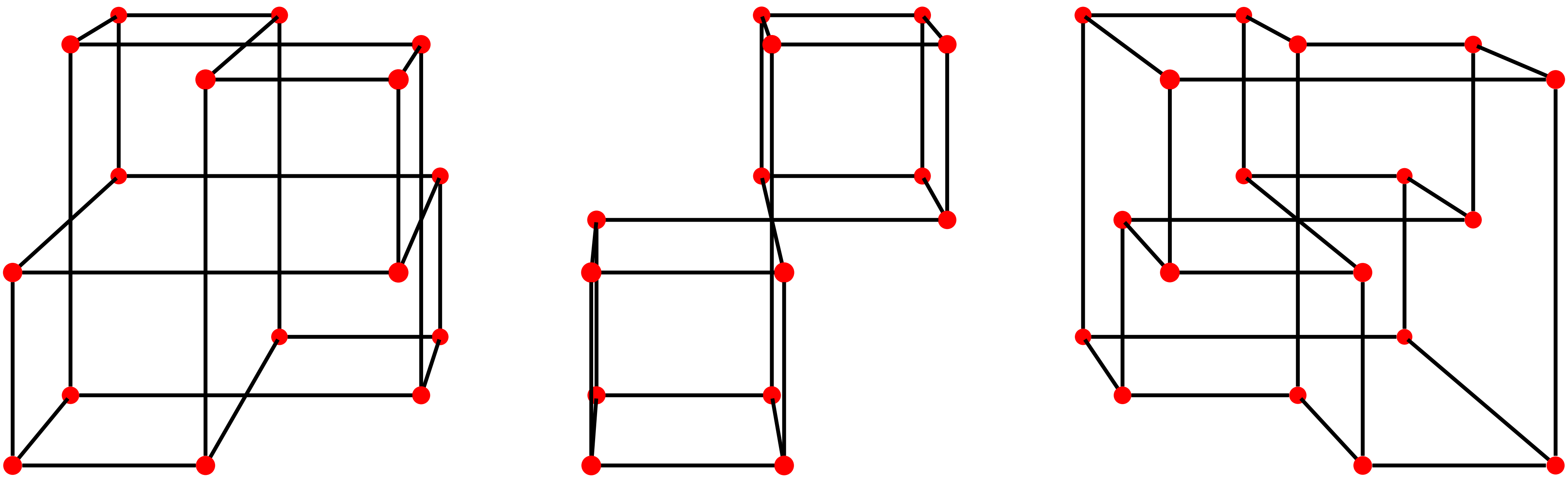
xyz graphs

Let S be a set of points in three dimensions
such that each axis-aligned line contains zero or two points of S

Draw an edge between any two points on an axis-aligned line



Three xyz graphs within a 3 x 3 x 3 grid

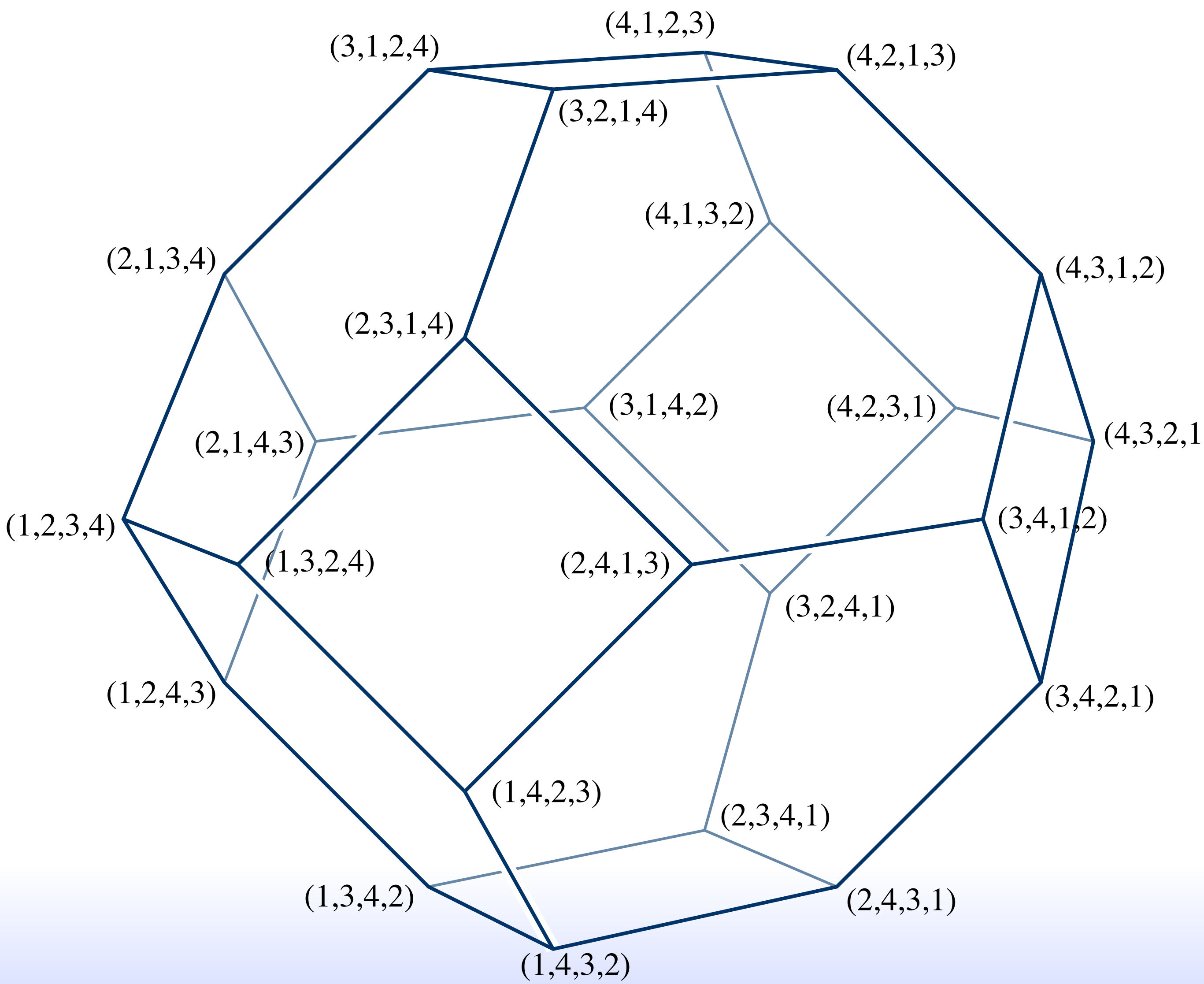


Note that edges are allowed to cross

Crossings differ visually from vertices as vertices never have two parallel edges

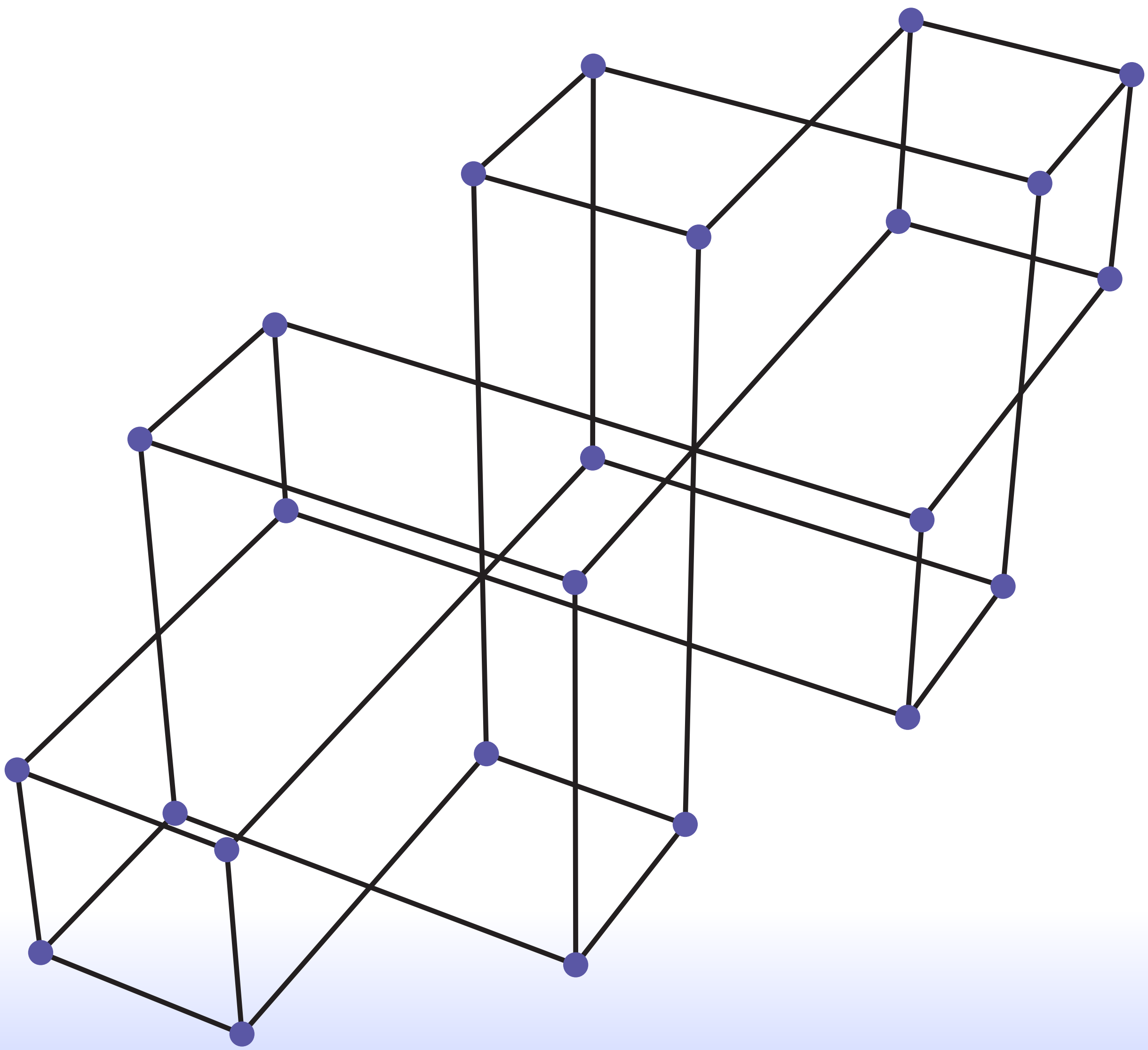
The permutohedron

Convex hull of all permutations of (1,2,3,4) in 3-space x+y+z+w=10
Forms a truncated octahedron



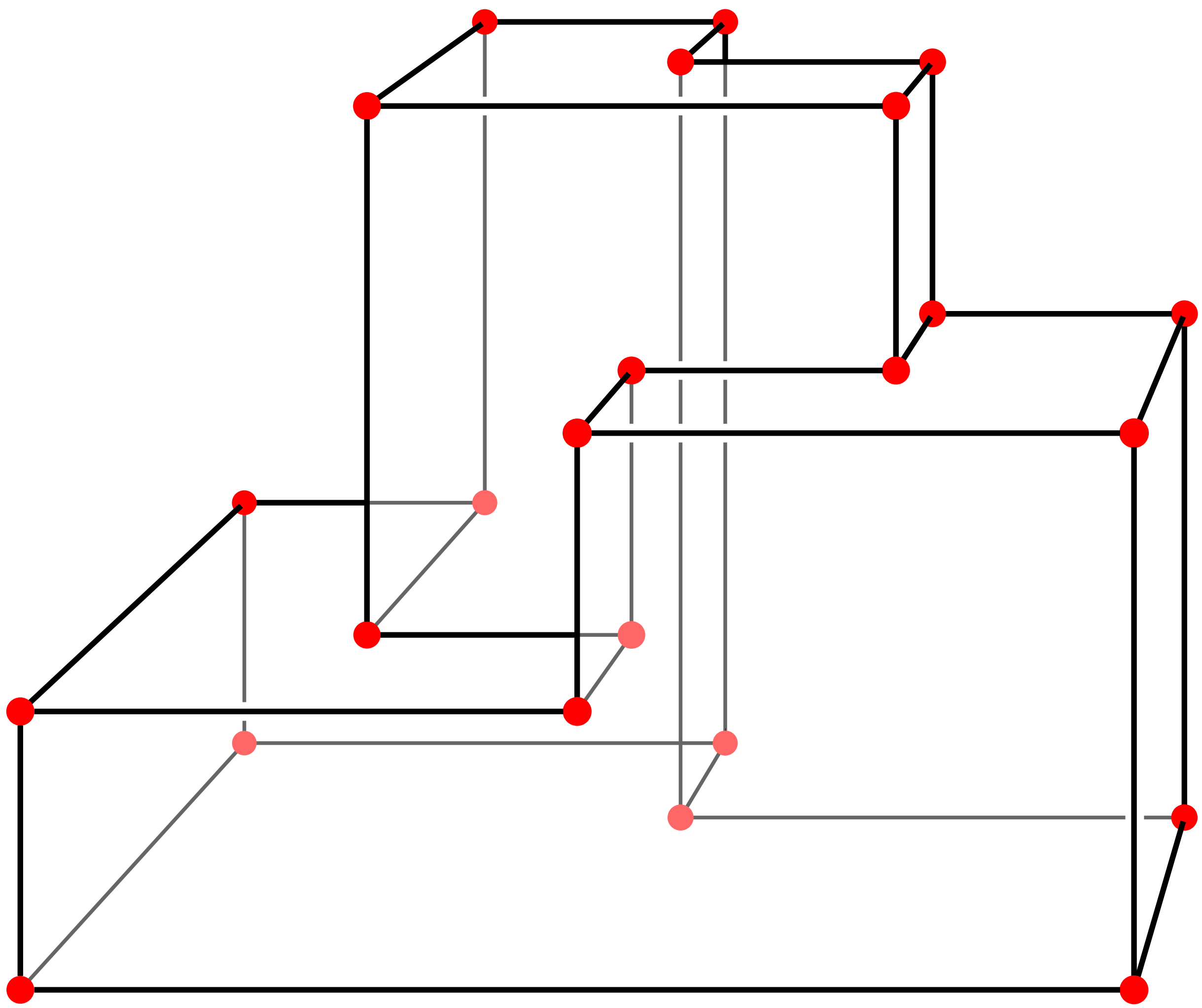
Inverting the permutohedron

Move each permutation vertex to its inverse permutation
affine transform so that the edges are axis-aligned



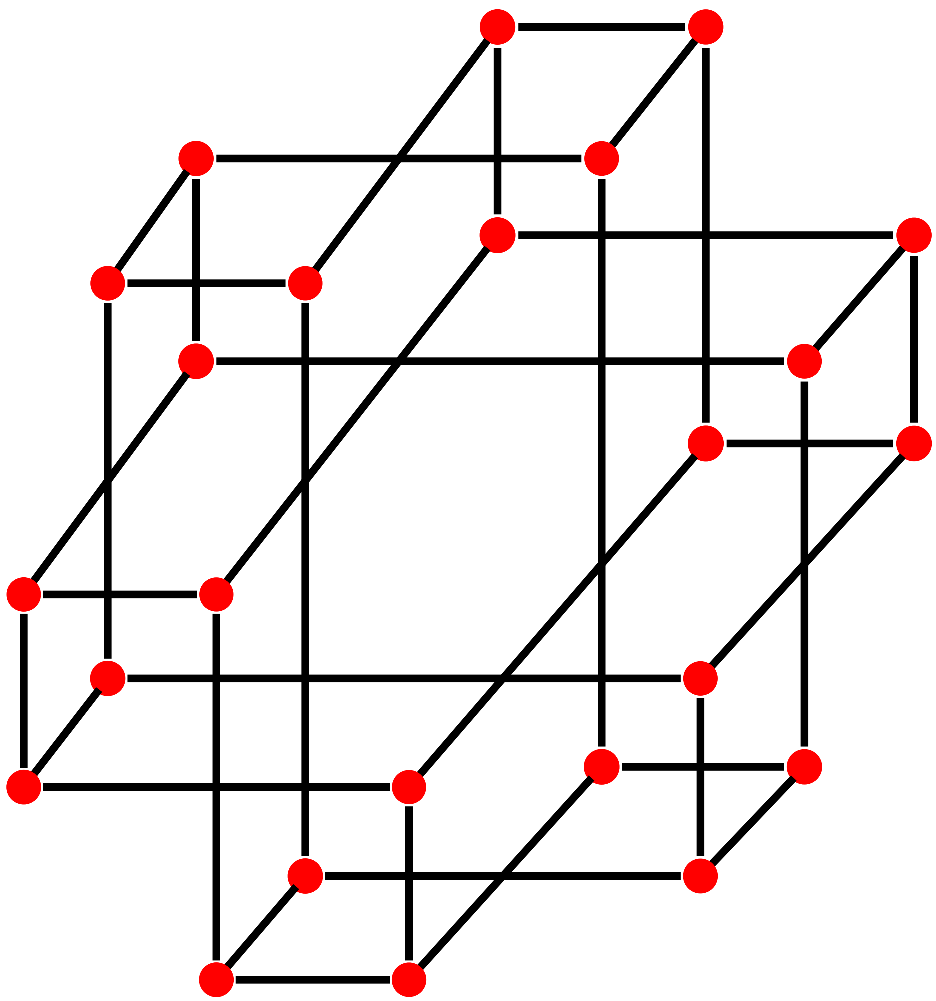
A polyhedron for the inverse permutohedron

Rearrange face planes to form nonconvex topological sphere



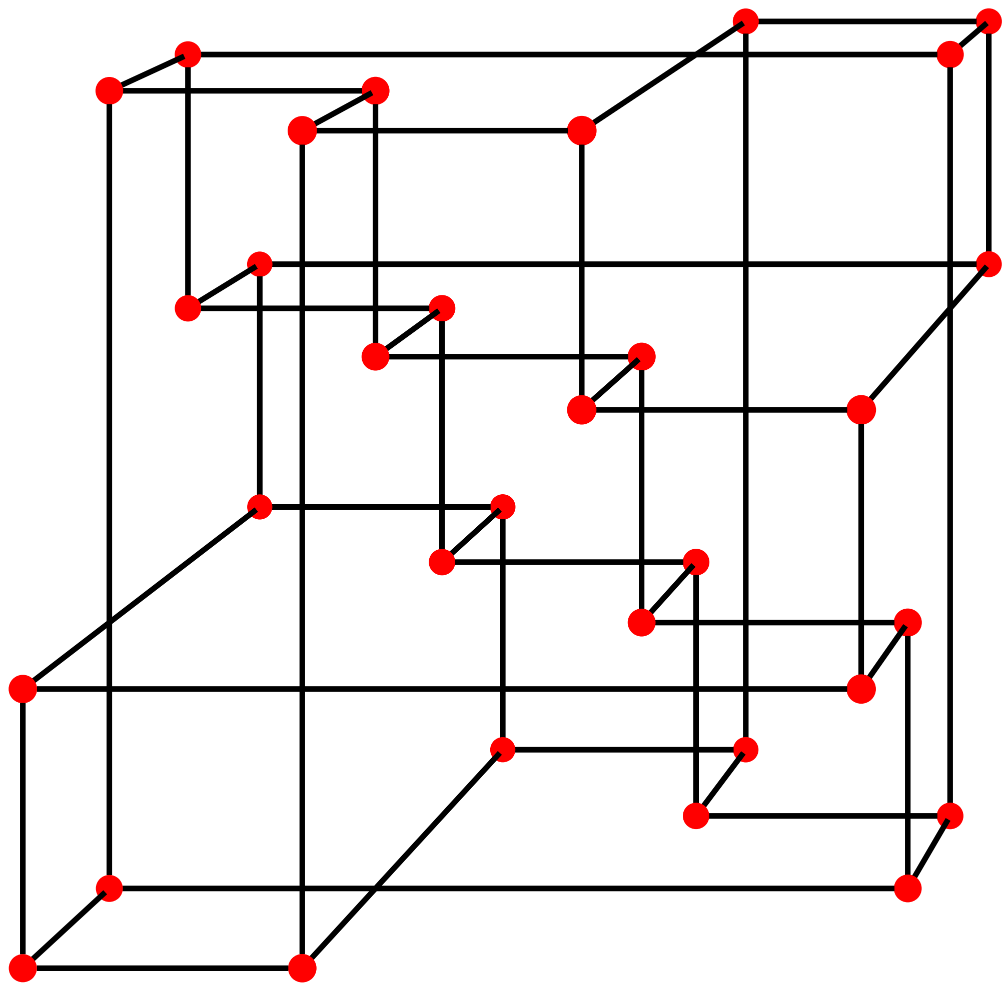
A different xyz graph on 4-element permutations

Project (x,y,z,w) to (x,y,z)



xyz graphs with many vertices in a small bounding box

In n x n x n box, place points such that x+y+z = 0 or 1 mod n



n = 4, the Dyck graph

Basic properties of xyz graphs

3-regular (each vertex has exactly three edges)

Triangle-free
and 5-cycle-free
(but may have longer odd cycles)

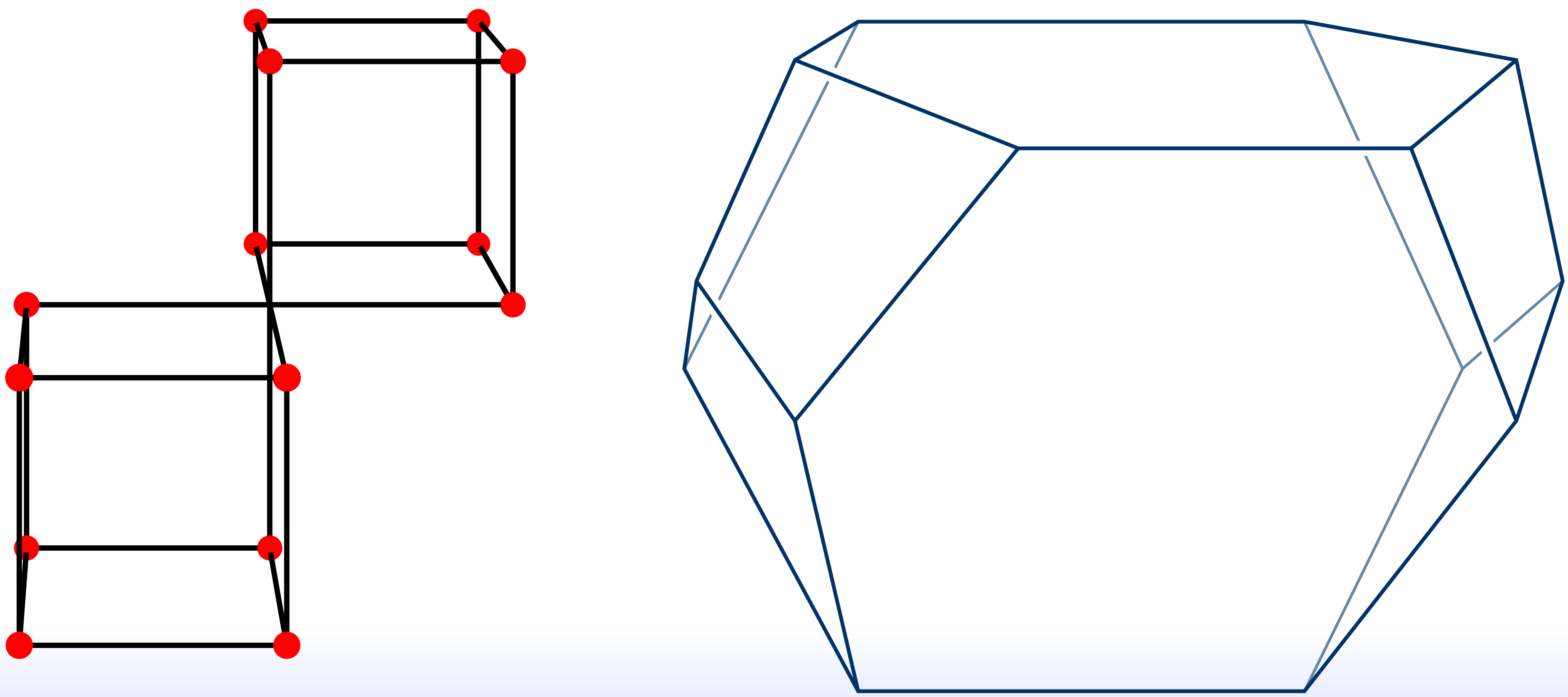
3-connected
(can replace any edge by paths of alternating parallel and perpendicular edges,
with two different choices of perpendicular direction)

Are these (or similar simple properties) sufficient to characterize them?

From xyz graphs to surface embeddings

Edges parallel to any coordinate plane
form degree-two subgraph (collection of cycles)

Form a face of a surface for each cycle



Basic properties of xyz surfaces

All faces are topological disks (by construction)

If two faces meet, they lie on perpendicular planes
the planes meet in a line
and the faces meet in an edge lieing on that line

The faces may be given three colors
(by the direction of the planes they lie in)
and are thus properly 3-colored

From xyz surfaces to xyz graphs

Le G be a 3-regular graph embedded on a surface, so that

faces are topological disks

any two intersecting faces meet in a single edge

the faces are properly 3-colored

(say, red, blue, and green)

Number the faces of each color

Assign coordinates of a vertex:

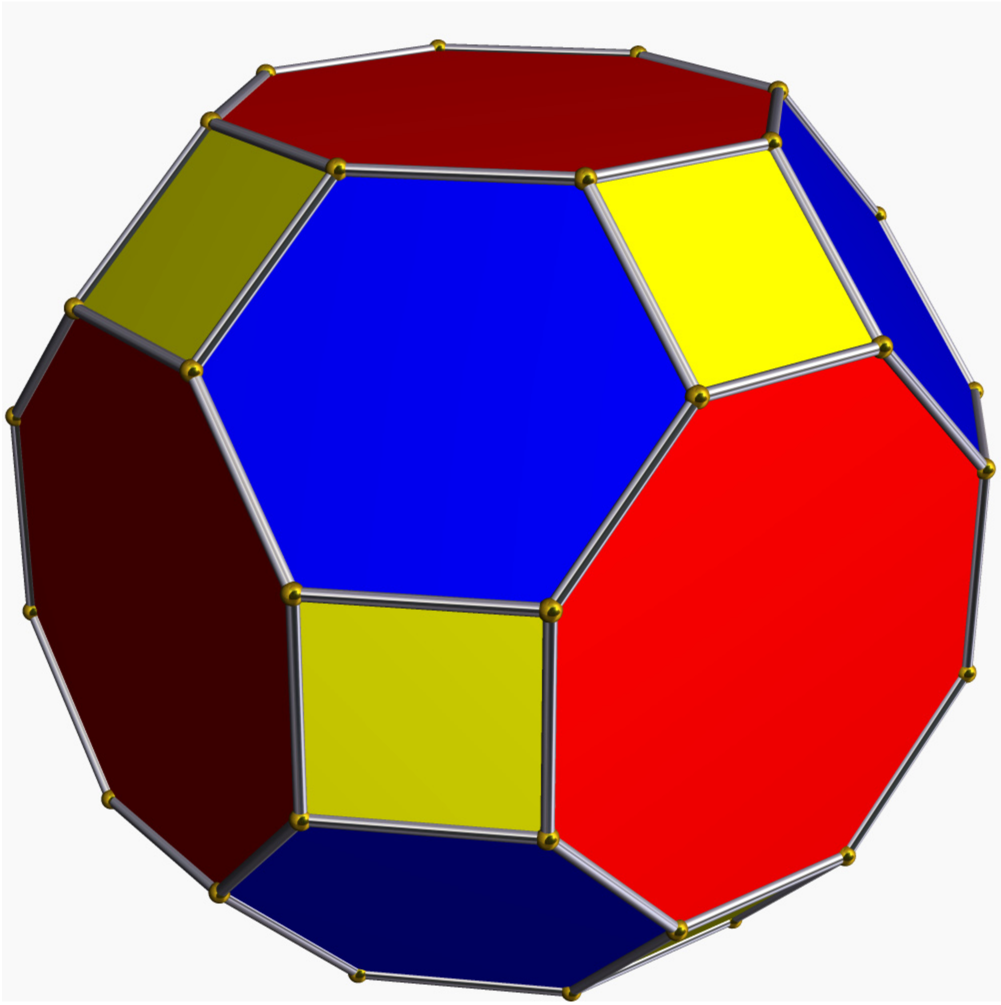
x = the number of its red face

y = the number of its blue face

z = the number of its green face

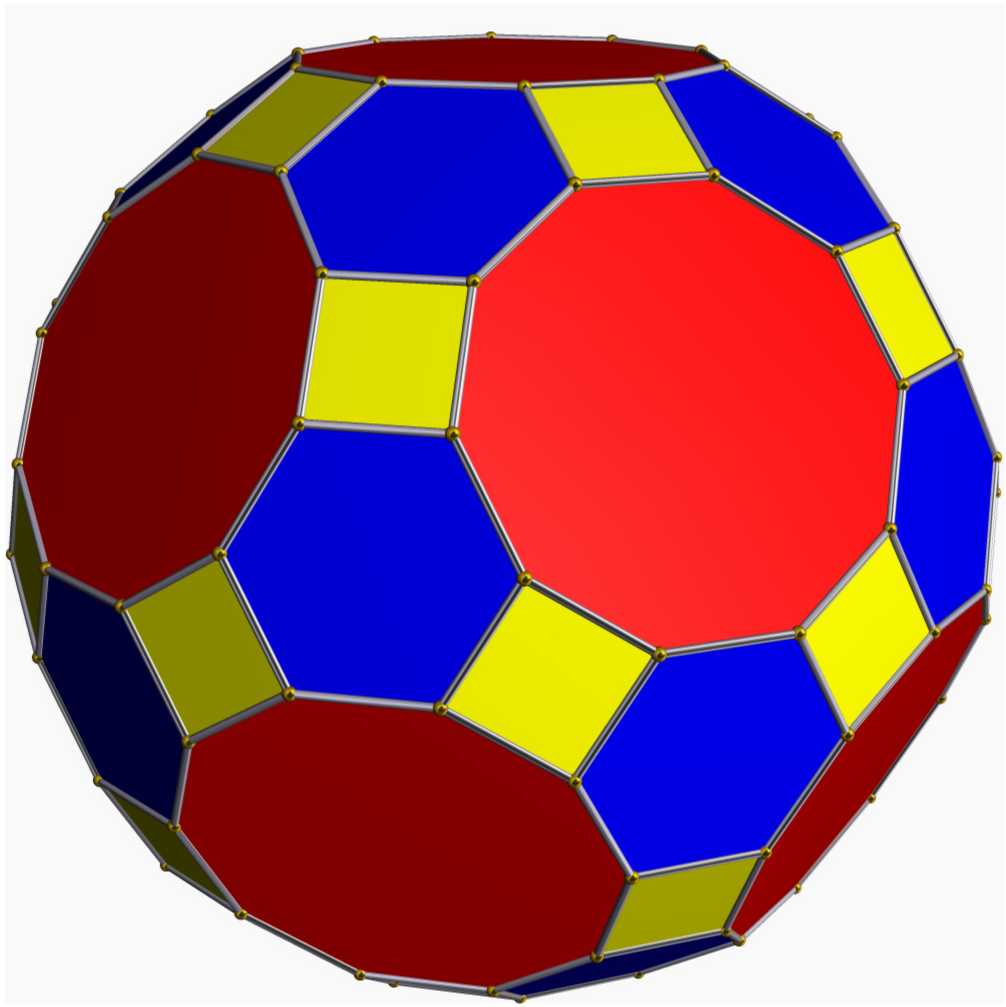
The result is an xyz graph embedding!

Great rhombicuboctahedron



By Robert Webb using Great Stella, http://www.software3d.com/Stella.html
image from http://commons.wikimedia.org/wiki/Image:Great_rhombicuboctahedron.png

Great rhombicosidodecahedron



By Robert Webb using Great Stella, http://www.software3d.com/Stella.html
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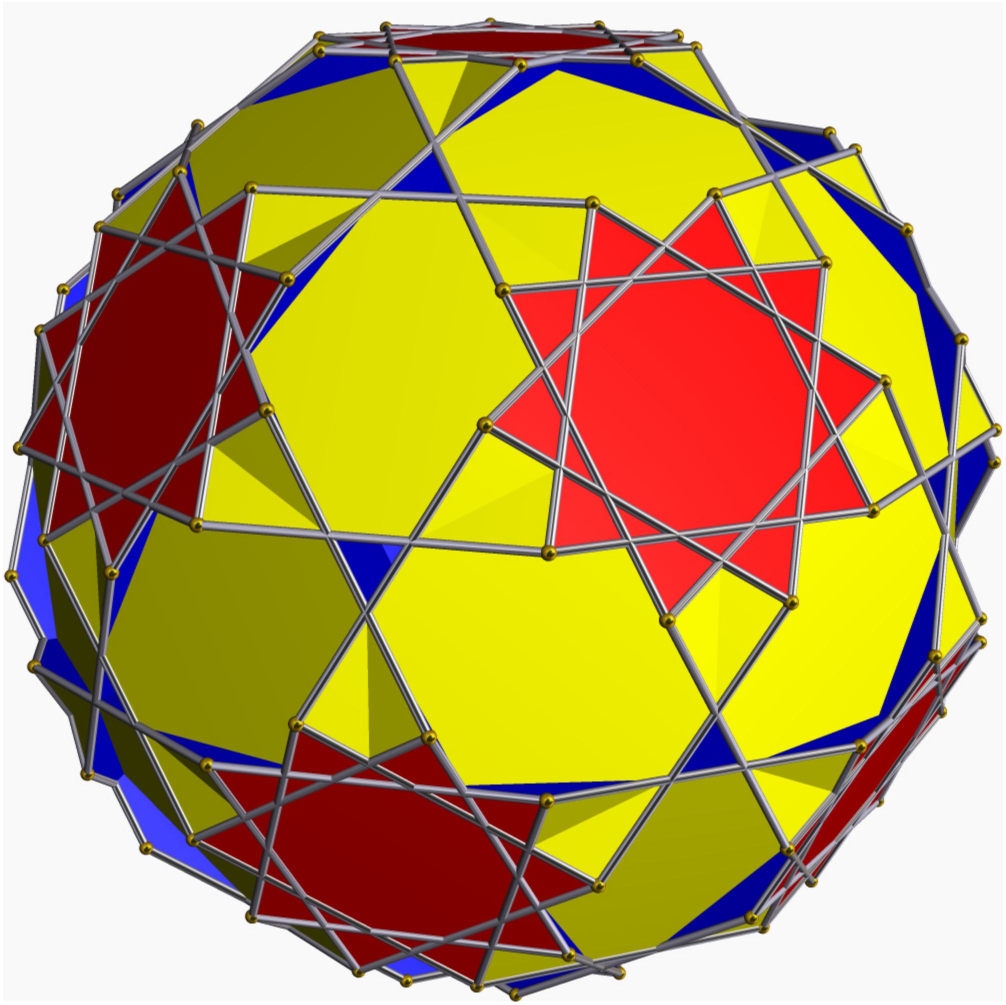
Truncated dodecadodecahedron

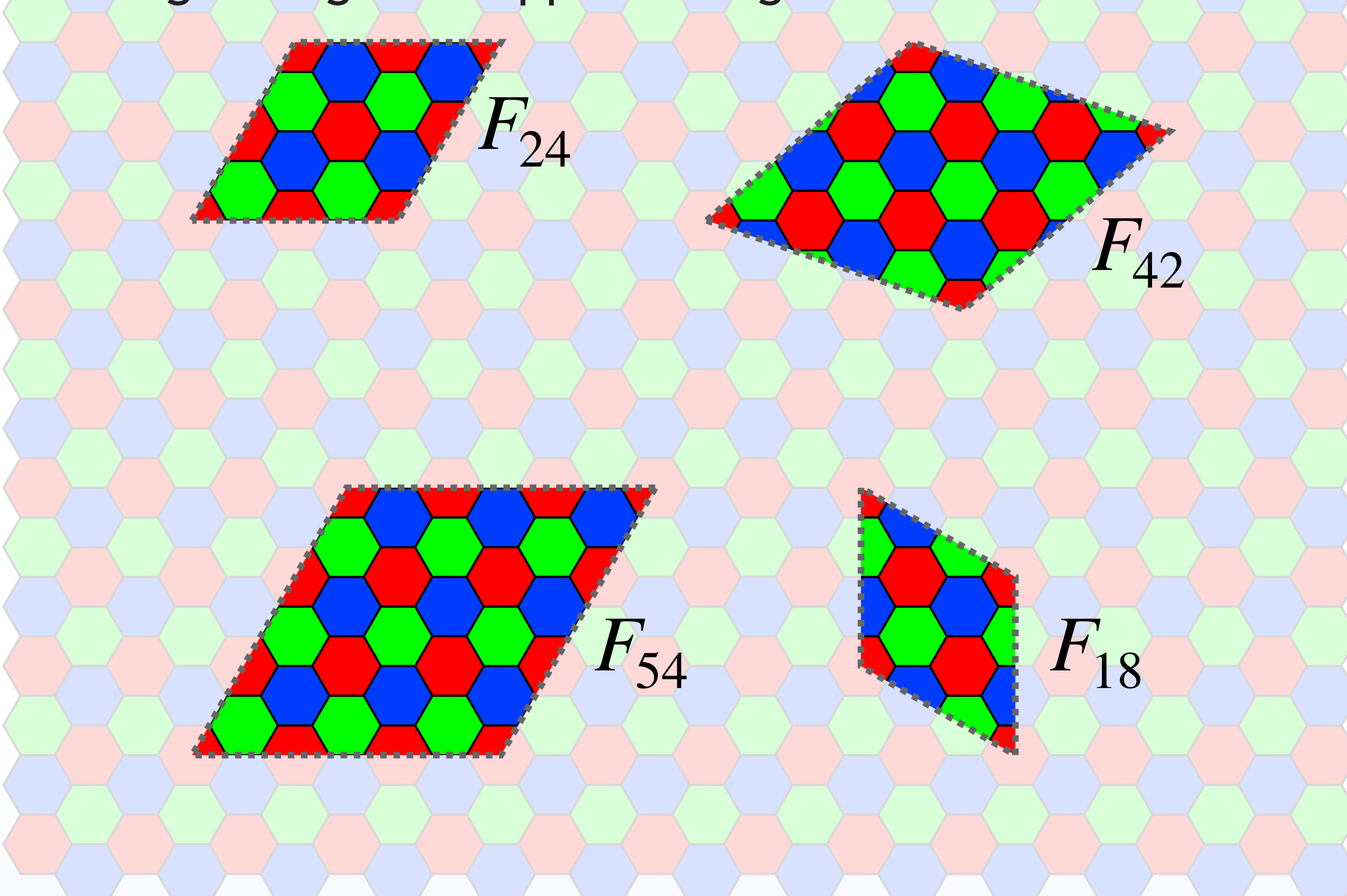


By Robert Webb using Great Stella, http://www.software3d.com/Stella.html
image from http://commons.wikimedia.org/wiki/Image:Truncated_dodecadodecahedron.png

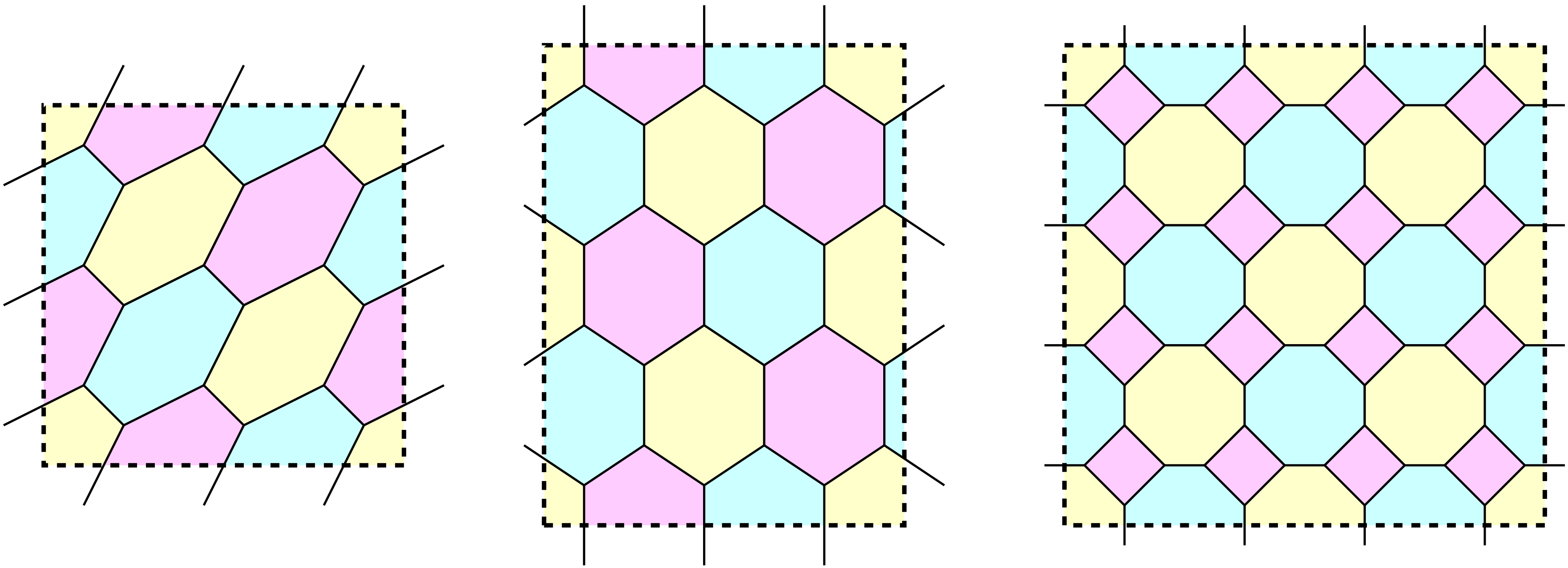
Symmetric graphs on the torus

Start with regular tiling of plane by 3-colored hexagons

Cut out a 60-120 rhombus with matching edge colorings
glue together opposite edges to form a torus



More xyz tori



Leftmost example is order-4 cube-connected cycles network
Embedding generalizes to any even order CCC

Bipartiteness and orientability

Theorem: Let G be an xyz graph.
Then G is bipartite if and only if the corresponding xyz surface is orientable

Orientable surfaces: sphere, torus, ...
Non-orientable surfaces: Möbius strip, projective plane, Klein bottle, ...

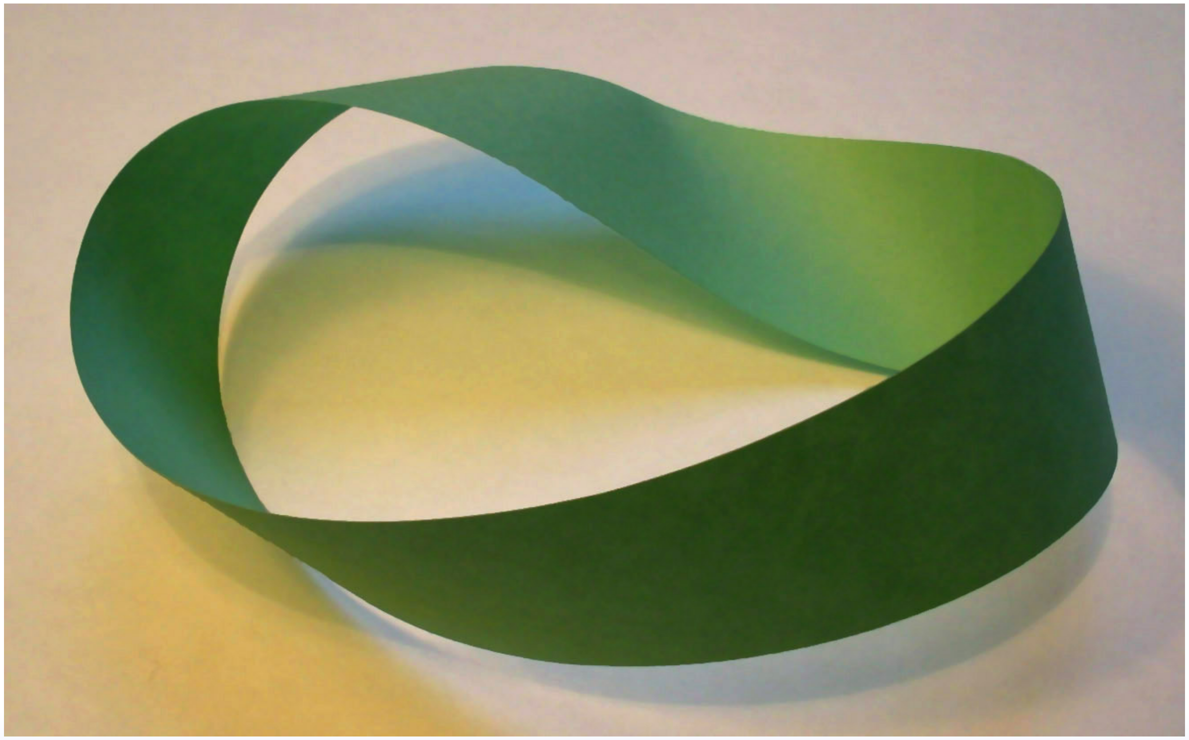
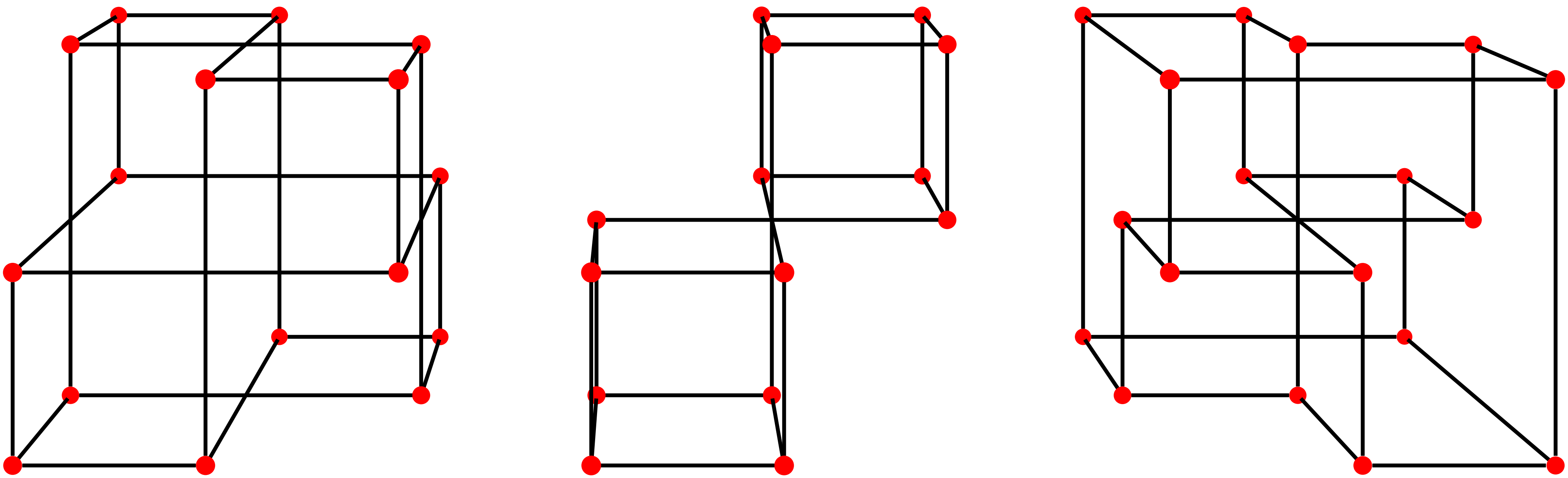
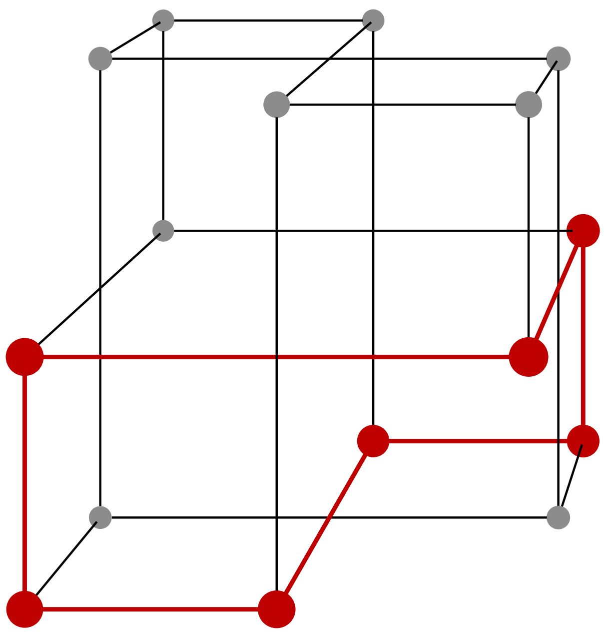


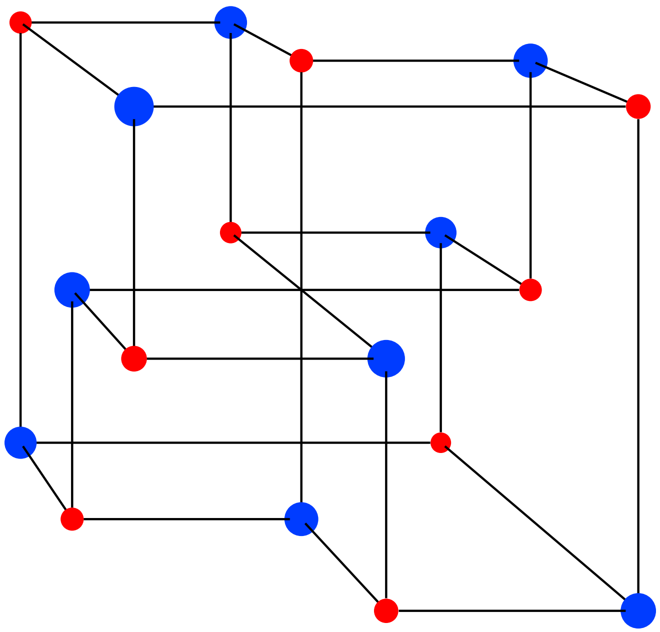
Photo by David Benbennick, http://commons.wikimedia.org/wiki/Image:M%C3%B6bius_strip.jpg

Puzzle: which of these three graphs is not bipartite?



Puzzle solution





Planar xyz graphs

Lemma: If G is a planar xyz graph, its xyz surface must be a topological sphere

Therefore, every planar xyz graph is 3-connected and bipartite

Known:

every 3-connected planar graph is the skeleton of a polyhedron
(so faces meet at most in single edges)

every bipartite polyhedron has 3-colorable faces

Therefore: a planar graph has an xyz embedding
if and only if it's 3-connected and bipartite

Testing if a surface embedding is xyz

Choose arbitrarily two colors for two adjacent faces

Propagate colors:
If some face has neighbors of two colors, assign it the third color

Must successfully color all faces of any xyz surface
(colors are forced by triples of faces along a path connecting any two faces)

So embedding is xyz iff faces intersect properly and coloring succeeds

Testing if a partition of edges into parallel classes is xyz

Find the xyz surface embedding that would correspond to the partition

Check that faces intersect properly and color it

Testing if a graph has an xyz embedding

Try all partitions of its edges into three matchings

Backtracking algorithm:

order vertices so all but two have both incoming and outgoing edges

assign edges of first vertex to matchings, arbitrarily

for each remaining vertex, in order:
try all assignments of its incident edges to matchings
that are consistent with previous choices

Vertex with two incoming edges has only one choice
Vertex with two outgoing edges has two choices
So number of search paths $\leq 2^{n/2-1}$ and total time = $O(2^{n/2})$

Implementation

123 lines of Python

http://www.ics.uci.edu/~eppstein/PADS/xyzGraph.py

Successfully run on graphs on up to 54 vertices

Could probably benefit from additional optimization:

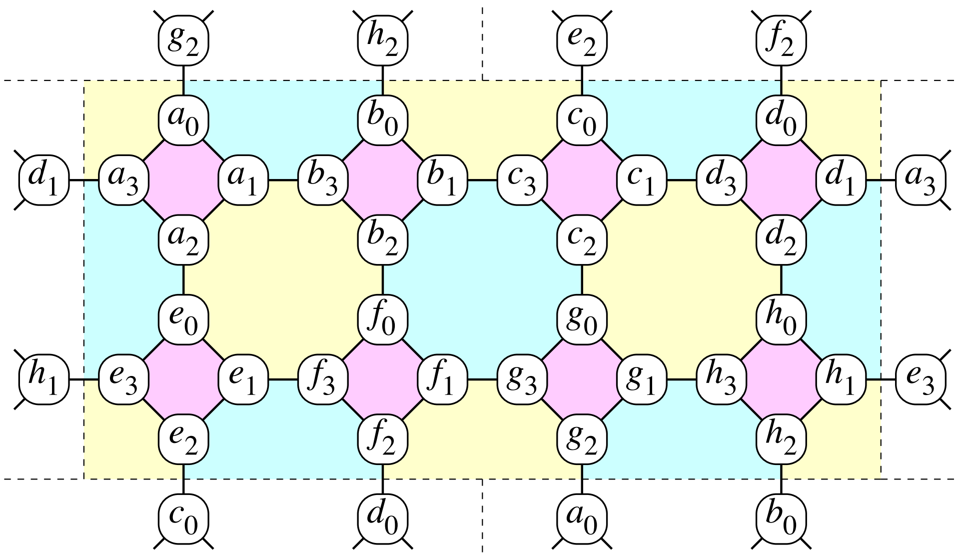
– Faster test for each edge partition

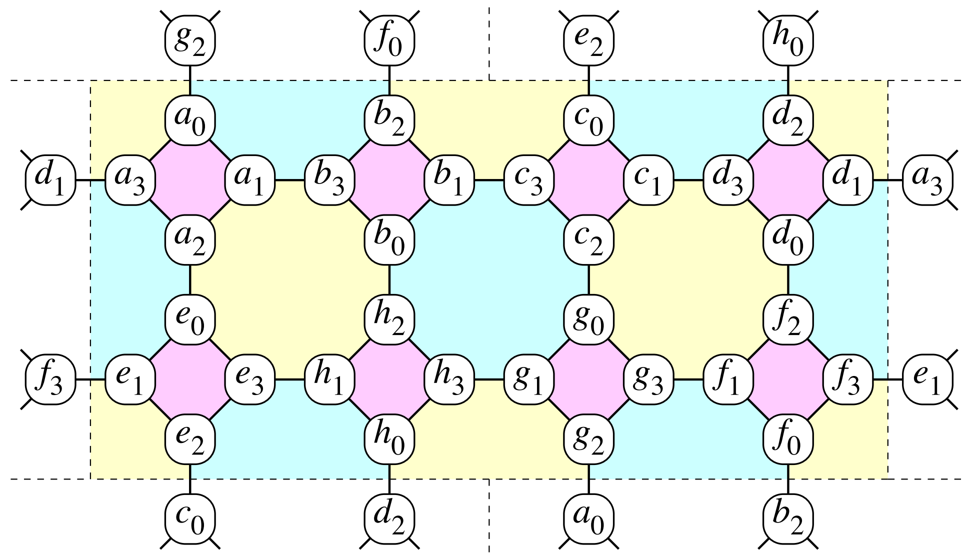
– Early backtrack for bad partial partitions

Uniqueness of xyz embeddings

Planar graphs have unique embeddings

But this 32-vertex graph has two (isomorphic torus) embeddings:





Similar "brick wall" patterns
give larger graphs with
multiple nonisomorphic embeddings

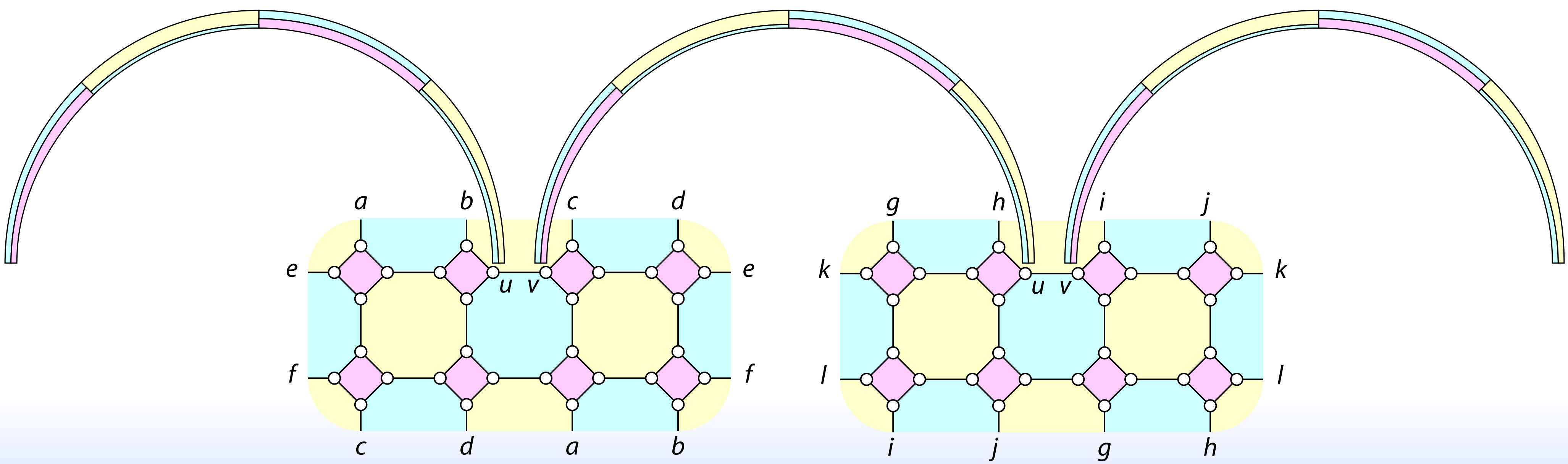
Recognizing xyz graphs is NP-complete

Proof idea: reduction from graph 3-coloring

Represent color as orientation of an edge of a connector gadget

Vertex of graph to be colored becomes planar graph in possible xyz graph

Edge in graph to be colored becomes edge gadget
formed from three connectors and two ambiguous tori



Conclusions

Interesting type of 3d graph drawing

Equivalence with 2d surface embedding leads to some deep theory

It's NP-complete

but...

that doesn't prevent us from implementing algorithms and finding drawings