Realization and Connectivity of the Graphs of Origami Flat Foldings

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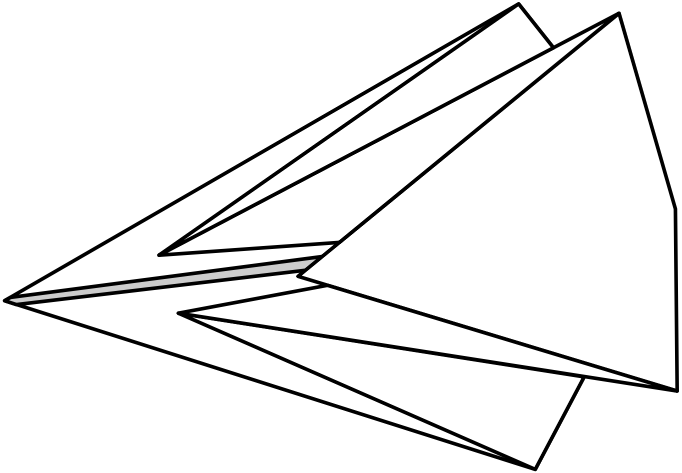
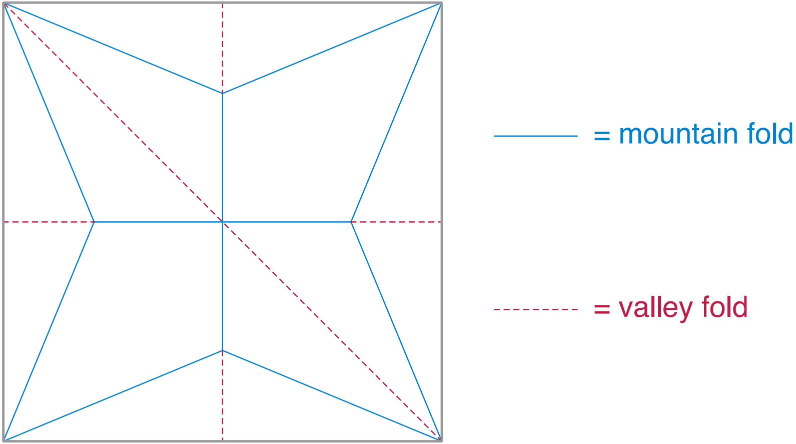
An experiment, I

Fold a piece of paper arbitrarily so that it lies flat again (without crumpling)



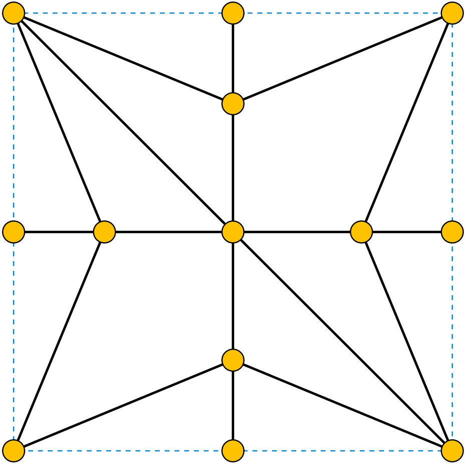
An experiment, II

Unfold it again and look at the creases from its folded state



An experiment, III

It looks like a graph!



An experiment, III

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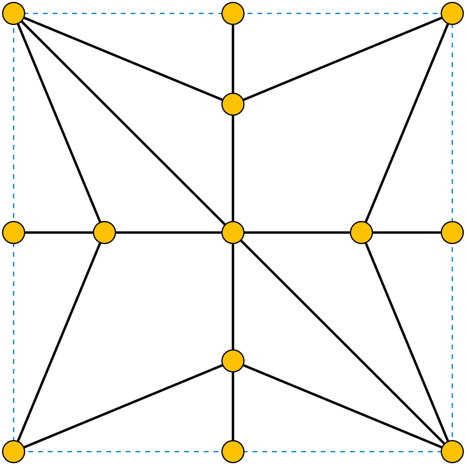
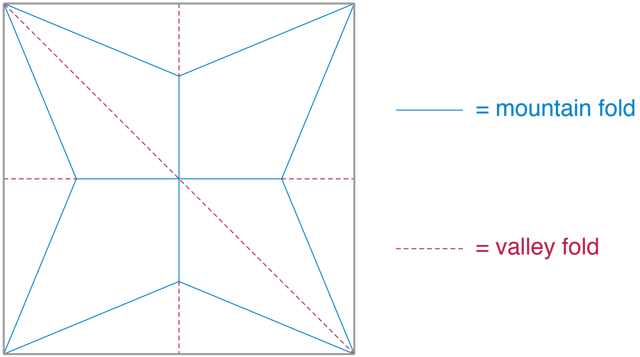


So, what graphs can you get in this way?

Local constraints at each vertex

Maekawa's theorem: at interior vertices,

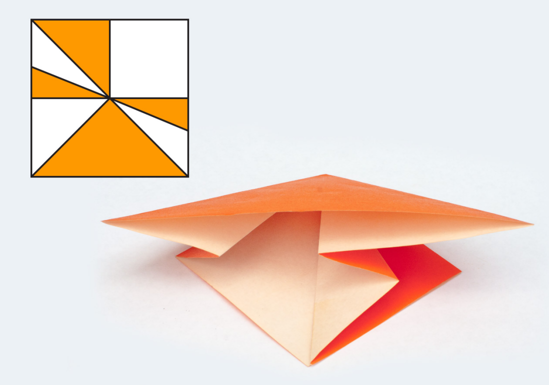
$$|\#\text{ mountain folds} - \#\text{ valley folds}| = 2$$



So all vertex degrees must be even and ≥ 4

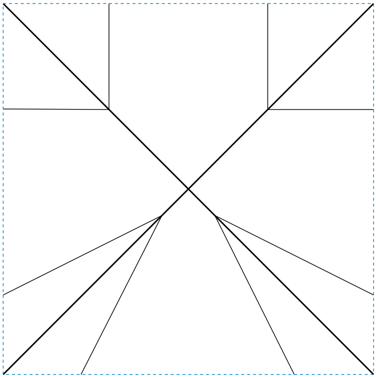
More local constraints at each vertex

Kawasaki's theorem: at interior vertices,
total angle facing up = total angle facing down
(alternating sum of angles must be zero)



Unclear what effect this has on combinatorial structure

Local constraints are not enough



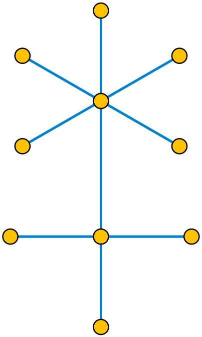
This pattern cannot be folded

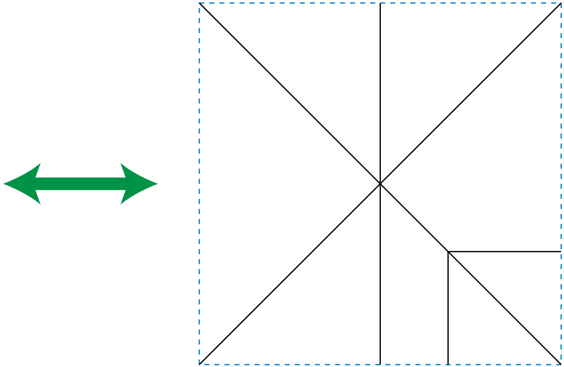
Central diagonal cross forces two opposite creases to nest tightly inside each other

Additional folds on the outer nested crease bump into the inner nested crease

Theorem 1

Tree T is realizable with internal vertices interior to paper and leaves on boundary $\iff$ all internal degrees are even and ≥ 4





Simplify by avoiding boundary conditions

Draw our tree on an infinite flat surface

Cropped from File:2007-04-27 15-01-24 Germany Baden-Württemberg Lausheim.jpg
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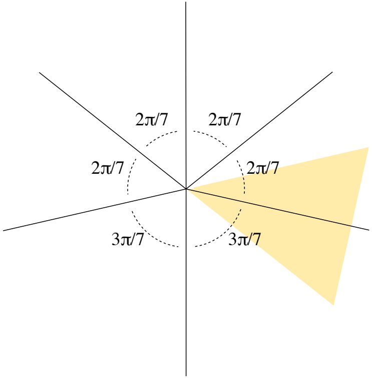
Leaves on boundary $\Longleftrightarrow$ diverging infinite rays

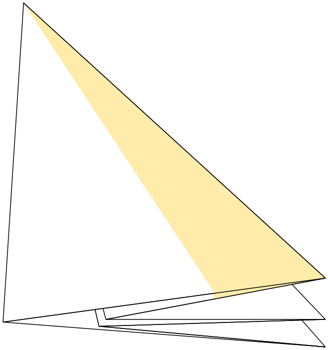
(Much like Voronoi realizations of trees;
cf. Liotta & Meijer, CGTA 2003)

Main idea of proof

Construct tree top-down from root

Maintain buffer zones to prevent creases from nearing each other





Alternative graph model for infinite paper

Instead of interpreting infinite rays as leaves,
add a special vertex at infinity as their shared endpoint

Cropped from File:Vanishing Point of Railway.jpg by Vespertunes on Wikimedia commons

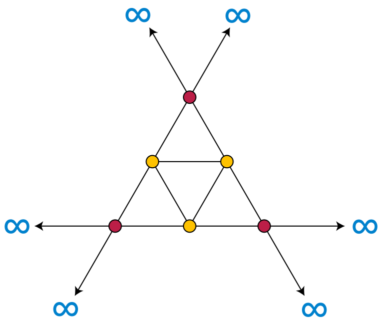
...so trees of Theorem 1 become series-parallel multigraphs

Theorem 2

The graphs of flat folding patterns with a vertex at infinity are:

- 2-vertex-connected
- 4-edge-connected
- not separable by removal of any 3 finite vertices

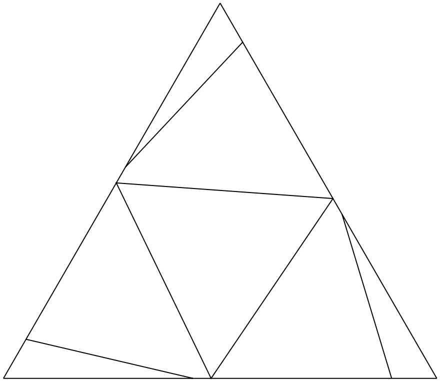
Proof ideas:
convexity of subdivision
rigidity of triangles



An unrealizable graph

Return to finite paper sizes

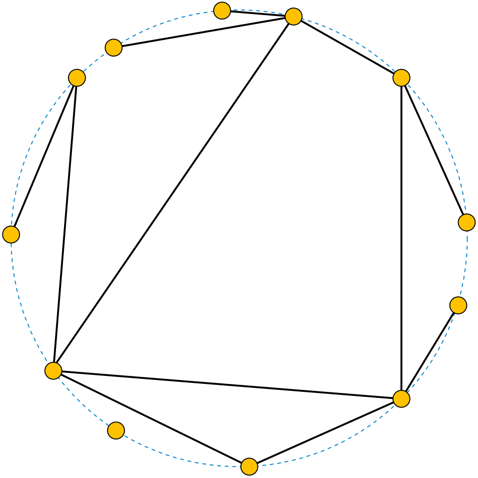
A different simplifying assumption:
All vertices are on the boundary of the paper



This triangle cannot be folded flat
(the three corners get in each others' way)

Theorem 3

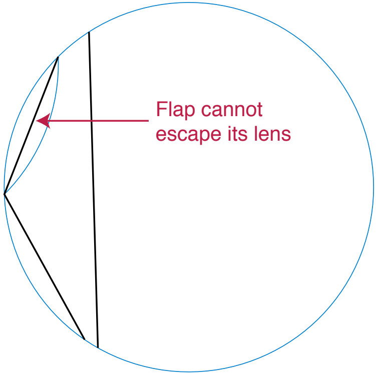
Every outerplanar graph can be realized as a flat-foldable crease pattern on circular paper, all vertices on the boundary of the paper



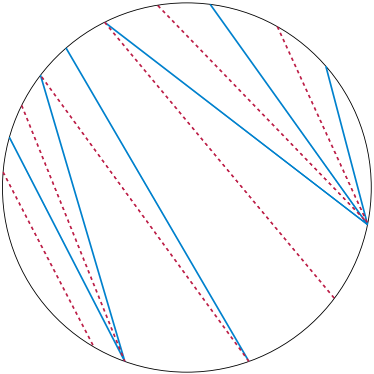
Theorem 3 (stronger variant)

If a crease pattern has all vertices on the boundary of a piece of circular paper, then it can always be folded flat

Region bounded by >2 creases has a crease whose flap cannot cross the other creases

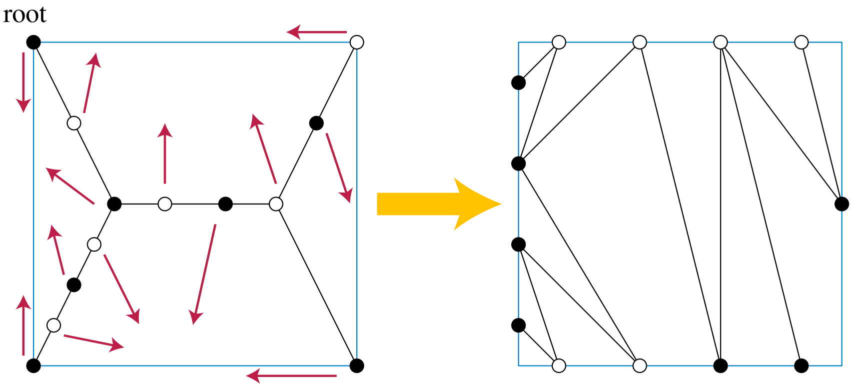


If all regions are bounded by ≤ 2 creases, we can accordion-fold the pattern



Theorem 4

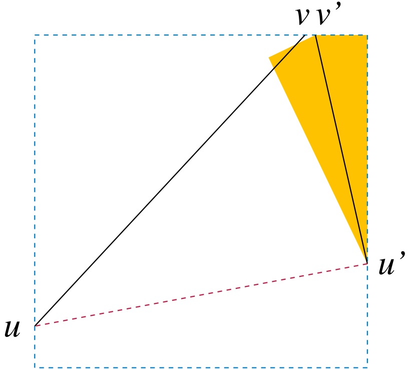
Tree T realizable, all vertices on the boundary of square paper $\Longleftrightarrow$ the subtree formed by removing all leaves from T has ≤ 4 leaves

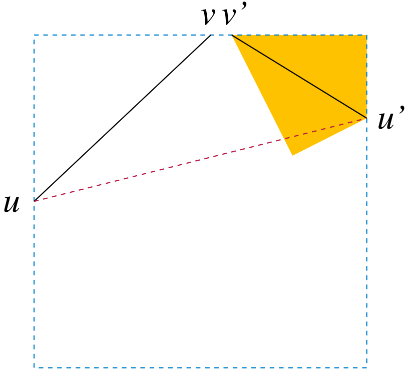


Mapping 4-leaf subtree to square, leaving room for removed leaves

The geometric part of Theorem 4

If a crease pattern has all vertices on the boundary of a piece of square paper, then it can always be folded flat





Proof involves finding *semi-safe* flaps that can interfere with only one of their neighboring creases

Put interfering pairs of flaps on opposite sides of paper

Conclusions

Several partial characterizations of the graphs of flat foldings in several different simplified models

- Trees with all internal vertices interior to the paper
- Connectivity of graphs with a vertex at infinity
- Graphs with all vertices on the boundary of circular paper
- Trees with all vertices on the boundary of square paper

Complete characterization still remains open