Improved Bounds for Intersecting Triangles and Halving Planes

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Abstract

If a configuration of m triangles in the plane has only n points as vertices, then there must be a set of

$$\max \begin{cases} \lceil m/(2n-5) \rceil \\ \Omega(m^3/(n^6 \log^2 n)) \end{cases}$$

triangles having a common intersection. As a consequence the number of halving planes for a three-dimensional point set is $O(n^{8/3} \log^{2/3} n)$. For all m and n there exist configurations of triangles in which the largest common intersection involves

$$\max \begin{cases} \lceil m/(2n-5) \rceil \\ O(m^2/n^3) \end{cases}$$

triangles; the upper and lower bounds match for $m = O(n^2)$. The best previous bounds were $\Omega(m^3/(n^6 \log^5 n))$ for intersecting triangles, and $O(n^{8/3} \log^{5/3} n)$ for halving planes.

1 Introduction

Suppose we have a set of triangles which share many of their corners. If there are n corners, and more than $2n$ triangles, then the graph of triangle edges must not be planar and so some two triangles must overlap.

A recent paper of Aranov et al. [1] generalized this observation. Suppose some m triangles again share n corners. The result proved in that paper was that some set of $\Omega(m^3/(n^6 \log^5 n))$ triangles have a common intersection. In other words, some point in the plane is covered by the interiors of that many triangles. As a consequence, the number of k-sets of a three-dimensional point set (i.e. the number of ways the point set can be divided by a plane into subsets of k and $n-k$ points) can be bounded by $O(n^{8/3} \log^{5/3} n)$.

There are several shortcomings with this result. First, it does not match the case $m = 2n$ that we started with: the first overlap is detected when $m = \Omega(n^2 \log^{5/3} n)$. Second, the exponent on the log seems too large to be true. And third, there is no indication how tight these bounds might be.

In this note we attempt to address these shortcomings. We show that the number of triangles that must have a common intersection is exactly $\lceil m/(2n-5) \rceil$ for $m = O(n^2)$, and $O(m/n + m^2/n^3)$ in general. We also slightly improve the general lower bound, to $\Omega(m/n + m^3/(n^6 \log^2 n))$.

Our new lower bound reduces the number of logarithmic factors for large m from five to two, and our upper bound indicates how far this technique can go in proving bounds on k-sets. If we could prove an $\Omega(m^2/n^3)$ bound for intersecting triangles, the number of three-dimensional k-sets would then be $O(n^{5/2})$, which essentially matches the best known bounds for the two-dimensional k-set problem [3]. Further improvements could not be based on intersecting triangles, and would also improve the planar k-set bounds.

2 Tight bounds for small m

We first consider $m = O(n^2)$, and show tight bounds for the number of overlapping triangles in this case. Our first bound holds for all m.

Theorem 1. *For any configuration of m triangles with n corners, there is a set of $\lceil m/2n \rceil$ triangles that has a common intersection.*

Proof: The inside angles of a single triangle add up to π. Therefore all angles add to $m\pi$, so for some corner x the angles add to at least $m\pi/n$. The

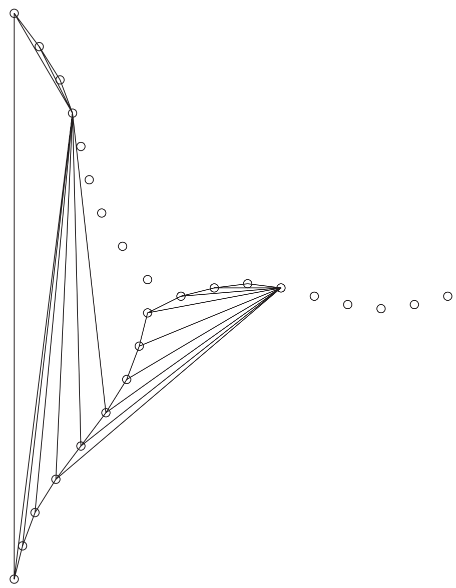

Figure 1. Upper bound construction: two of six fans in G_4 for $n = 27$.

total angle around x is 2π, so the weighted average of the overlap number around x is $m/2n$; the maximum overlap is then at least that average. $\square$

We now give an upper bound on the size of the largest intersection that, for $m = O(n^2)$, matches the lower bound above to first order.

Theorem 2. *For any n and m with $m < n^2/3 - O(n)$, there is a configuration of m triangles with n corners, in which the largest common intersection involves $\lceil m/(2n-10) \rceil$ triangles.*

Proof: Assume for the moment that n is a multiple of 3. We place $n/3$ points on each of three rays with a common center point. The rays are curved slightly in a "S"-shape. Let the points be labeled x_i, y_i, and z_i, where smaller indices denote points closer to the center point, and where the names of the rays are chosen so that the points x_i near the center are concave towards z_i, and convex towards y_i; this is reversed for points near the ends of the rays, on the other half of the "S". We then form a sequence of planar graphs G_k as follows. Figure 1 depicts part of G_4 for $n = 27$.

G_k will have 3-way rotational symmetry, so we only describe the edges connecting points x_i. Let $k' = (n/3) - k + 1$; we assume $k \le k'$. All points x_i are connected to x_{i-1} and x_{i+1}. Point x_k is connected to all $\{x_i, i < k\}$ and all $\{y_i, i \leq k'\}$. Point $x_{k'}$ is connected to all $\{x_i, i > k'\}$ and all $\{z_i, i \geq k\}$. In other words we form two fans, emanating from points x_k and $x_{k'}$.

Each G_k is planar, and has $2n-6$ triangular faces, none of which occur in any other graph G_j. If we choose some value x, and take the set of triangles in all G_k for $k \leq x$, the maximum intersection size will be x. The number of triangles is $(2n-6)x$. So for m triangles, we achieve intersection size $\lceil m/(2n-6) \rceil$. If n was not a multiple of 3, we ignore at most two points, giving a bound of $\lceil m/(2n-10) \rceil$. This construction works as long as $x \leq n/6$, i.e. $m \leq n^2/3 - O(n)$. □

Both bounds can be made to match exactly. In the lower bound, at least three corners are on the convex hull. If only three such vertices exist, they form a triangle, and can only meet other triangles within a total angle less than π. If four or more such vertices exist, the situation is even better. So the total bound is $\lceil m/(2n-5) \rceil$. In the upper bound, we first merge x_1, y_1, and z_1 into a common center point, saving two points but losing three triangles. If $n \neq 1 \bmod 3$, we simply add a point to one or two of the rays; this does not cause any difficulty in triangulation and only changes the $O(n)$ term in the restriction $m < n^2/3 - O(n)$. Again the bound is $\lceil m/(2n-5) \rceil$.

The techniques used in Theorem 1 also lead to a novel proof of the Euler characteristic formula $V + F - E = 2$, for planar straight line graphs. By the Fáry embedding theorem (e.g. see [4]) this can be extended to arbitrary planar graphs, matching the generality of the usual proof of Euler's formula.

3 Upper bounds for large m

In the previous section we proved an upper bound of $O(m/n)$ on the intersection size, but our construction only worked for $m = O(n^2)$. We extend this to a bound for all values of m, by giving a different construction with intersection size $O(m^2/n^3)$ when $m = \Omega(n^2)$. Therefore the intersection size is always $O(m/n + m^2/n^3)$.

As a consequence, $O(n^{5/2})$ is the best bound on the number of 3-dimensional k-sets that could possibly be proved using the intersecting triangle method. Such a bound would essentially match the best known bound on the number of planar k-sets [3].

Our proof is based on the following:

Lemma 1 (Aronov et al. [1]). *For any n and $m = \Omega(n)$, there exists a set of m intervals on the real line, sharing n endpoints, such that at most*

$O(m^2/n^2)$ intervals have a common intersection.

Proof: Choose those intervals containing fewer than $O(m/n)$ points. $\square$

This result is tight: Aronov et al. prove that for any such set of intervals, there is a subset of $\Omega(m^2/n^2)$ having a point of common intersection.

Theorem 3. *For any n points in the plane, and any $m = \Omega(n^2)$, there exist sets of m triangles having the points as corners, such that at most $O(m^2/n^3)$ triangles have a common intersection.*

Proof: Partition the points into two sets A and B of $n/2$ points each. For each point x in A, we consider the points of B in sorted order around x, and form $2m/n$ triangles by connecting x with pairs of points in B corresponding to the intervals in Lemma 1. These triangles have largest common intersection size $O((2m/n)^2/(n/2)^2) = O(m^2/n^4)$. Therefore the union of the $n/2$ sets of triangles has largest common intersection $O(m^2/n^3)$. $\square$

4 Lower bounds for large m

We finally modify the bound of Aronov et al. [1] to have fewer logarithmic factors. We first outline the method of Aronov et al., as our bound is a simple modification of theirs. The method is best expressed as an algorithm performing the following steps:

1. For each triangle, let the *long edge* be the one with longest projection onto a horizontal line. For each possible edge e, count the number of triangles for which e is the long edge.

2. Find a subset T of the triangles and a number y, such that each long edges is the long edge for $\Theta(y)$ triangles. If there are x long edges, there are then $xy = \epsilon m$ triangles in T. In [1], this step is done by dividing the long edges into $O(\log n)$ buckets according to the number of triangles they touch; some bucket must have $m/O(\log n)$ triangles, so here $\epsilon = 1/O(\log n)$.

3. For each long edge, form y^2 *quadrilaterals* by pairing up each triangle. $O(xy^2)$ quadrilaterals are formed.

4. Project onto a horizontal line the intervals formed by the middle two vertices of each quadrilateral (some intervals will be formed by more than one quadrilateral). Using *weighted interval selection* (see [1]) find a subset of intervals all containing some common point v. As shown in [1], this can be done so that, if z denotes the number of intervals and w denotes the number of interval endpoints, $z/w = \Omega(xy^2/(n \log n))$.

5. Reverse the projection, turning v into a vertical line cutting between the middle points of z quadrilaterals. For each cut quadrilateral, define an interval on v, defined by the lines connecting the two middle points with the two opposite outside points. There are z intervals, having a total of $O(wn)$ endpoints.

6. Using *unweighted interval selection* [1] find a point p in $\Omega(z^2/(wn)^2)$ intervals, and hence that many quadrilaterals.

7. Each triangle contributes to at most y quadrilaterals. So the number of triangles containing p is

$$\begin{aligned}
\Omega(z^2/((wn)^2 y) &= \Omega\left((z/w)^2/(n^2 y)\right) \\
&= \Omega\left((xy^2/(n \log n))^2/(n^2 y)\right) \\
&= \Omega\left((xy)^3/(xn^4 \log^2 n)\right) \\
&= \Omega\left((\epsilon m)^3/(xn^4 \log^2 n)\right)
\end{aligned}$$

For the choices described in step 2, $x = O(n^2)$, $\epsilon = 1/O(\log n)$, and the total bound is $\Omega(m^3/n^6 \log^5 n)$.

As hinted above, we improve the bound by modifying the selection of long edges touching many triangles performed in step 2.

Theorem 4. *For any configuration of m triangles with n corners, there is a set of $\Omega(m^3/n^6 \log^2 n)$ triangles that has a common intersection.*

Proof: We find a set of x long edges, touching $\Theta(y)$ triangles each, so that if $xy = \epsilon m$, then $x/\epsilon^3 = O(n^2)$. This can be done as follows.

We divide the long edges into $O(\log n)$ buckets as before. Bucket 0 contains the edges touching between 1 and 4 triangles. Bucket 1 contains the edges touching between 4 and 16 triangles. And in general bucket i contains the edges touching between 4^i and 4^{i+1} triangles.

Then if there are at least $m/2$ triangles in bucket 0, we choose that bucket. If there are at least $m/4$ triangles in bucket 1, we choose that bucket. In general, we choose bucket i if there are at least $m/2^{i+1}$ triangles having long edges in that bucket. The sum of all these bounds is simply m, so some bucket must have the desired number of points.

Therefore we can find a bucket containing long edges touching between 4^i and 4^{i+1} triangles each, with at least $m/2^{i+1}$ triangles total. So $\epsilon = 1/2^{i+1}$, and $x = m\epsilon/y = O(m/8^i)$, from which it follows that $x/\epsilon^3 = O(n^2)$. □

5 Conclusions

We have demonstrated bounds on the common intersections of m triangles sharing n corners, of $O(m/n+m^2/n^3)$ and $\Omega(m/n+m^3/(n^6\log^2 n))$. This is tight for $m = O(n^2)$, but leaves open the question of finding a tight bound for $m = \Omega(n^2)$. Logical candidates for such a bound are $\Theta(m/n + m^2/n^3)$ and $\Theta(m/n + m^3/n^6)$. The former seems more likely, for the following reason. For small m the true bound is $\lceil m/(2n-5)\rceil$, because we can cover the point set with a collection of maximal planar graphs, no two graphs using the same triangle. But this only works for $m = O(n^2)$, because we run out of triangles touching any given convex hull edge and can no longer find a new maximal planar graph. So something happens at $m = \Theta(n^2)$. If the bound were $\Theta(m/n+m^3/n^6)$, there would be another breakpoint at $m = \Theta(n^{5/2})$. So the former hypothesis involves fewer breakpoints than the latter.

We note that the proof of Theorem 4 works for a slightly more general situation in which the triangles may have curved sides that are portions of an arrangement of *pseudo-lines* (curves intersecting at most once per pair). In this case horizontal and vertical are defined relative to a *topological sweep* of the pseudo-line arrangement [2]. However Theorem 1 is proved using angles, which do not behave as well for pseudo-lines as they do for lines. Can we extend Theorem 1 to pseudo-lines? Or is this an example of a fundamental difference between lines and pseudo-lines?

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