The Parametric Closure Problem

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The closure problem

Find max-weight downward-closed subset of a partial order

Classical example: open-pit gold mining

Sunrise Dam Gold Mine, Australia.

Elements = blocks of ore

Partial order = must remove higher block to access lower one

Weight = value of extracted gold − extraction cost

Bicriterion closure problem

Optimize a nonlinear combination of two different sums of element values

E.g. return on investment:

Find downward-closed subset of partial order achieving max profit/cost, where

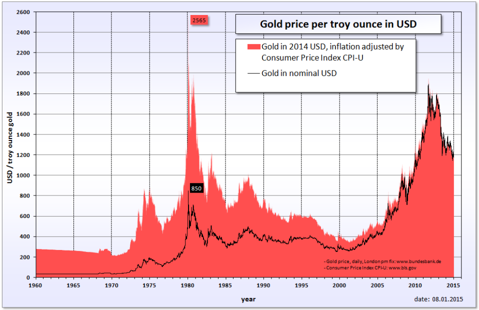
- Profit is sum of extracted values of chosen ore blocks
- Cost is sum of extraction costs of chosen ore blocks

Parametric closure problem

Element value is a linear function of an unknown parameter:

$$\text{value} = \text{amount of gold extracted} \times \text{price of gold} - \text{extraction cost}$$

Goal: construct the (convex piecewise linear) function mapping each parameter value to its optimum closure

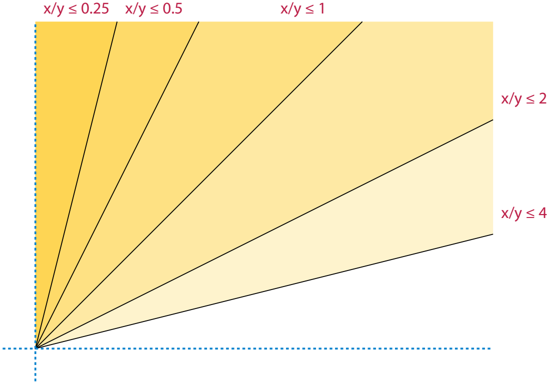


PD image "Gold price in USD" by Realterm on Wikimedia commons

Converting bicriterion to parametric problems

Whenever a bicriterion problem maximizes a quasiconvex function of its two arguments x and y, its optimum can be found as one of the parametric optima for $\lambda x + y$

Quasiconvex: all lower sets $\{(x, y) \mid f(x, y) \leq \theta\}$ are convex



Example: return on investment $f(x, y) = x/y$, with $x, y > 0$

Classical results

- Closure problem has a polynomial time solution (reduction to min cut)
- Parametric and bicriterion solutions for other problems (especially minimum spanning trees and shortest paths)
- Several isolated problems that in retrospect fit into this framework

Detail of Raphael's *School of Athens*, with Plato and Aristotle

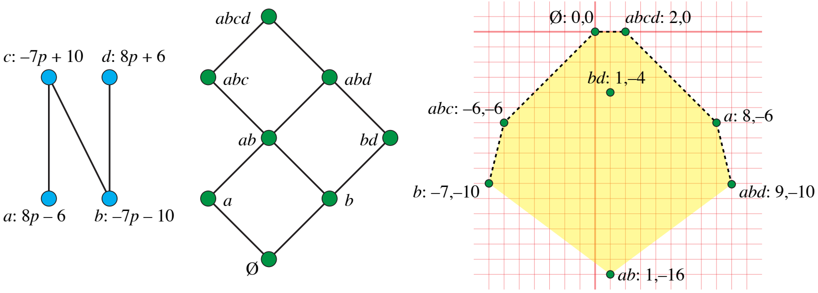
New contributions

- Formulation of parametric and bicriterion closure problems
- Equivalence to natural problems in combinatorial geometry (complexity of polytopes constructed by combinations of Minkowski sums and convex hulls of unions)
- Efficient solutions for several important classes of partial orders, but not for the general problem

Ukiyo-e image of blind monks examining an elephant, by Itcho Hanabusa

A geometric point of view

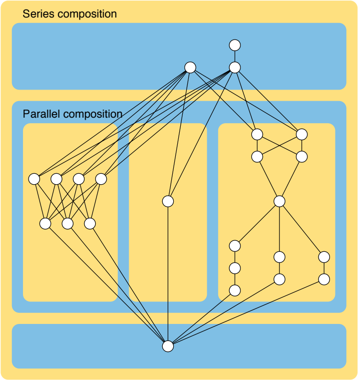
Replace parametric weight $f(p) = ap + b$ by geometric point (a, b)



Given a partial order (left), the candidate solutions (center) map to points in the plane (right)

Then parametric optima form the upper part of the convex hull (the value of p determines the slope of a tangent to the hull)

Series-parallel partial orders



Series composition of orders: all elements of one order are below all elements of the other

Parallel composition: no relation between the elements of the two orders

Corresponding operations on convex hulls: convex hull of union, Minkowski sum

Neither operation increases #vertices $\Leftrightarrow$ parametric problem has $O(n)$ solutions

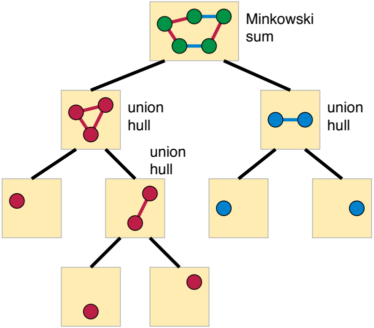
Series-parallel parametric closure algorithm

Geometric problem: evaluate expression tree of polygons in which each operation is either hull of union or Minkowski sum

Hull of union: merge sorted sequences of vertex x-coords, local fixup for nonconvexities

Minkowski sum: merge sequences of edge slopes

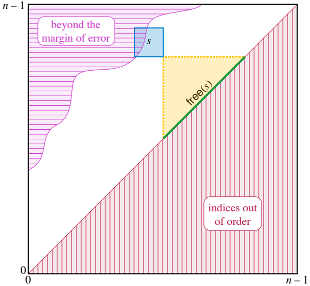
Using dynamic finger trees, logs telescope $\Rightarrow O(n \log n)$ time

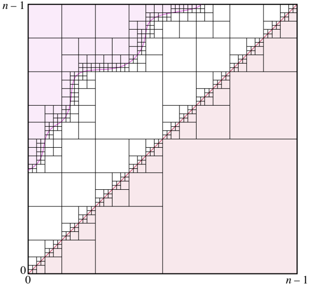


Semiorders: preferences with uncertainty

Each element has a numeric utility value

Partial order = numeric order, unless values are too close





We use a quadtree to form series-parallel subproblems, showing:

Parametric closure problem has $O(n \log n)$ solutions

They can be found in time $O(n \log^2 n)$

More partial orders with good solutions

Bounded width (no large antichains)

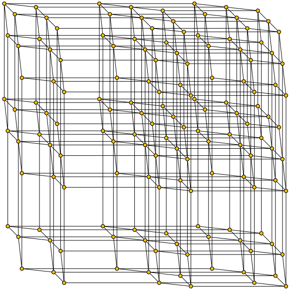
- typical version control edit histories

Transitive reduction has bounded treewidth

- fence poset reduction is a path

Incidence posets of graphs

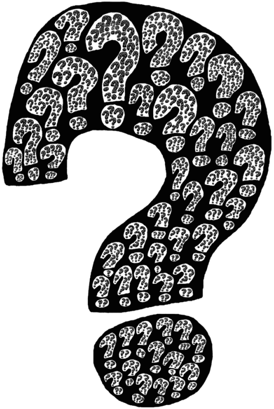
- elements are vertices and edges
- each edge $\geq$ its endpoints
- used to model depot location as closure



A Fibonacci cube, the family of closures of a fence poset

Conclusions

Some answers, but more questions:



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- Do general posets have polynomially many parametric closures?
- Is there any family of posets with more than linearly many parametric closures?
- What is the complexity of higher-dimensional expression trees of union-hulls and Minkowski sums?