Widths of Geometric Graphs

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Workshop on Parameterized Algorithms for Geometric Problems
(Part of Computational Geometry Week 2023)

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Parameterized geometric algorithms

Parameterized complexity

Idea: Measure algorithm performance as a function of both input size n and some other parameter

Fixed-parameter tractable: time is polynomial of $n \times$ function of param

XP: time is n^x where x is a function of the parameter

Various forms of hardness $\Rightarrow$ some problems may not have such bounds

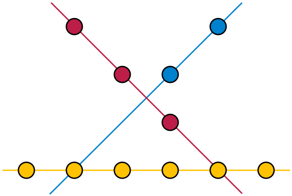
Our main topic:

Overview of what parameters to use, focusing on graph width for geometric graphs

Natural parameters

Parameterize by solution value, a small integer

Example: Covering n points by min # lines



For solution $= k$, repeatedly choose lines that cover $> k$ points, and discard points covered by the chosen lines

(Must be in solution else covering those points would be too expensive)

If $\leq k^2$ points left, solve by brute force, else give up

Time: poly(n) + exponential(k^2)

[Langerman and Morin 2005]

Geometric parameters

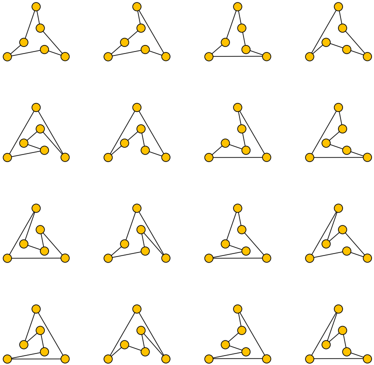
Use a parameter directly related to the input geometry that is not the output value

Example: [Eppstein 2023b]

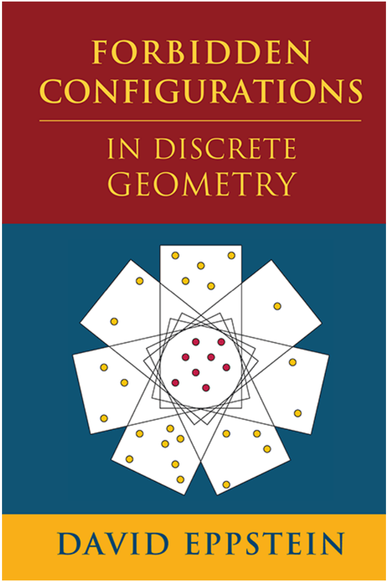
Lists all polygonalizations of a set of points

XP in two parameters:

- # points interior to convex hull
- # points to delete to make input collinear



Gratuitous plug



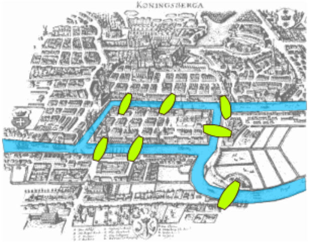
Studies geometric parameters that are

- Defined using only the order-type of a point set
- Hereditary (monotonic under point deletions)

Many famous problems in discrete and computational geometry fit this framework!

Graph-based parameters

Relate the geometric problem you want to solve
to some kind of graph defined from its geometry



Use a parameter defined from the graph
instead of defined directly from the geometry

Advantage: Much more research into parameterized graph
algorithms than into parameterized geometric algorithms

What is graph width?

A maze of equivalent definitions

Bounded treewidth $\iff$

Hierarchical clustering of edges by vertex separators of size $O(1)$

No large grid minor

Subgraph of chordal graph with no large cliques

Tree decomposition with no large bags

$O(1)$ cops can win "cops with helicopters" pursuit-evasion game

No "bramble", touching subgraphs with high hitting number

No "haven" assigning "large component" to small vertex deletions

No "tangle" assigning "large side" to small vertex separators

A maze of different parameters

Treedepth, shrubdepth, etc: star-like graphs

Bandwidth, cutwidth, pathwidth, etc: path-like graphs

Treewidth, branchwidth, carving width, etc: tree-like graphs

Row treewidth, row pathwidth, etc: grid-like product structure

Bounded expansion, polynomial expansion, clique-width, rank-width, matching-width, twin-width, flip-width, monadic dependence, etc:

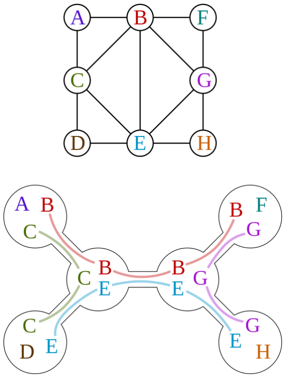
generalizations to sparse graphs and well-structured dense graphs

Algorithms from recursive decompositions

For many types of width, bounded width $\Rightarrow$ recursive structure

E.g. treewidth: tree decomposition

- Tree labeled with small subsets of vertices ("bags")
- Each vertex belongs to a connected subtree
- Each edge has a bag containing its endpoints

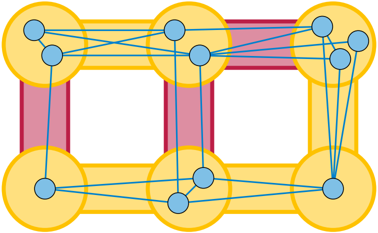


Dynamic program: bottom-up tree traversal, finding optimal solution for each local configuration ("state") in each bag, and the subgraph represented by its subtree

E.g. weighted maximum independent set: state = independent subset of the bag. 2^w states per bag, total time $O(2^w n)$

Algorithms from construction sequences

Twin-width: Repeatedly merge two clusters of vertices into larger superclusters, maintaining bounded-degree "red graph" of pairs of clusters with inconsistent cluster–cluster adjacencies



Approximate coloring: repeatedly un-merge, starting from a single supercluster, using greedy coloring for each newly un-merged cluster

Gives coloring with width $+ 2$ colors for triangle-free graphs

Can be generalized to color k-chromatic graphs with $f(k, w)$ colors
[Bonnet et al. 2021a; Pilipczuk and Sokołowski 2023]

Algorithms from forbidden subgraphs

For graphs closed under minors (subgraphs and edge contractions, including bounded pathwidth, treewidth, and tree-depth) every minor-closed graph property has finitely many forbidden minors

For graphs of bounded tree-depth, every property closed under induced subgraphs has finitely many forbidden induced subgraphs

To test if the property is true, just look for these forbidden graphs!

Famous example [Wagner 1937]: forbidden minors for testing whether a graph is planar are K_5 and $K_{3,3}$

Algorithms from logical descriptions

Logic of graphs:

- Variables are vertices, edges, or sets of vertices or edges
- Predicates are equality, membership, incidence, or adjacency

Example: Is there a universal vertex? $\exists v \forall w (v \neq w \Rightarrow v \sim w)$

Messier example: Is there a Hamiltonian cycle?

- Does there exist a set C of edges, such that
- Every proper subset X of vertices has an edge in C with exactly one endpoint in X ($\Rightarrow$ C connects the graph), and
- Every vertex has exactly two incident edges in C?

Algorithms from logical descriptions

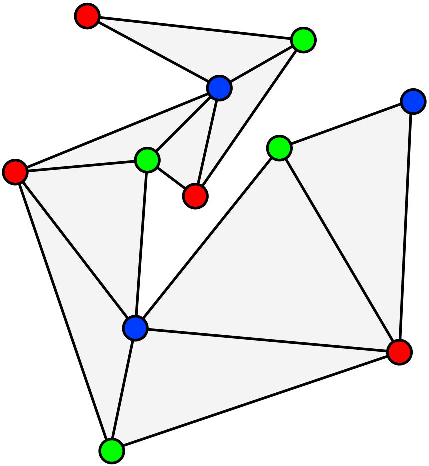
Model checking: Is this formula true of this graph?

Fixed-parameter tractable for many combinations of variable type and graph width:

- Formulas with sets of vertices or edges and bounded tree-width
- Sets of vertices (but not edges) and bounded clique-width
- Individual vertices (but not sets) and nowhere dense, bounded twin-width or beyond

Geometric graphs with low width

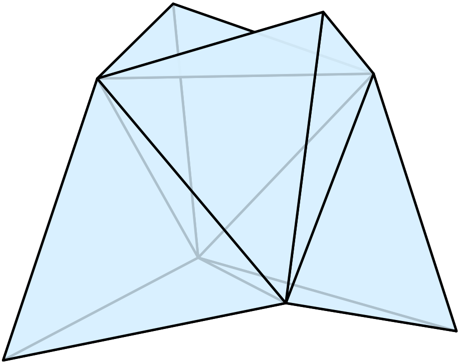
Polygon triangulation



Treewidth is 2 $\Rightarrow$ weighted independent set $O(n)$, etc.

Stacked polyhedra

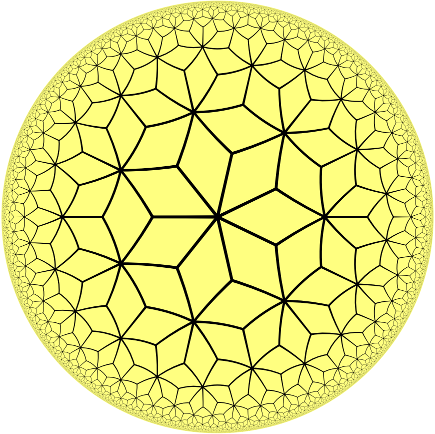
Glue tetrahedra (or higher-dim simplices) face-to-face in a tree



Treewidth = dimension

Hyperbolic tilings

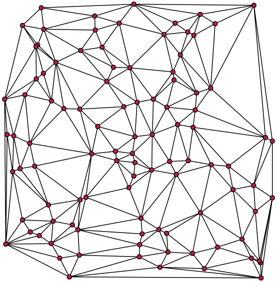
Finite subgraphs have logarithmic treewidth



Used by [Kopczyński 2021] to solve hyperbolic minesweeper in polynomial time

Planar graphs

For instance, 2d Delaunay triangulations



Treewidth is $O(\sqrt{n}) \Rightarrow$ subexponential for NP-hard problems

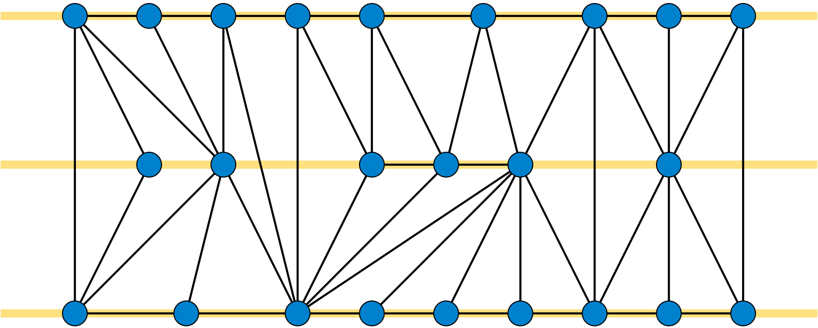
(Treewidth n^c for $c < 1$ = "polynomial expansion")

Row-treewidth is $O(1) \Rightarrow$ many algorithmic consequences

Twin-width is $O(1)$

[Lipton and Tarjan 1979; Dujmović et al. 2020; Bonnet et al. 2021b]

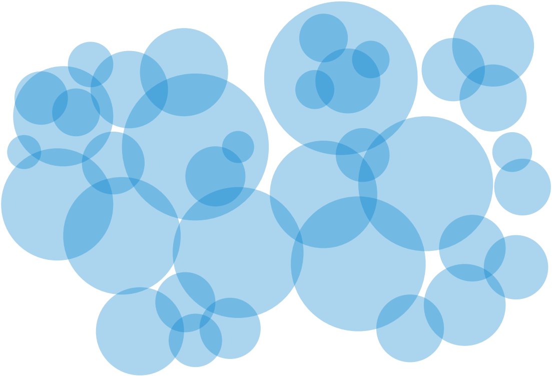
Planar graphs on parallel lines



Pathwidth is $\leq$ # lines $\Rightarrow$ FPT recognition [Dujmović et al. 2008]

Neighborhood systems

Intersection graph of balls in d-dimensional Euclidean space s.t. each point of common intersection belongs to $O(1)$ balls



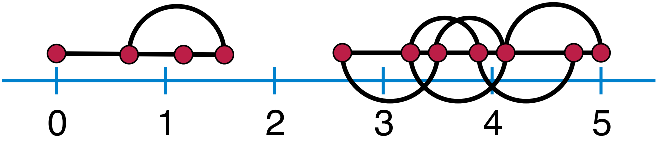
Treewidth is $O(n^{1-\frac{1}{d}})$; twin-width is $O(1)$

Applications in finite element meshing and mesh partitioning

[Miller et al. 1997; Bonnet et al. 2022]

Unit interval graphs

Equivalently: Connect 1d points at distance ≤ 1



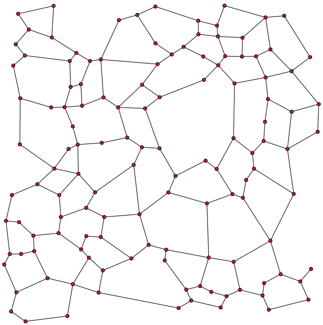
Can be dense!

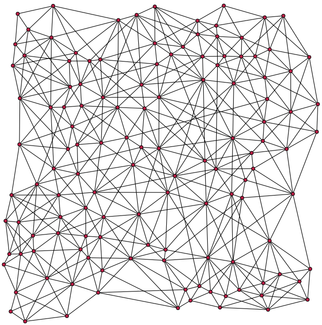
Unbounded clique-width [Lozin 2011]

Twin-width ≤ 2 [Bonnet et al. 2021a]

Greedy spanners

Add line segments in order by length whenever current best path between segment endpoints is longer by a factor of t





Treewidth is $O(n^{1-\frac{1}{d}})$

[Eppstein and Khodabande 2021; Le and Than 2022]

Unknown whether twin-width or flip-width is bounded

Geometric graphs with high width

Width from pursuit–evasion games

Cops with helicopters occupy vertices, trying to catch a robber moving on graph paths

Each turn:

- Cops announce where they will move
- Robber moves
- Cops move as announced

$O(1)$ cops catch unlimited-speed robber $\Rightarrow$ bounded treewidth

$O_s(1)$ cops catch speed-s robber $\Rightarrow$ nowhere dense

More powerful cops that can flip edges in induced subgraphs instead of occupying single vertices $\Rightarrow$ bounded flip-width

Bounded flip-width also implies bounded twin-width

[Toruńczyk 2023]

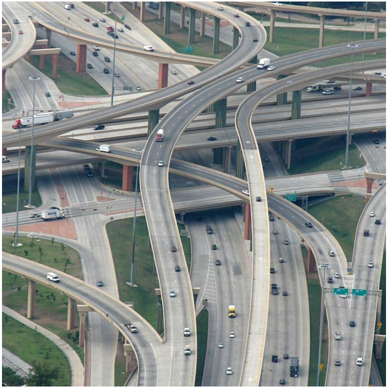
Method for proving high width

Use a special subgraph called an interchange to find a winning strategy for the robber in the flip-width game

Many dense geometric graphs contain interchanges

Therefore they do not have bounded flip-width, or any other width encompassing twin-width, clique-width, tree-width, nowhere density, etc

[Eppstein and McCarty 2023]

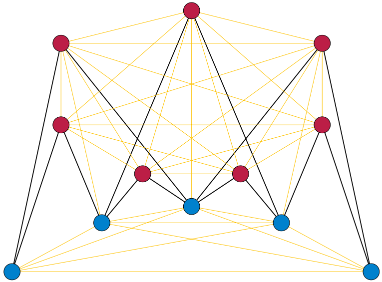


Definition of an interchange

Intuitively: like a subdivision of a complete graph

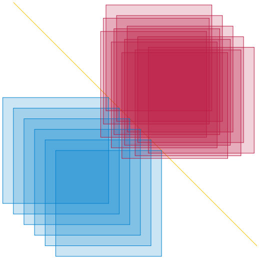
Interchange of order n contains:

- Ordered sequence of n "lane" vertices (blue)
- More "ramp" vertices connecting pairs of lanes $\leq n-3$ steps apart (red)
- Optional ramp–lane edges (yellow) only when the lane is between the two connections for the ramp
- Optional ramp–ramp and lane–lane edges

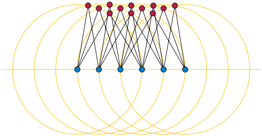


Robber escape strategy: Move to a lane with two-edge paths to many other lanes

Geometric graphs with unbounded flip-width

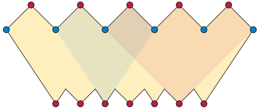


Intersection graphs of axis-aligned unit squares



Unit distance graphs

Unit disk graphs



Visibility graphs of simple polygons

Geometric graphs with unbounded flip-width

Interval graphs

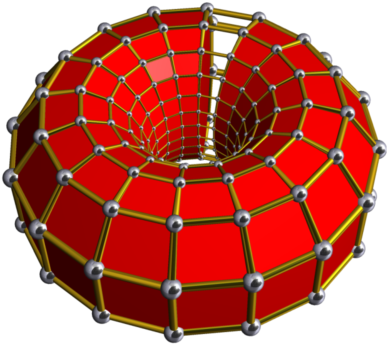
Permutation graphs

Circle graphs

Intersection graphs of axis-aligned line segments

Geometric graphs with unbounded flip-width

3d Delaunay triangulations and 4d convex polytopes

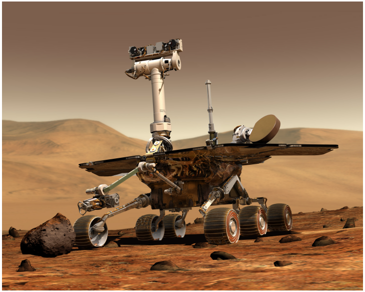


Augment $n \times n$ toroidal grid by
n points on central axis
n points on center circle of torus

Parameterizing by width

General principle

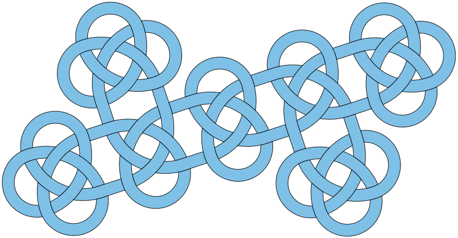
When the graphs underlying a given problem can have high width, this is an opportunity, not an obstacle



It means we can use parameterized complexity to explore the landscape of easy instances and hard instances

Knot theory

Knot diagram: 4-regular planar multigraph, with casing (over-under relation at each crossing)



Some important knot-theoretic computations can be done directly from the diagram and are fast when diagram has low treewidth

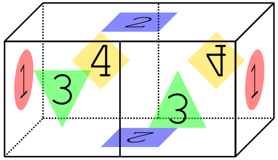
E.g. Tutte/Jones polynomial [Noble 1997]

One of this year's SoCG papers concerned treewidth of knots, [Lunel and de Mesmay 2023]

Low-dimensional computational topology

3-manifold: each point has neighborhood with $\mathbb{R}^3$-topology

Often defined by gluing together cubes or tetrahedra



Many hard computational problems can be solved in FPT time when the gluing pattern has low treewidth

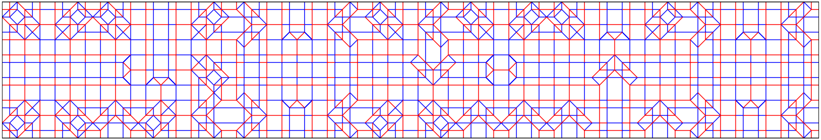
Example: Taut angle structure: assign $(0, 0, 0, 0, \pi, \pi)$ to edges of tetrahedra, with π angles opposite, so each edge is surrounded by tetrahedra with total angle 2π [Burton and Spreer 2013]

Many other papers by Benjamin Burton

Another paper from this year's SoCG [Huszár and Spreer 2023]

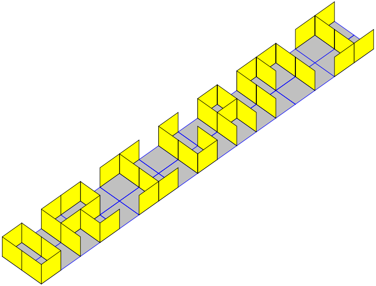
Computational origami

Problem: Given an origami folding pattern, can it fold flat?



NP-complete [Bern and Hayes 1996]

FPT in treewidth + max # layers in folded result [Eppstein 2023a]



Conclusions and references

Conclusions

Graph theory has a wealth of widths that we can use in geometry

Many natural sparse geometric graphs have bounded width, for various types of width

Many natural dense geometric graphs do not

When width is bounded, can be used for efficient algorithms

Even when the width can be unbounded, it can be used as a parameter in efficient parameterized algorithms

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