The Parameterized Complexity of Finding Point Sets with Hereditary Properties

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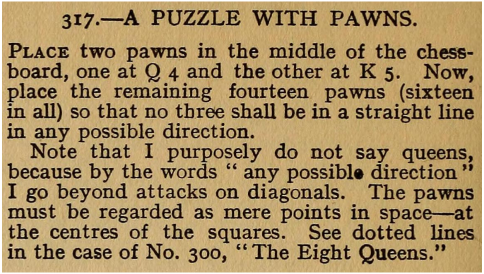
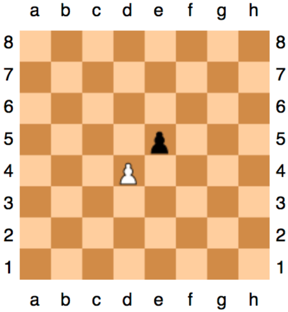
A puzzle from Dudeney (1917)



317.—A PUZZLE WITH PAWNS.

PLACE two pawns in the middle of the chessboard, one at Q 4 and the other at K 5. Now, place the remaining fourteen pawns (sixteen in all) so that no three shall be in a straight line in any possible direction.

Note that I purposely do not say queens, because by the words "any possible direction" I go beyond attacks on diagonals. The pawns must be regarded as mere points in space—at the centres of the squares. See dotted lines in the case of No. 300, "The Eight Queens."



Later known as the "no three-in-line puzzle"

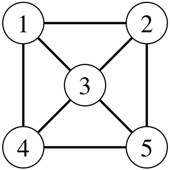
Every $n \times n$ grid has $\Omega(n)$ points with no three in line (Erdős) but it's unknown whether $2n$ is always achievable

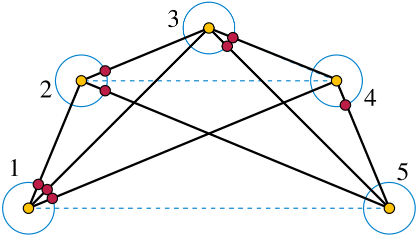
Dudeney's puzzle as a parameterized problem

Given a finite set of points in the Euclidean plane
(the 64 points of an 8×8 grid

Find a subset of k points in general position
(no three collinear; in Dudeney's puzzle, $k = 16$)

NP-hard and APX-hard but fixed-parameter tractable
(Eppstein 2018, Theorems 9.3 and 9.5)

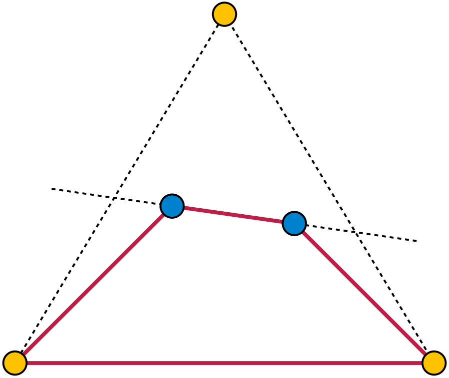




Hardness reduction from maximum independent set
Blows up the parameter so does not show parameterized hardness

The happy ending theorem

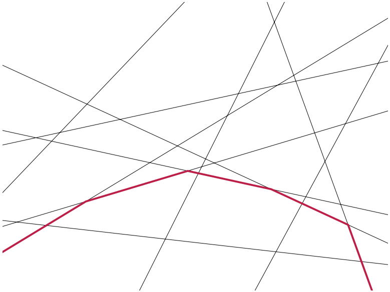
Every five points in general position include the vertices of a convex quadrilateral (Klein)



More generally every n points in general position include a convex k-gon for $k = (1 + o(1)) \log_2 n$ (Erdős & Szekeres; Suk)

Happy endings as a parameterized problem

Given a finite set of points in the Euclidean plane
Find a subset of k points in convex position



Solvable in cubic time (independent of parameter) and linear space

Guess bottom point, topologically sweep line arrangement dual to points above it (Chvátal & Klincsek; Edelsbrunner & Guibas)

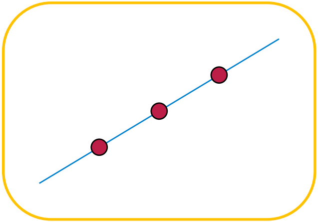
The bigger picture

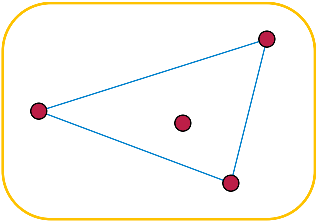
Both general position and convex position are *hereditary properties*:

They depend only on the order type

(which triples are clockwise, counterclockwise, or collinear)

They remain true if we remove points





Every hereditary property can be defined by *forbidden patterns*

(minimal order types that do not have the property)

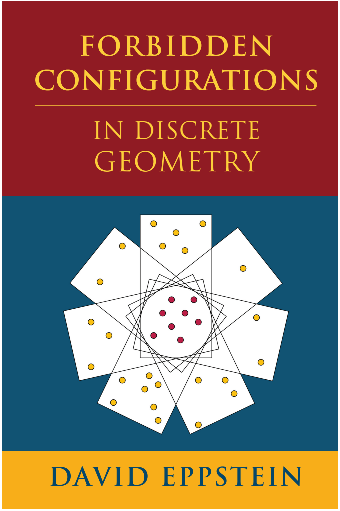
Our starting problem

My new book:

Forbidden Configurations in Discrete Geometry
(Cambridge, 2018)

Takes a unified view of discrete geometry via hereditary properties

Open Problem 7.6:
If property Π has finitely many forbidden patterns, is it FPT to find k points with property Π?

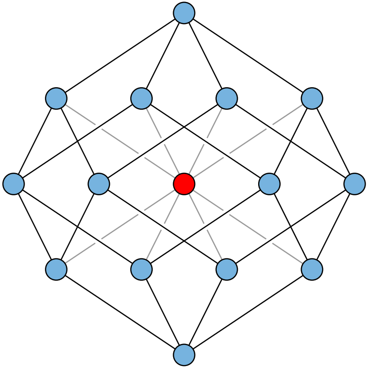


Analogy with hereditary properties of graphs

Hereditary: Closed under induced subgraphs

Many classical parameterized problems seek hereditary induced subgraphs:

- k-vertex clique or independent set
- k-vertex induced path
- k- or $(n-k)$-vertex planar (planarization, apex)
- $(n-k)$-vertex forest (feedback vertex set)
- $(n-k)$-vertex bipartite (odd cycle transversal)



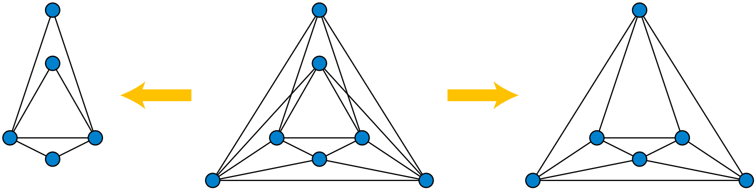
Dichotomy for hereditary properties of graphs

Khot and Raman:

If true for all cliques and all independent sets ⇒ (Ramsey)
all large graphs have k-vertex subgraph with property ⇒
finding a k-vertex subgraph is trivially FPT

If false for large-enough cliques and independent sets ⇒ (Ramsey)
k-vertex subgraph can only exist for $k = O(1)$ ⇒
finding a k-vertex subgraph is trivially FPT

Otherwise it's $\mathrm{W}[1]$-complete



E.g. chordal subgraph is FPT; planar subgraph is hard

Our results (I)

Finding a subset of k points with a hereditary property is:

FPT when all collinear sets and all convex sets have the property
(trivial from happy ending)

FPT when a collinear set and a convex set don't have the property
(trivial from happy ending)

So far, completely analogous to Khot and Raman
(with happy ending theorem replacing graph Ramsey theorem)

Could it be $\mathrm{W}[1]$-complete in all remaining cases?

Our results (II)

Finding a subset of k points with a hereditary property is:

FPT when all collinear sets and all convex sets have the property
(trivial from happy ending)

FPT when a collinear set and a convex set don't have the property
(trivial from happy ending)

FPT when all collinear sets have it but a convex set doesn't
(polynomial number of parameter-bounded brute force searches)

FPT when there is only one forbidden pattern (kernelization)

$\mathrm{W}[1]$-complete for a (contrived) property
(plus nearly ETH-tight time lower bounds)

Unknown in the remaining cases!

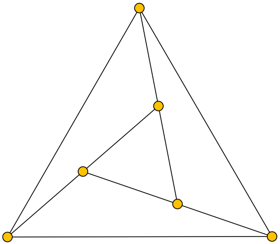
Typical example of nontrivial FPT case

Find a subset of k points not containing any convex q-gon

k is the parameter; q is a constant

- If a line contains k points $\Rightarrow$ done
- Lemma: solution can be covered by an $O_q(1)$-tuple of lines
- Try all (polynomially many) tuples of lines through pairs of points
- For each tuple, do a brute force search among $O_q(k)$ points

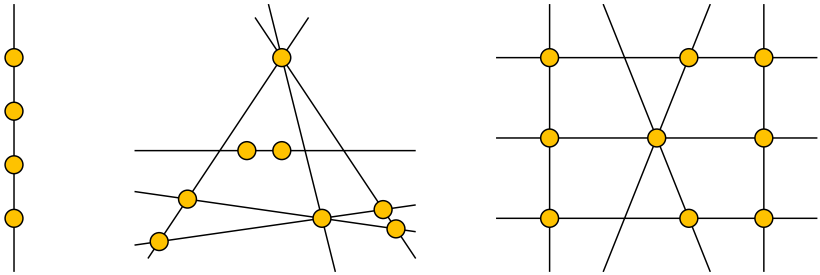
(Unknown: Is this hard when k and q are both variables?)



Six points with no convex quadrilateral

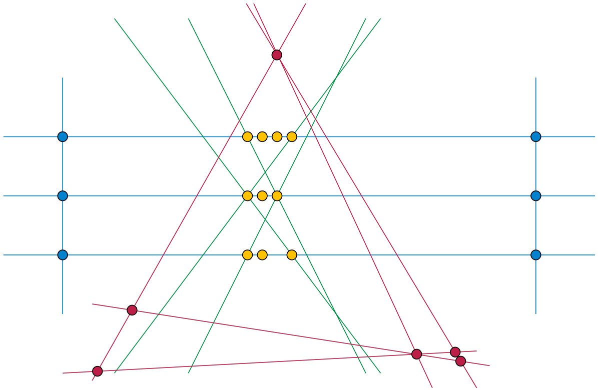
The hard property

Find k points avoiding the following three forbidden patterns:



Hard inputs for this property

Six points in a complete quadrilateral (red, the six pairwise crossings of four lines) plus other points on several horizontal lines

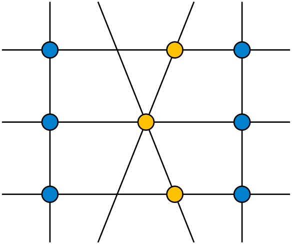


Set k so solution must include all red and blue points, plus one yellow point per horizontal line

(>3 points/line or >1 yellow/line $\Rightarrow$ forbidden pattern)

Why are these inputs hard?

When triples of blue points are collinear, we must choose the yellow points in order to make their corresponding triples also be collinear, else we get a forbidden pattern



Reduce from (partitioned) subgraph isomorphism
Blue-triple collinearity ⇔ pattern graph edge-vertex incidence
Yellow-triple collinearity ⇔ host graph edge-vertex incidence

The tricky part: finding points with integer coordinates that have exactly the right pattern of collinearities

Conclusions

Some problems of finding k-point subsets with hereditary properties are hard (so the answer to the open problem is no)

We don't have a nice dichotomy like in the induced subgraph case

Many more problems in this area remain open!

E.g. partition into general position subsets is hard in its natural parameter but FPT in the min number of lines that cover the input. Does it have a polynomial kernel? (Eppstein 2018, Open Problem 10.30)