Number Representation

ICS 6D

Sandy Irani

Number representation

- Our number system represents numbers in base 10 (also called decimal notation)
  - Each place represents a power of 10:

    $3045 = \mathbf{3}\cdot 10^3 + \mathbf{0}\cdot 10^2 + \mathbf{4}\cdot 10^1 + \mathbf{5}\cdot 10^0$
- Computers are limited to two digits (0 and 1) and therefore represent numbers base 2 (also called binary)
  - Each place represents a power of 2:

    $1011 = \mathbf{1}\cdot 2^3 + \mathbf{0}\cdot 2^2 + \mathbf{1}\cdot 2^1 + \mathbf{1}\cdot 2^0$

Number representation

- Can represent the value of a number in any base.
  - Representation in base b uses digits {0, 1,2,…,b-1}
  - Example: representing a number in base 7 uses digits {0, 1, 2, 3, 4, 5, 6}.
  - $(612)_7 = \mathbf{6}\cdot 7^2 + \mathbf{1}\cdot 7^1 + \mathbf{2}\cdot 7^0 =$

Number representation theorem

- Theorem: Fix an integer b > 1. Any non-negative integer n can be expressed uniquely as:

$$n = a_k \cdot b^k + a_{k-1} \cdot b^{k-1} + ... + a_1 \cdot b^1 + a_0 \cdot b^0$$

Each a_j is an integer digit in the range {0,1,...,b-1} and $a_k \neq 0$.

From base b to decimal

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- $(452)_6$
- $(101101)_2$
- $(107)_9$

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Hexadecimal Notation

- If b > 10, then need to use letters to represent digits larger than 9.
- Base 16: hexadecimal notation
  - Digits = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F}

- $(6D)_{16}$
- $(E1F)_{16}$

Decimal to base b conversion

- $(6532)_7 = 2306 = 7 \cdot 329 + 2$

Decimal to base b conversion

- Represent n base b:

  [**base be representation of (n div b)**][**n mod b**]

What is the representation of 743 base 5?

What is the representation of 47 base 2?

What is the representation of 165 base 16?

Recursive algorithm to convert from decimal to base b *

- Input: integer n and b, $b > 1$, $n \geq 1$
- Output: Base b representation of n

- Base-b-Rep(n, b)
  - If (n = 0) return
  - Base-b-Rep(n DIV b, b)
  - Output digit for n MOD b

- If b’ = b^m for some integer m, then m digits base b correspond to one digit base b’ in the conversion between base b’ and base b.

- Example: Hexadecimal (base 16)

  Binary (base 2)

  $16 = 2^4$

In converting between binary and hex, 4 bits translate directly into 1 hex digit.

Binary to Hex conversion

HEX	0	1	2	3	4	5	6	7
Decimal	0	1	2	3	4	5	6	7
Binary	0000	0001	0010	0011	0100	0101	0110	0111

HEX	8	9	A	B	C	D	E	F
Decimal	8	9	10	11	12	13	14	15
Binary	1000	1001	1010	1011	1100	1101	1110	1111

Binary to Hex conversion

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HEX	0	1	2	3	4	5	6	7
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Binary	1000	1001	1010	1011	1100	1101	1110	1111

Number of digits in a base b representation

- The largest number that can be represented with 5 digits base 8 is:

- If n is represented with 5 digits base 8 then:

Number of digits in a base b representation

- The largest number that can be represented with d digits base b is b^d-1.

- If n is represented with d digits base b then:

  $n \leq b^d-1$

- The number of digits needed to represent n in base b is at least:

  $\log_b(n+1)$

Another view of fast exponentiation

- Compute 7^{32} mod 11

7 $\quad 7^2$ mod 11 $\quad 7^4$ mod 11 $\quad 7^8$ mod 11 $\quad 7^{16}$ mod 11 $\quad 7^{32}$ mod 11

- Compute 7^{33} mod 11

- Compute 7^{35} mod 11

Another view of fast exponentiation

- $a^b \bmod N$ $\quad b_k \cdot 2^k + b_{k-1} \cdot 2^{k-1} + ... + b_1 \cdot 2^1 + b_0 \cdot 2^0 \quad b_j = 0/1$

- $a^b \bmod N =$

Compute $(52)^{46}$ mod 7

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FastExp(x, y, n) = x^y mod n

$(52)^{46}$ mod 7

```
Initialize: p = 1, s = x, r = y
While (r > 0)
        If (r mod 2 = 1)
                p = p*s mod n
        s = s*s mod n
        r = r DIV 2
End-while
Return(p)
```

RecFastExp(x, y, n) = x^y mod n

$(52)^{46}$ mod 7

```
If (y = 0) Return(1)
p = RecFastExp(x, y DIV 2, n)
p = p * p mod n
If (y mod 2 = 1)
        p = p*x mod n
Return(p)
```