Applied Bayesian Nonparametrics

5. Spatial Models via Gaussian Processes, not MRFs

Tutorial at CVPR 2012

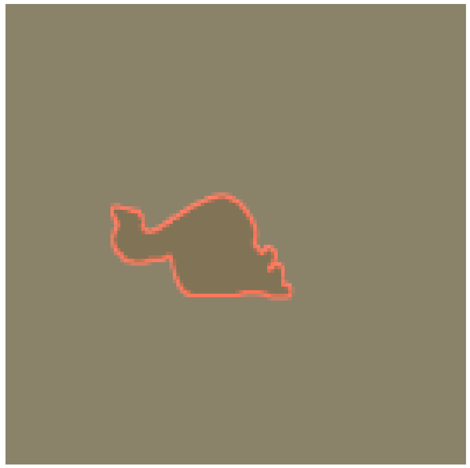
Erik Sudderth

Brown University

NIPS 2008: *E. Sudderth & M. Jordan, Shared Segmentation of Natural Scenes using Dependent Pitman-Yor Processes.*
CVPR 2012: *S. Ghosh & E. Sudderth, Nonparametric Learning for Layered Segmentation of Natural Images.*

Human Image Segmentation

BNP Image Segmentation

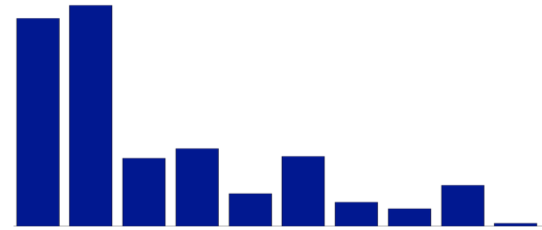

Segmentation as Partitioning

- How many regions does this image contain?
- What are the sizes of these regions?

Why Bayesian Nonparametrics?

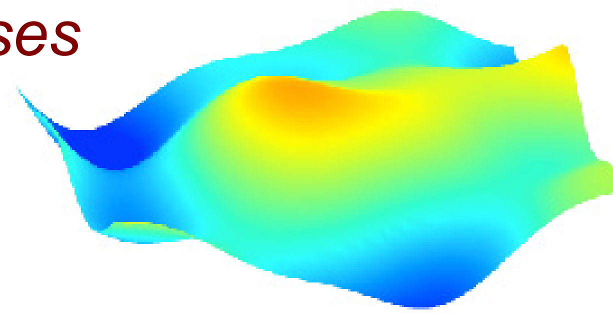
- Huge variability in segmentations across images
- Want multiple interpretations, ranked by probability

BNP Image Segmentation



Model

- Dependent *Pitman-Yor processes*
- Spatial coupling via *Gaussian processes*



Inference

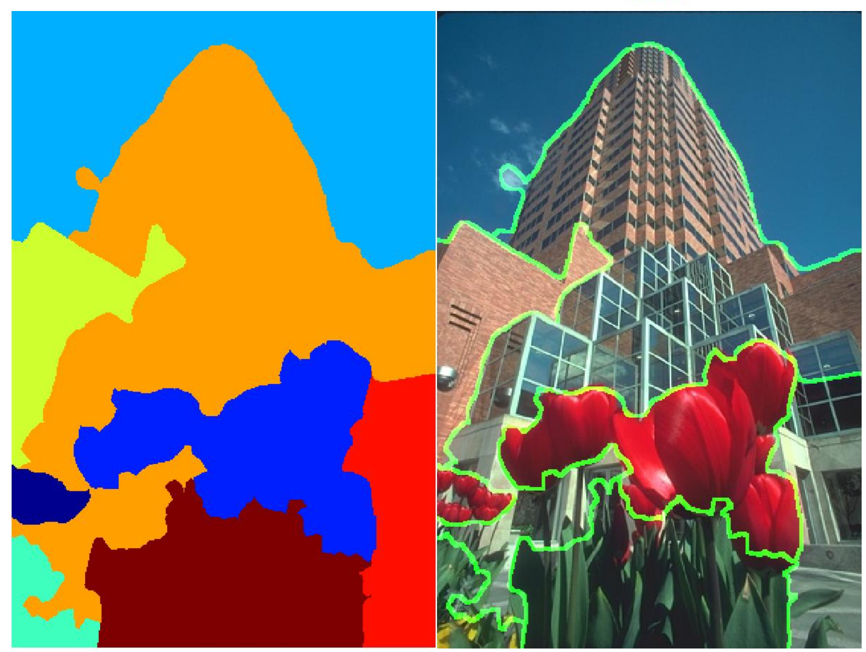
- Stochastic search & *expectation propagation*

Learning

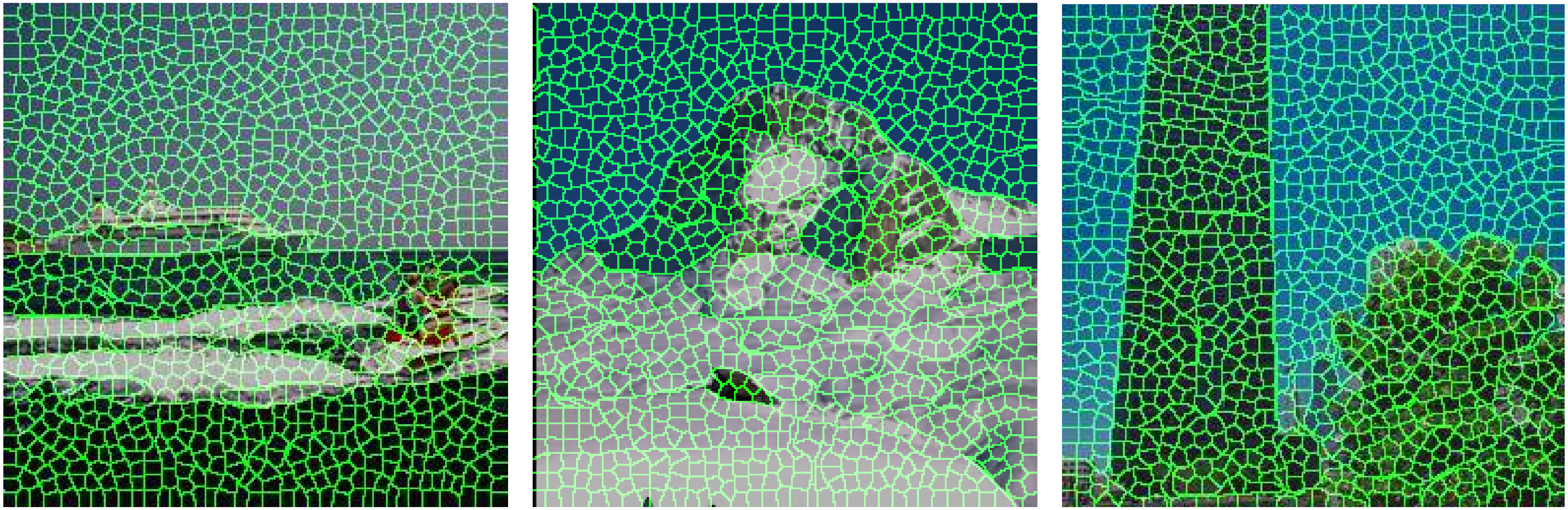
- Conditional covariance calibration

Results

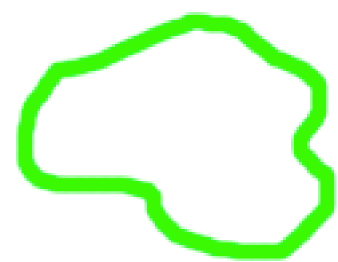
- Multiple segmentations of natural images



Feature Extraction



- Partition image into ~1,000 *superpixels*
- Compute *texture* and *color* features:
  Texton Histograms (VQ 13-channel filter bank)
  Hue-Saturation-Value (HSV) Color Histograms
- Around 100 bins for each histogram



Pitman-Yor Mixture Model

PY segment size prior

$$\pi_k = v_k \prod_{\ell=1}^{k-1} (1 - v_\ell)$$

$$v_k \sim \mathrm{Beta}(1 - a, b + ka)$$

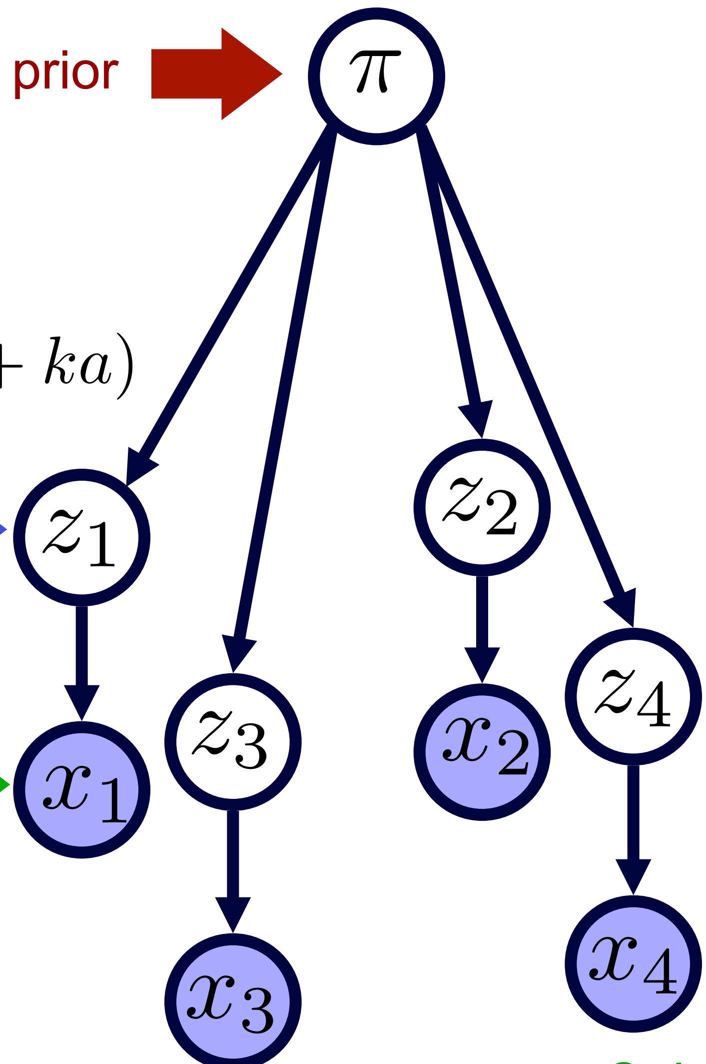
Assign features to segments

$$z_i \sim \mathrm{Mult}(\pi)$$

Observed features (color & texture)

$$x_i^c \sim \mathrm{Mult}(\theta_{z_i}^c)$$

$$x_i^s \sim \mathrm{Mult}(\theta_{z_i}^s)$$



Visual segment appearance model

Color: $\theta_k^c \sim \mathrm{Dir}(\rho^c)$

Texture: $\theta_k^s \sim \mathrm{Dir}(\rho^s)$

Dependent DP&PY Mixtures

Some dependent prior with DP/PY "like" marginals

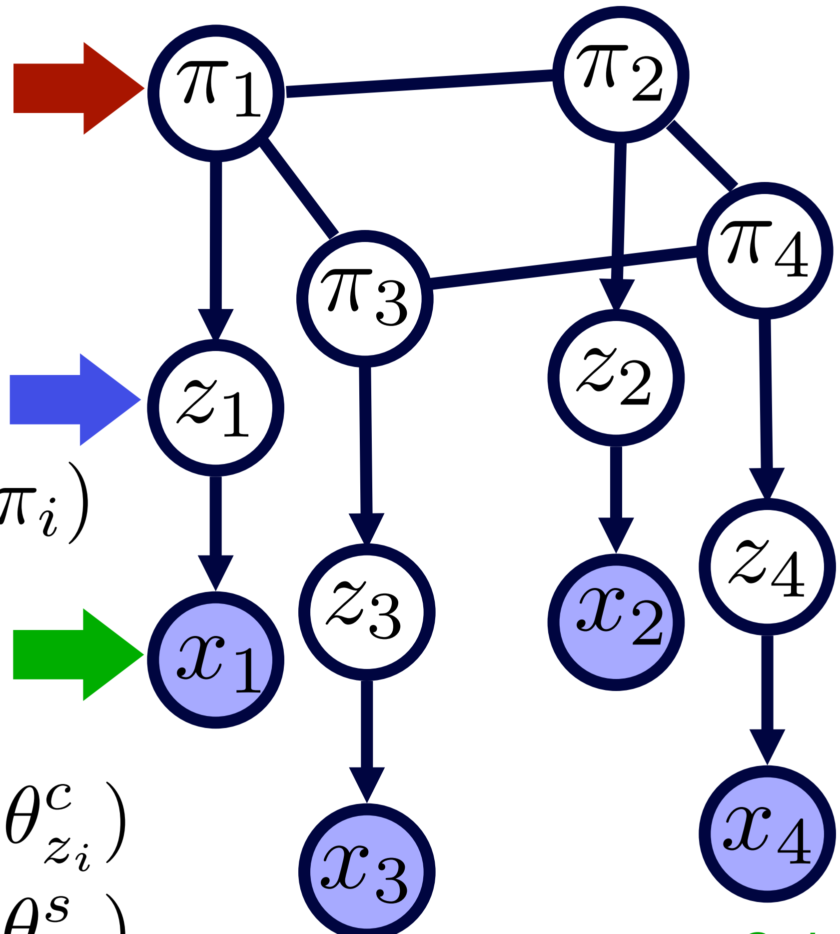
Assign features to segments

$$z_i \sim \mathrm{Mult}(\pi_i)$$

Observed features (color & texture)

$$x_i^c \sim \mathrm{Mult}(\theta_{z_i}^c)$$

$$x_i^s \sim \mathrm{Mult}(\theta_{z_i}^s)$$



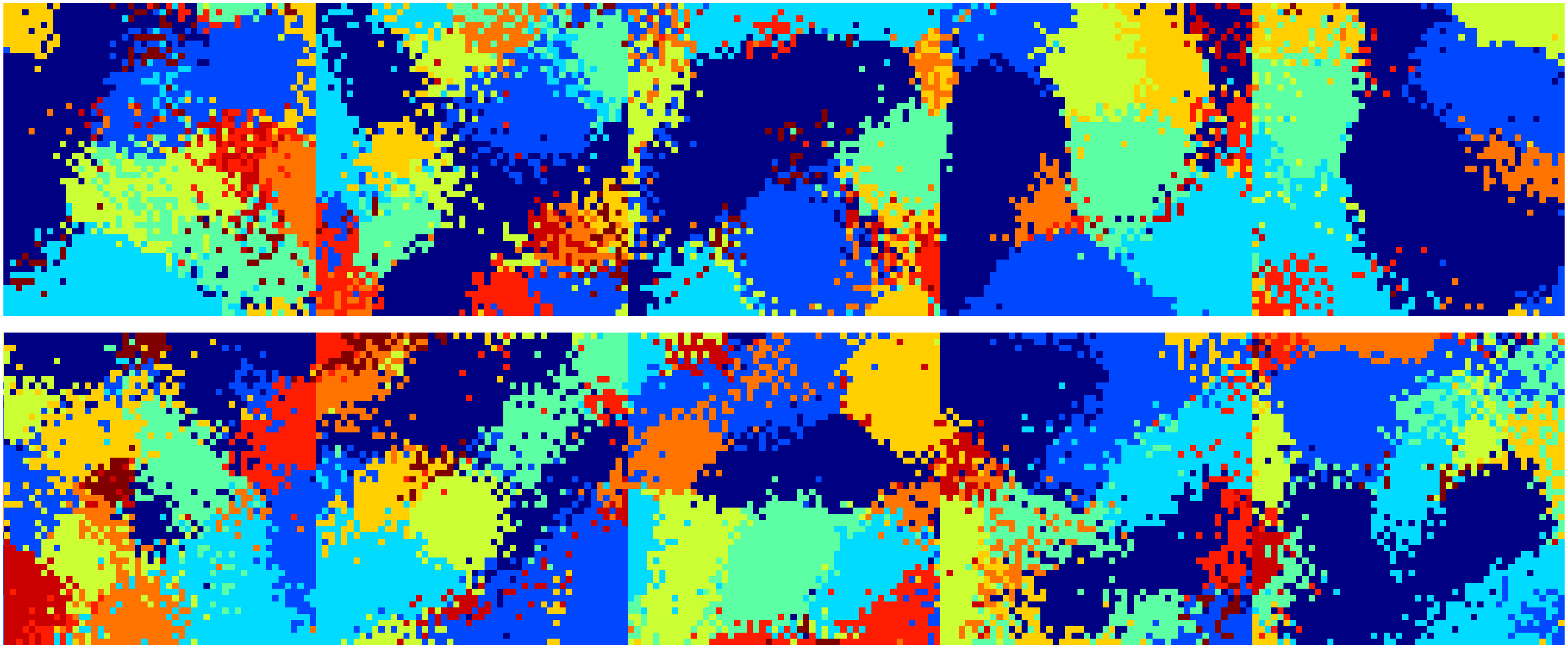
Kernel/logistic/probit stick-breaking process, order-based DDP, …

Visual segment appearance model

Color: $\theta_k^c \sim \mathrm{Dir}(\rho^c)$

Texture: $\theta_k^s \sim \mathrm{Dir}(\rho^s)$

Example: Logistic of Gaussians



- Pass set of Gaussian processes through softmax to get *probabilities* of *independent* segment assignments

  Fernandez & Green, 2002

  Figueiredo et. al., 2005, 2007

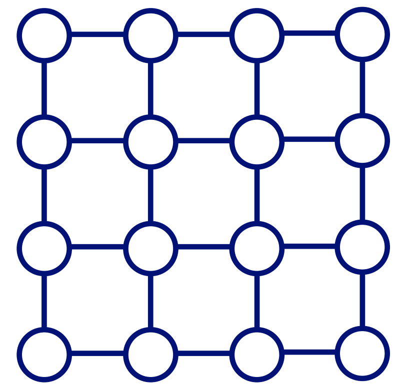
  Woolrich & Behrens, 2006

  Blei & Lafferty, 2006

- Nonparametric analogs have similar properties

Discrete Markov Random Fields

Ising and Potts Models



$$p(z) = \frac{1}{Z(\beta)} \prod_{(s,t)\in E} \psi_{st}(z_s, z_t)$$

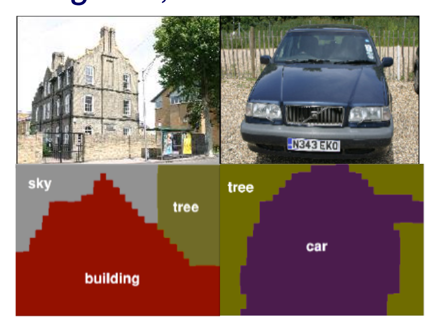
$$\log \psi_{st}(z_s, z_t) = \begin{cases} \beta_{st} > 0 & z_s = z_t \\ 0 & \text{otherwise} \end{cases}$$

GrabCut: *Rother, Kolmogorov, & Blake 2004*

Previous Applications

- Interactive foreground segmentation
- Supervised training for known categories

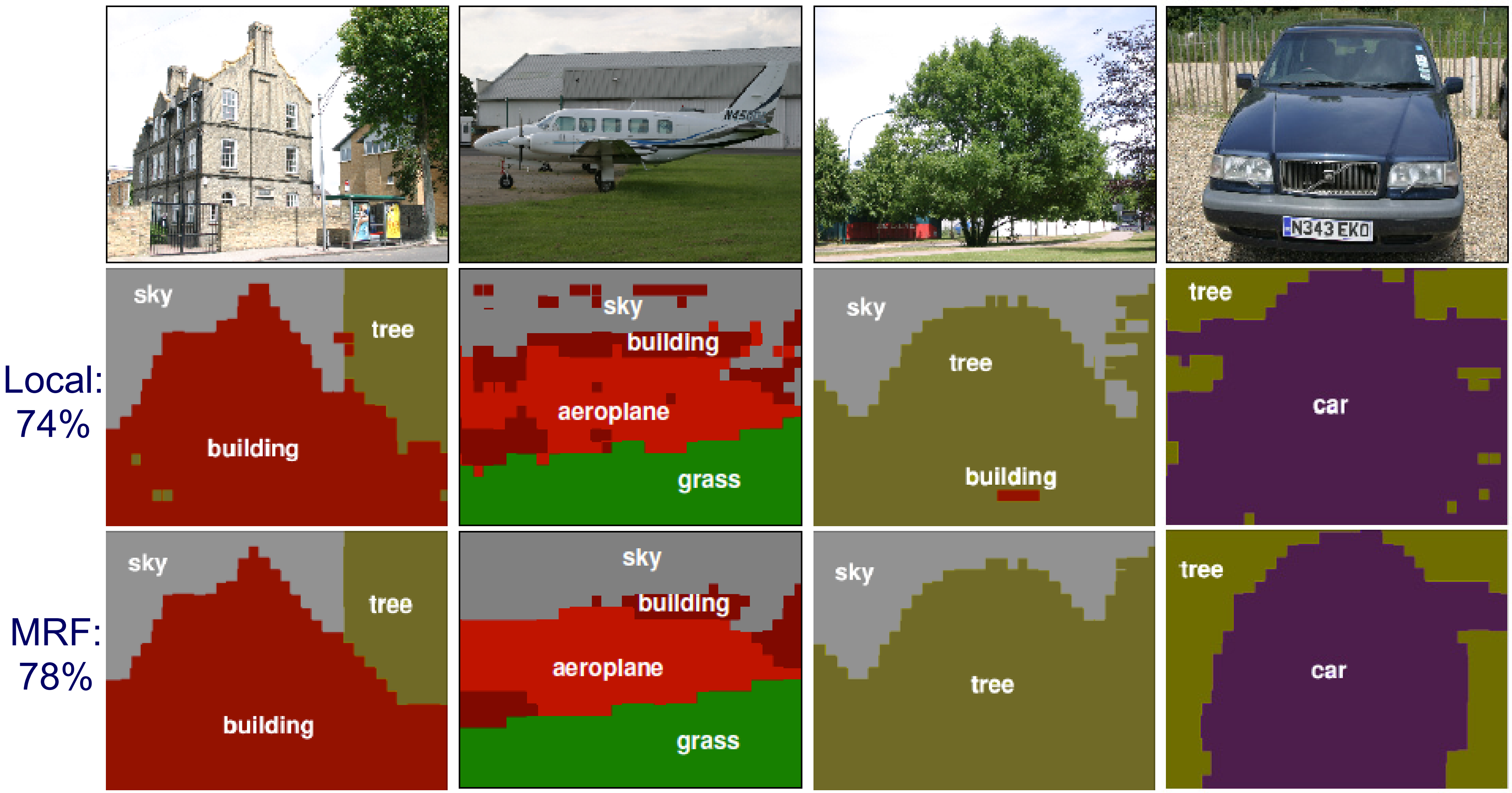
…but learning is challenging, and little success at unsupervised segmentation.



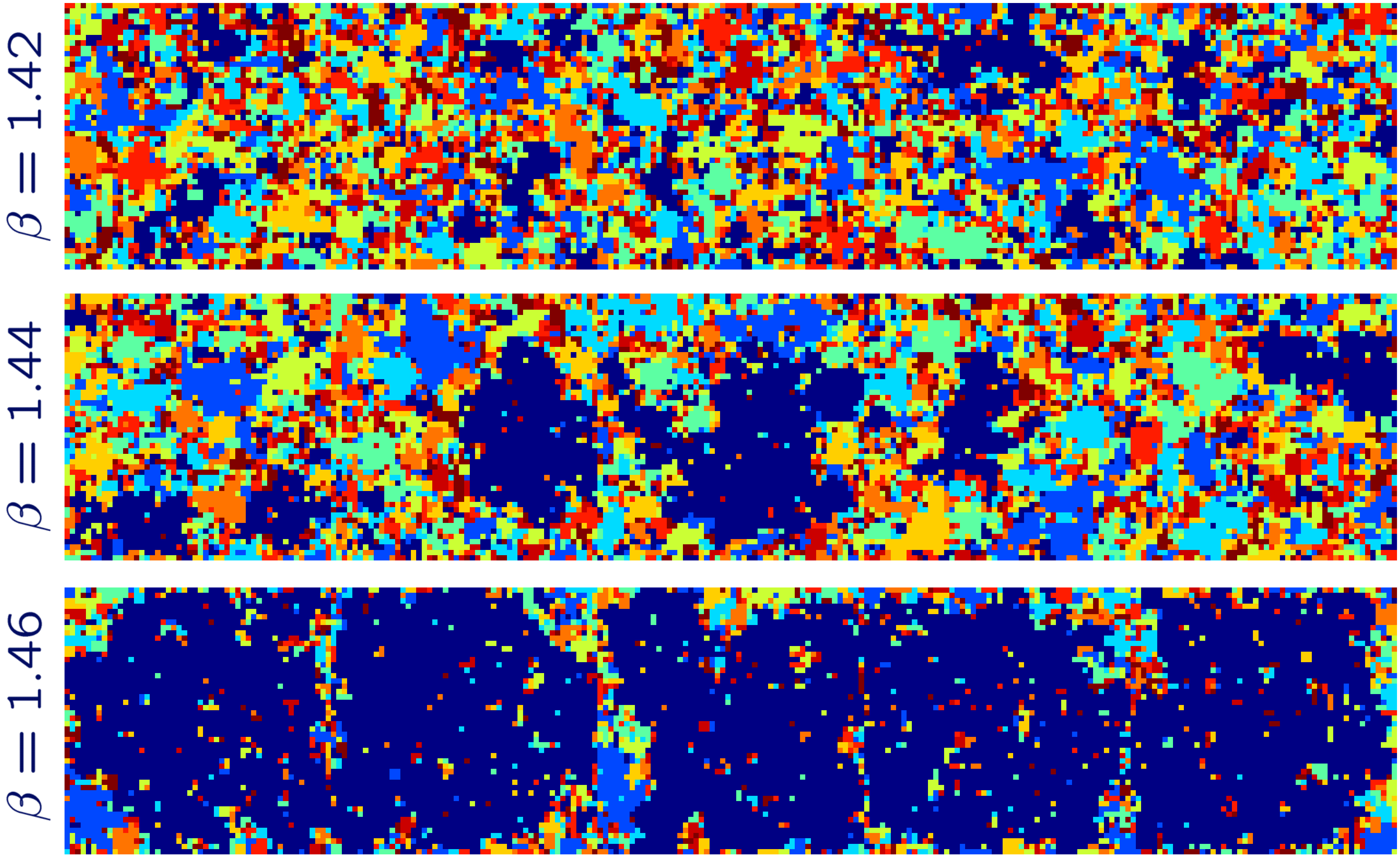
Verbeek & Triggs, 2007

Region Classification with Markov Field Aspect Models

Verbeek & Triggs, CVPR 2007



10-State Potts Samples



States sorted by size: largest in blue, smallest in red

1996 IEEE DSP Workshop

The Ising/Potts model is not well suited to segmentation tasks

R.D. Morris *X. Descombes*
J. Zerubia
INRIA, 2004, route des Lucioles, BP93, Sophia Antipolis Cedex, France.

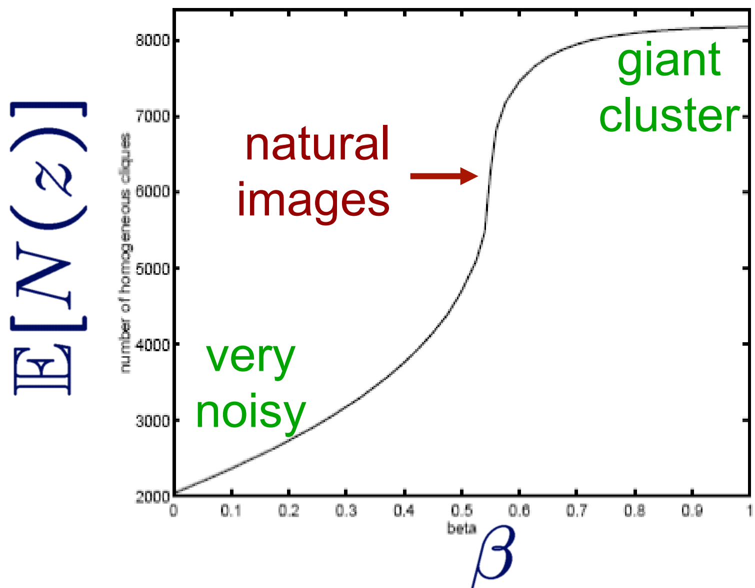


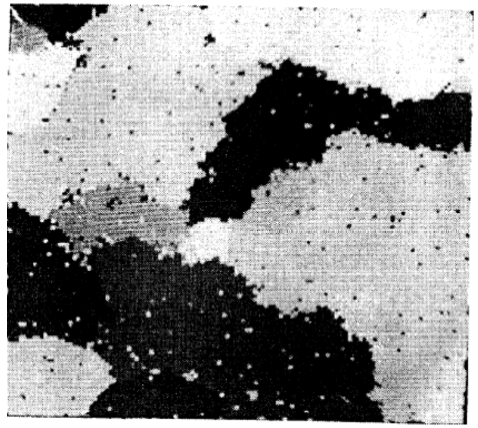
Figure 1. $< N(\mathbf{x}) >$ vs β for $64 \times 64 \times 4$-state Potts model

$N(z)$ → number of edges on which states take same value

β → edge strength

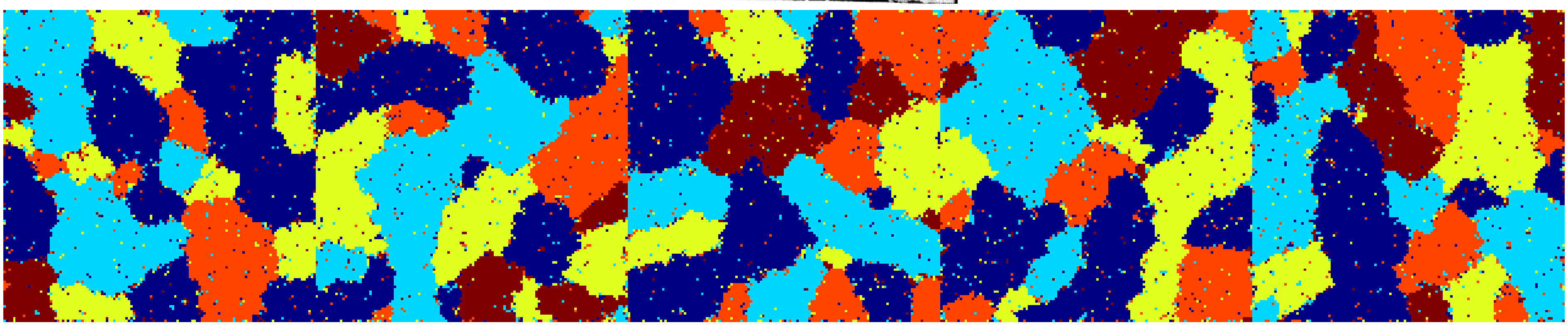
Even within the *phase transition* region, samples lack the *size distribution* and *spatial coherence* of real image segments

Geman & Geman, 1984

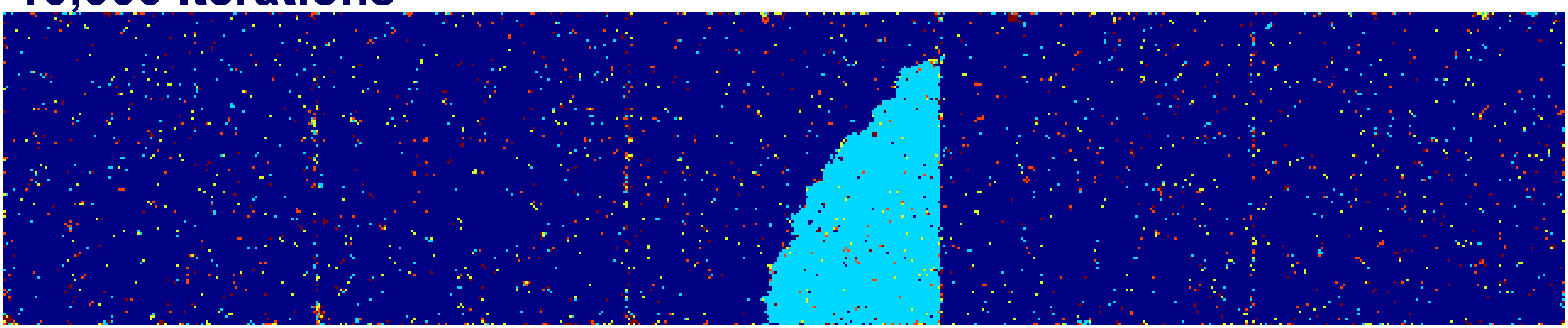


128 x128 grid
8 nearest neighbor edges
K = 5 states
Potts potentials: $\beta = 2/3$

200 Iterations



10,000 Iterations



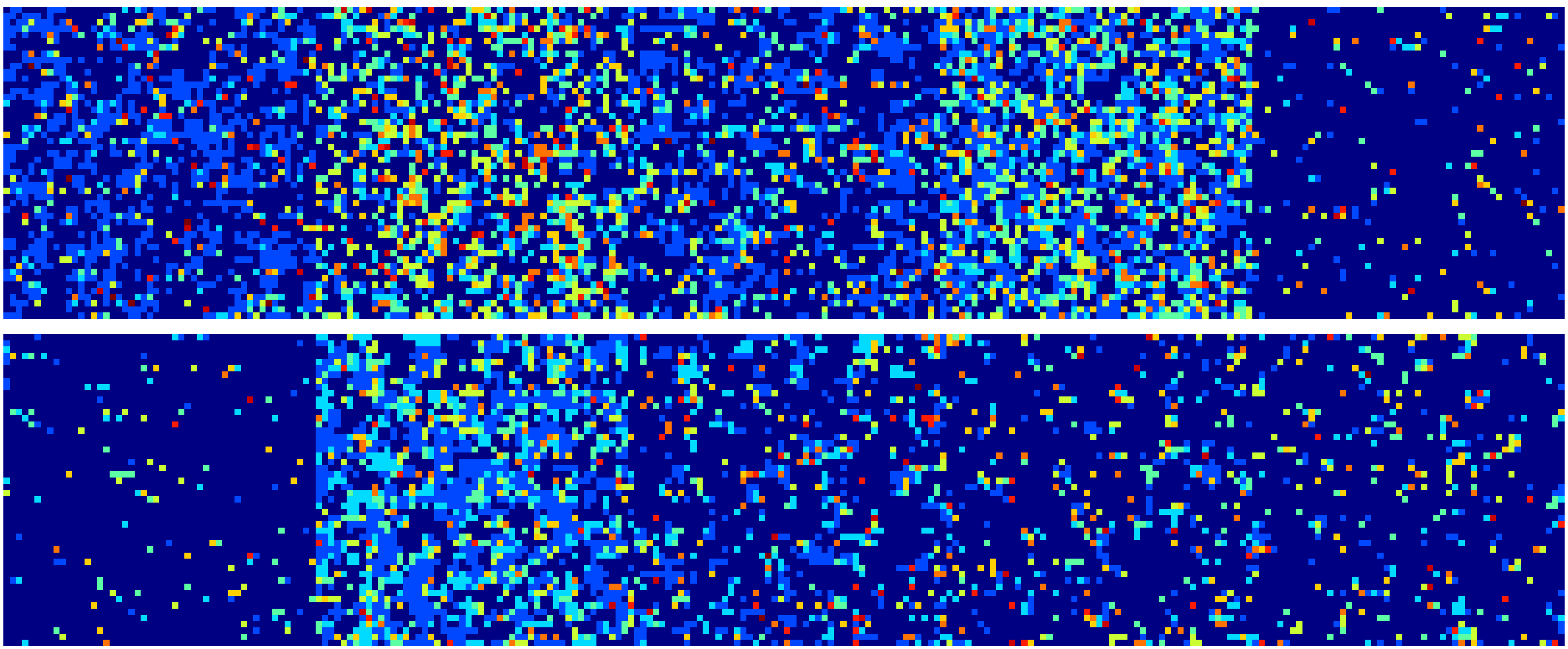
Product of Potts and DP?

Orbanz & Buhmann 2006

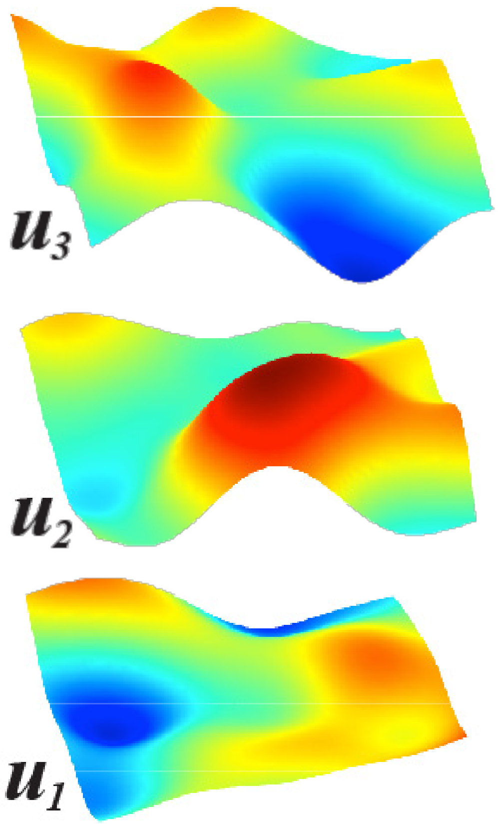
$$p(z) = \frac{1}{Z(\beta, \pi)} \prod_{(s,t)\in E} \psi_{st}(z_s, z_t) \prod_{s \in V} \pi(z_s)$$

Potts Potentials

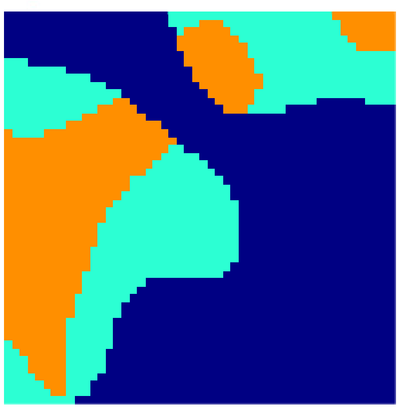
DP Bias: $\pi \sim \text{Stick}(\alpha)$

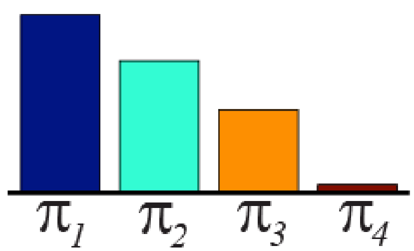


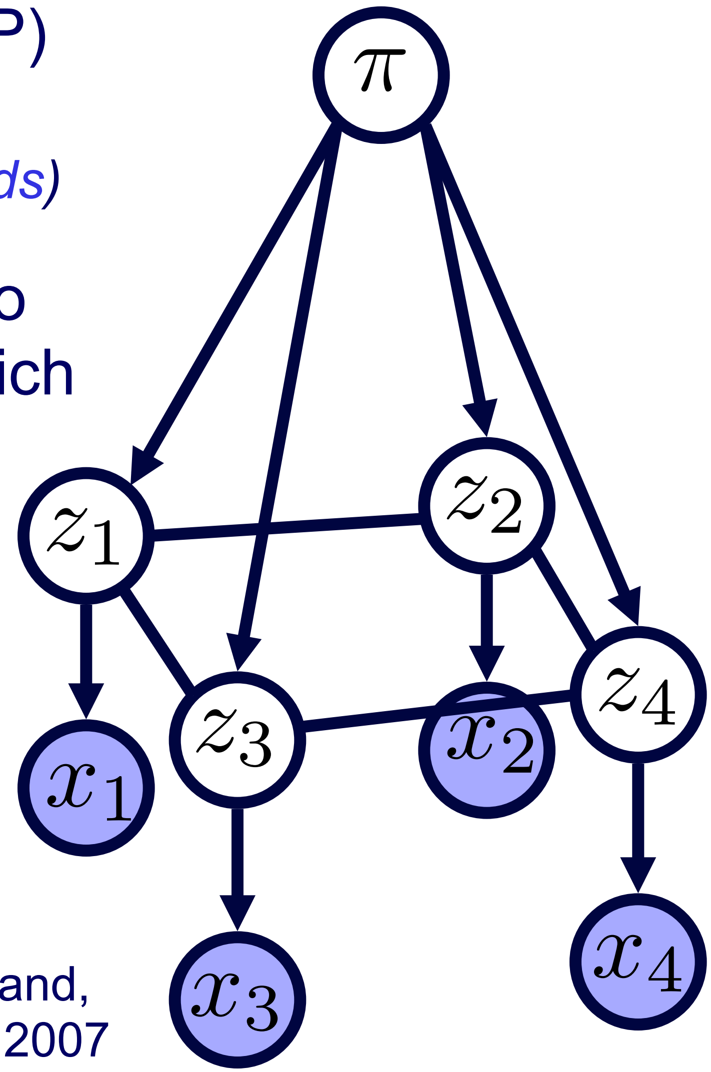
Spatially Dependent Pitman-Yor



- Cut random *surfaces* (samples from a GP) with *thresholds*
  (as in Level Set Methods)
- Assign each pixel to the *first* surface which exceeds threshold
  (as in Layered Models)

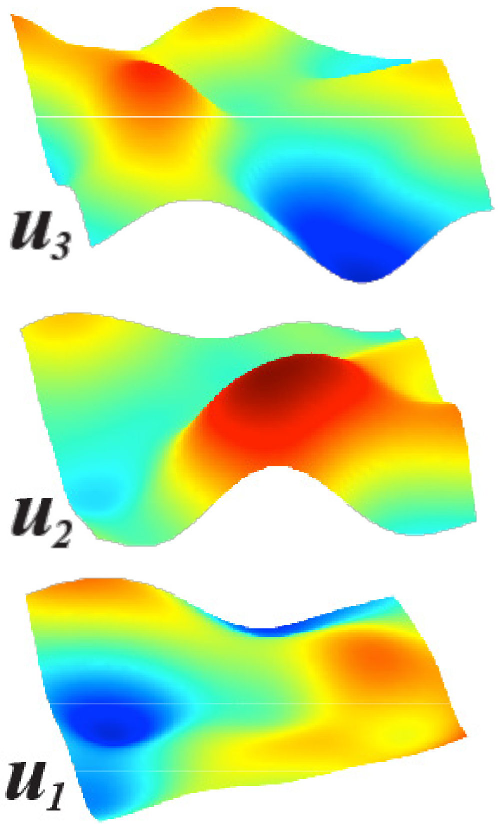




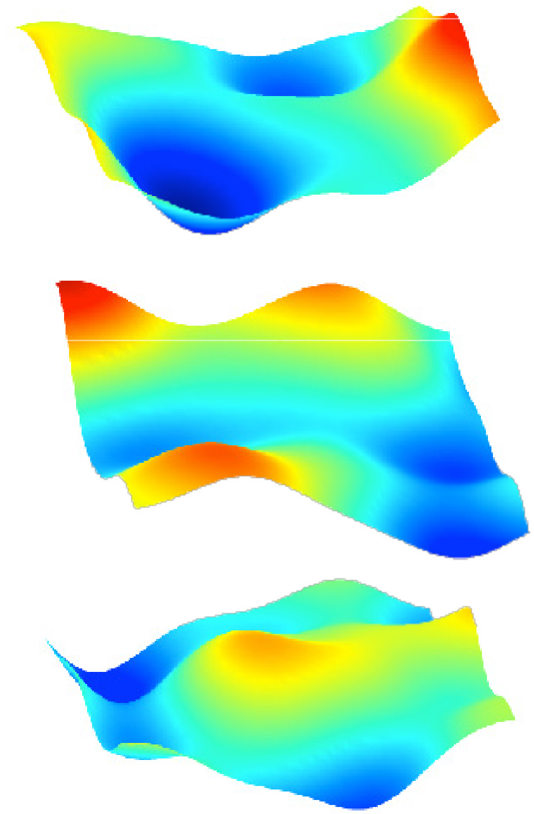


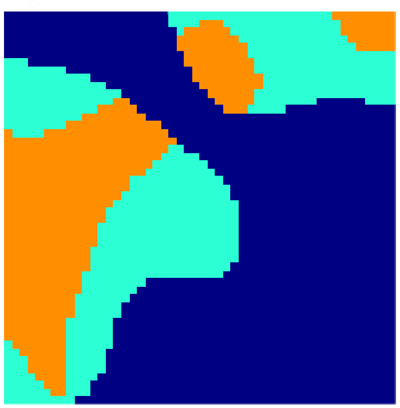
Duan, Guindani, & Gelfand,
Generalized Spatial DP, 2007

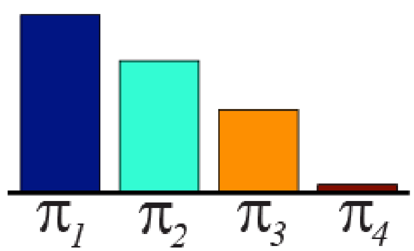
Spatially Dependent Pitman-Yor

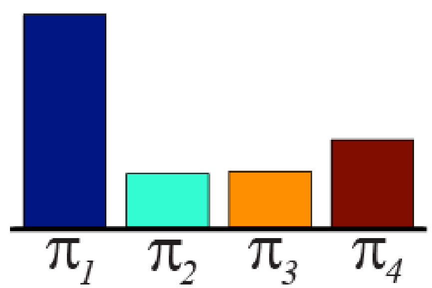


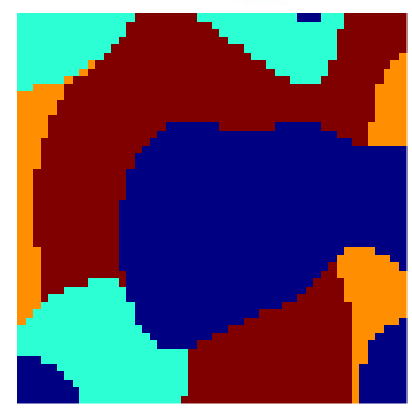
- Cut random *surfaces* (samples from a GP) with *thresholds* *(as in Level Set Methods)*
- Assign each pixel to the *first* surface which exceeds threshold *(as in Layered Models)*





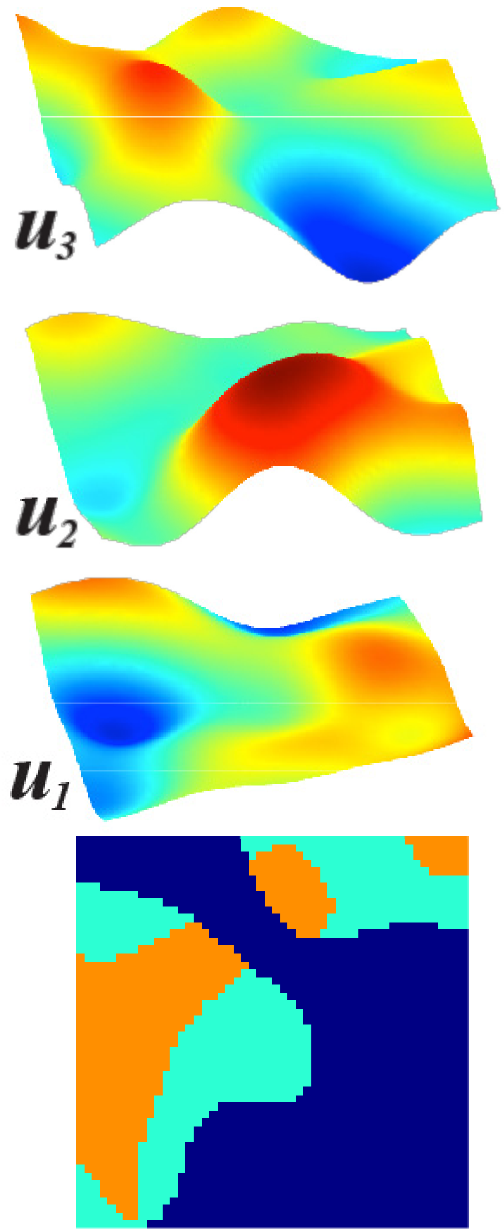




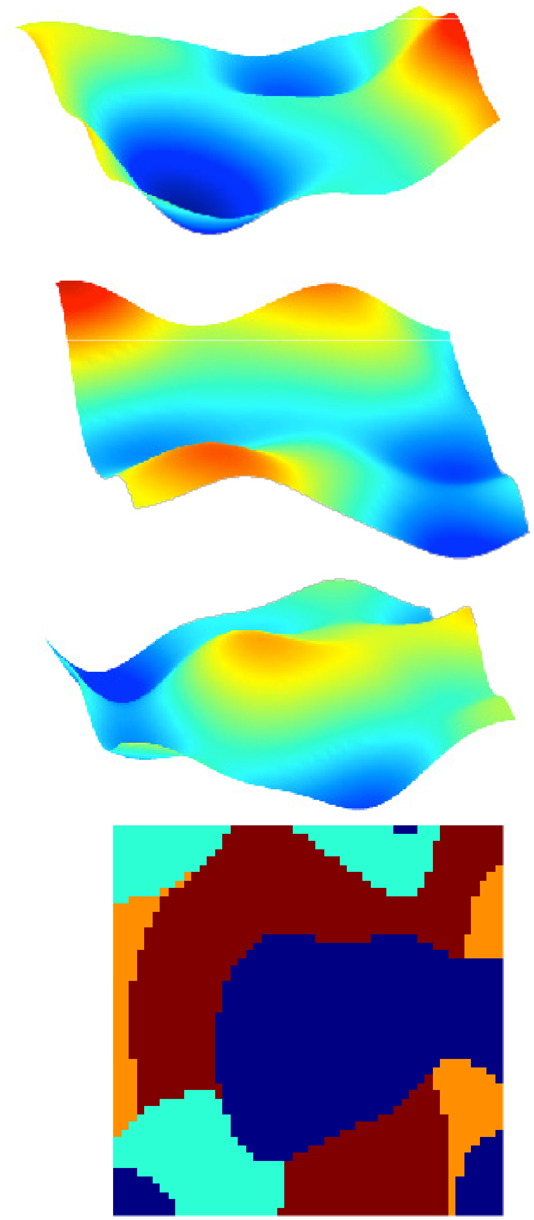


Duan, Guindani, & Gelfand, *Generalized Spatial DP*, 2007

Spatially Dependent Pitman-Yor

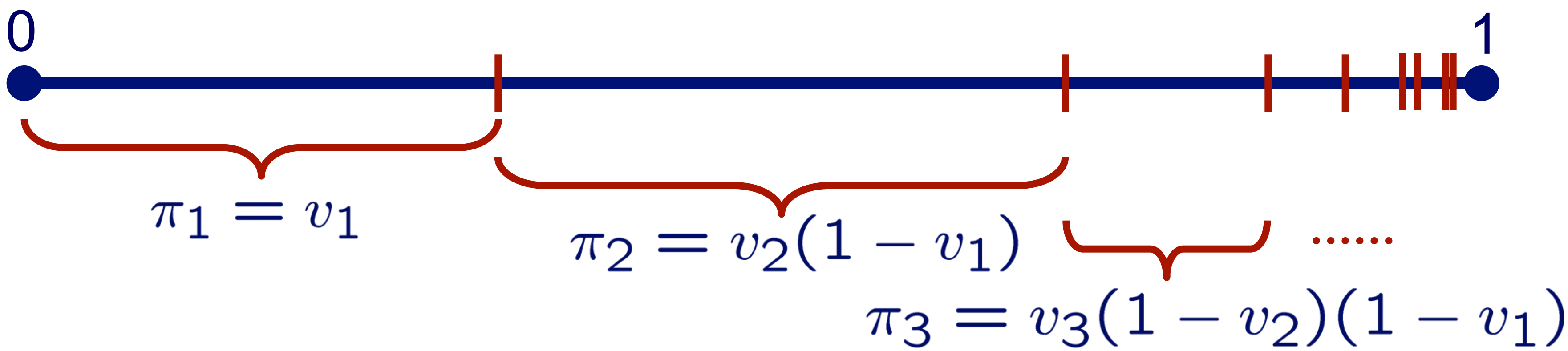


- Cut random *surfaces* (samples from a GP) with *thresholds*
  (as in Level Set Methods)
- Assign each pixel to the *first* surface which exceeds threshold
  (as in Layered Models)
- Retains *Pitman-Yor marginals* while jointly modeling rich *spatial dependencies*
  (as in Copula Models)



Stick-Breaking Revisited

$$\pi_k = v_k \prod_{\ell=1}^{k-1} (1 - v_\ell) \qquad v_k \sim \text{Beta}(1 - a, b + ka)$$



$$v_k = \mathbb{P}(z_i = k \mid z_i \neq k-1, \ldots, 1)$$

Multinomial Sampler:

$$u_i \sim \text{Unif}(0, 1)$$

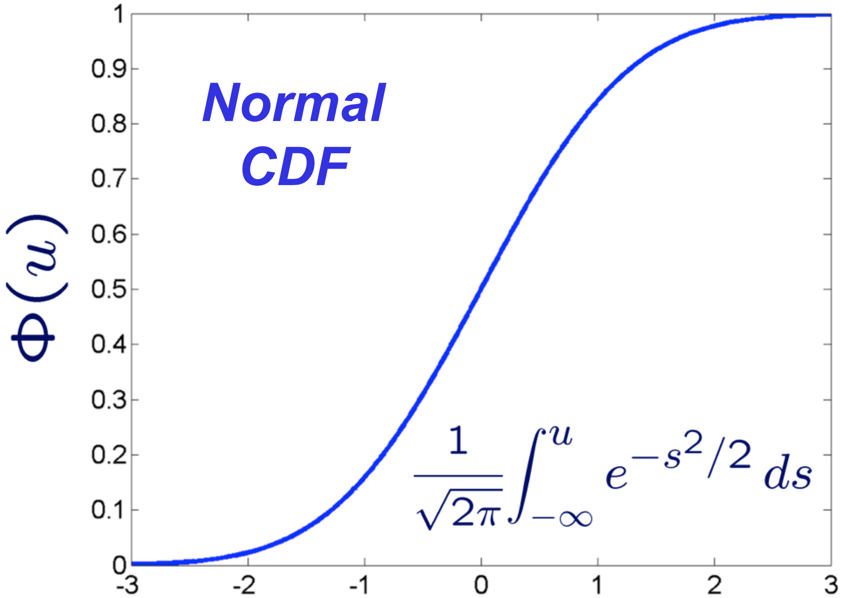
$$z_i = \text{CDF}_\pi^{-1}(u_i)$$

Sequential Binary Sampler:

$$b_{ki} \sim \text{Bernoulli}(v_k)$$

$$z_i = \min\{k \mid b_{ki} = 1\}$$

PY Gaussian Thresholds



$$\mathbb{P}[\Phi(u_{ki}) < v_k] = v_k$$

because

$$\Phi(u_{ki}) \sim \text{Unif}(0,1)$$

Gaussian Sampler:

$$u_{ki} \sim \mathcal{N}(0,1)$$

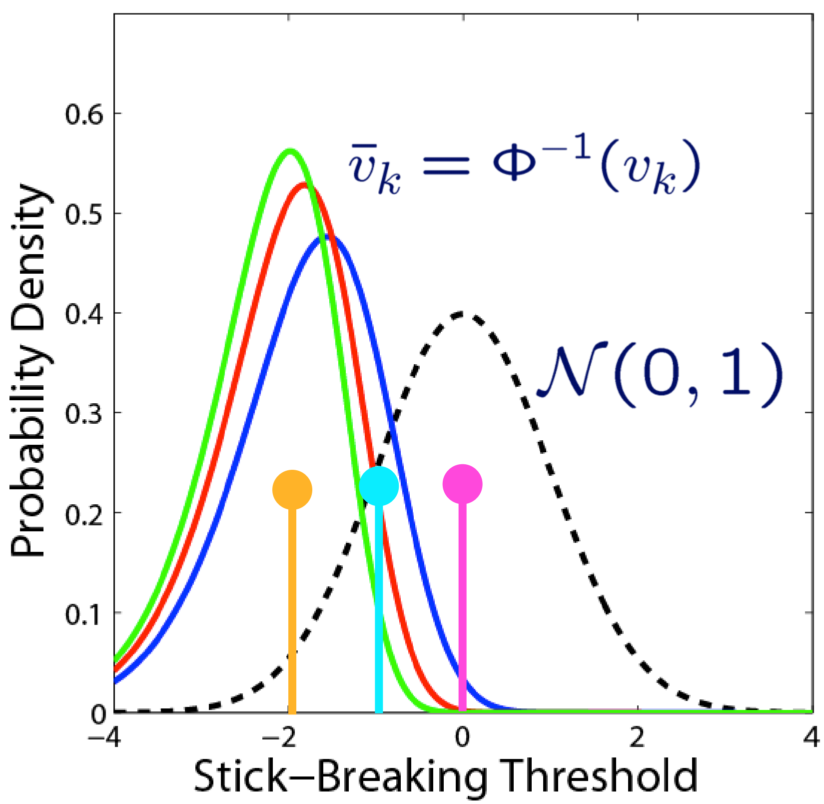
$$z_i = \min\{k \mid u_{ki} < \Phi^{-1}(v_k)\}$$

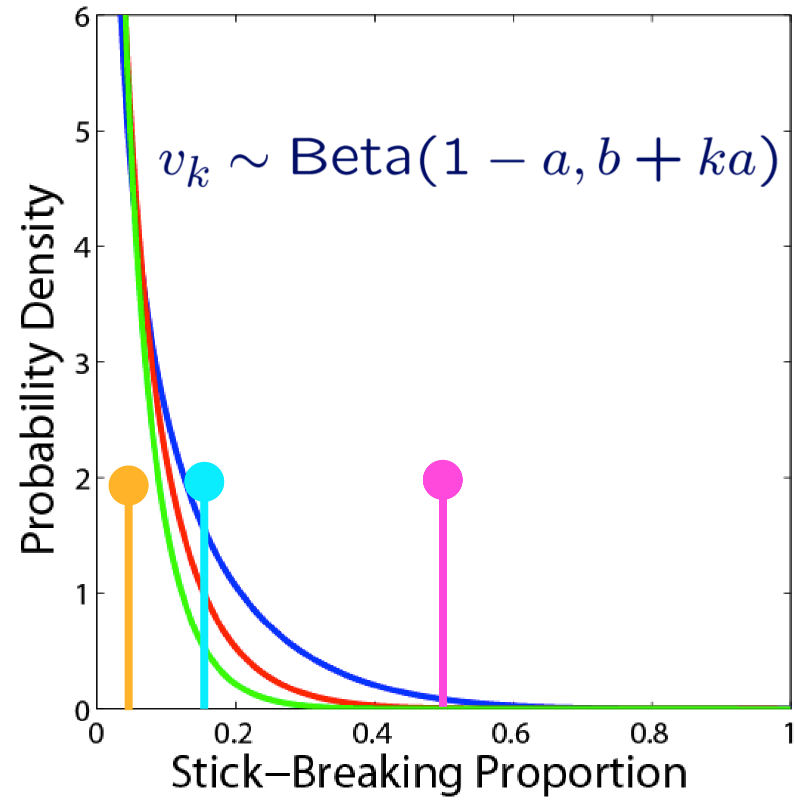
Sequential Binary Sampler:

$$b_{ki} \sim \text{Bernoulli}(v_k)$$

$$z_i = \min\{k \mid b_{ki} = 1\}$$

PY Gaussian Thresholds





Gaussian Sampler:

$$u_{ki} \sim \mathcal{N}(0,1)$$

$$z_i = \min\{k \mid u_{ki} < \Phi^{-1}(v_k)\}$$

Sequential Binary Sampler:

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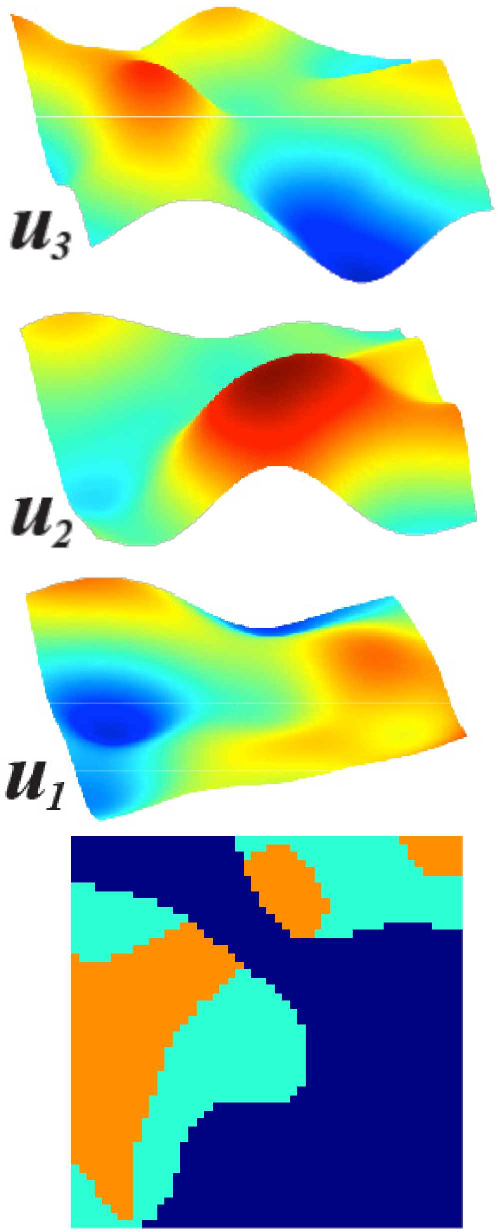
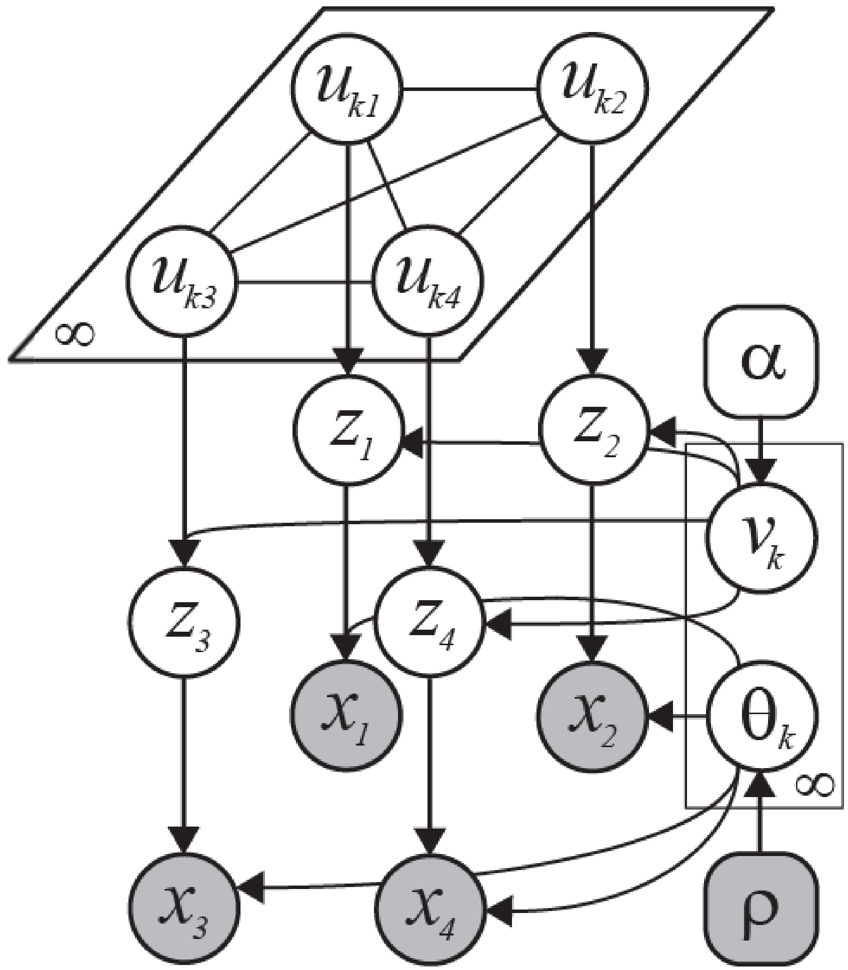
Spatially Dependent Pitman-Yor





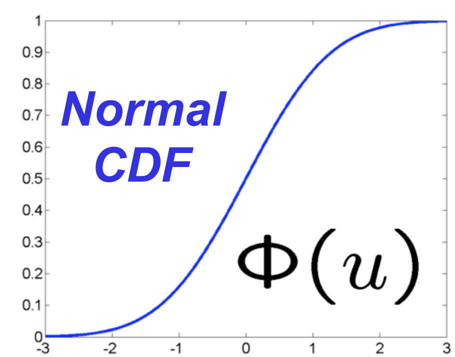
Non-Markov Gaussian Processes:

$$u_{ki} \sim \mathcal{N}(0,1)$$

$$u_{ki} \perp u_{\ell i}$$

PY prior: Segment size

$$v_k \sim \text{Beta}(1-a, b+ka)$$

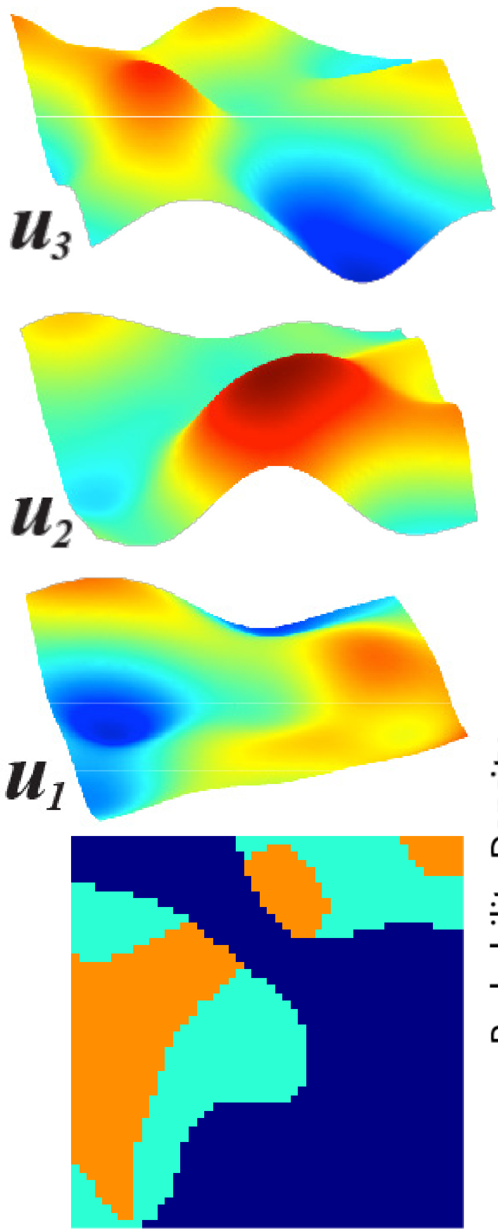


$$z_i = \min\{k \mid u_{ki} < \Phi^{-1}(v_k)\}$$

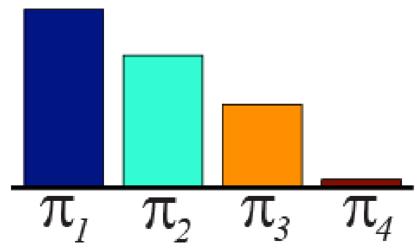
$$x_i \sim \text{Mult}(\theta_{z_i})$$

Feature Assignments

Preservation of PY Marginals



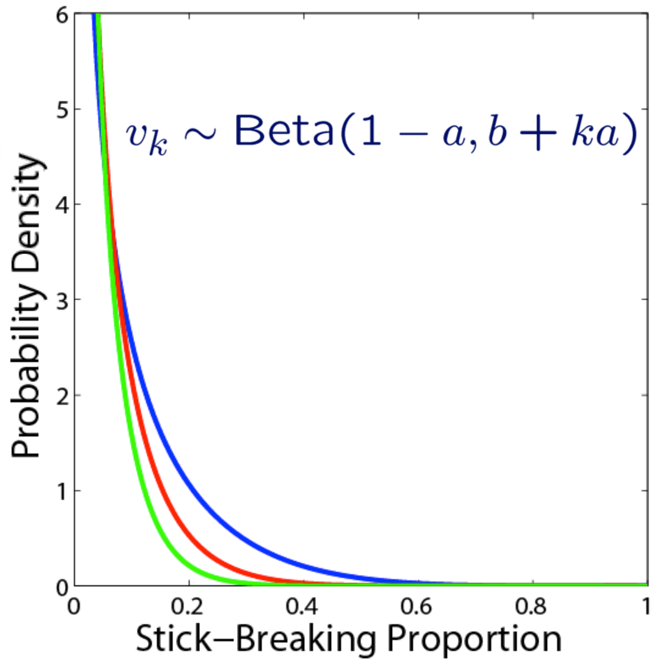
Why Ordered Layer Assignments?

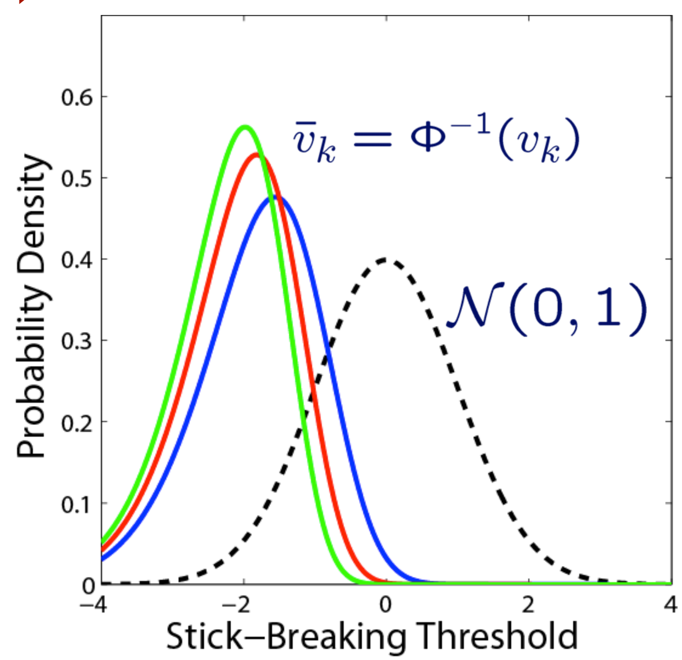


$$\pi_k = v_k \prod_{\ell=1}^{k-1} (1 - v_\ell)$$

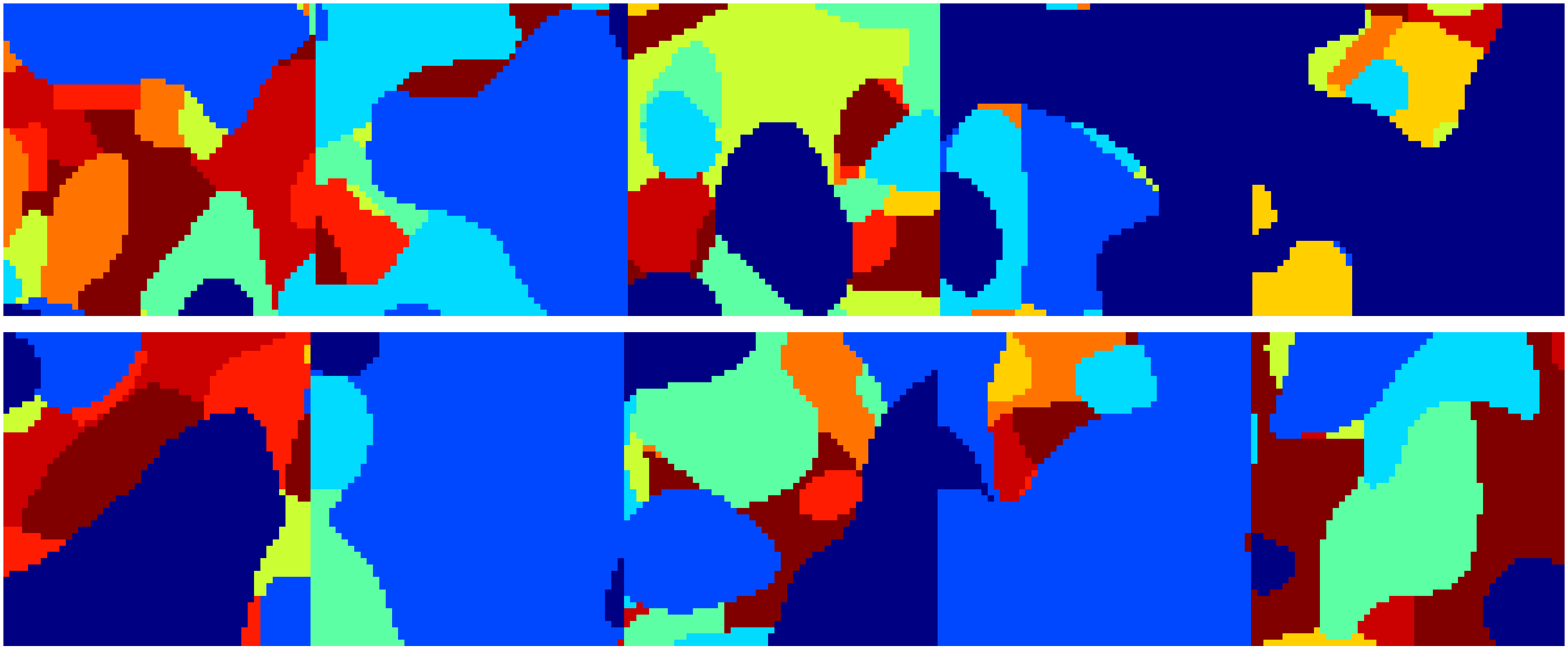
$$v_k = \mathbb{P}(z_i = k \mid z_i \neq k-1, \ldots, 1)$$

Stick Size Prior → *Random Thresholds*

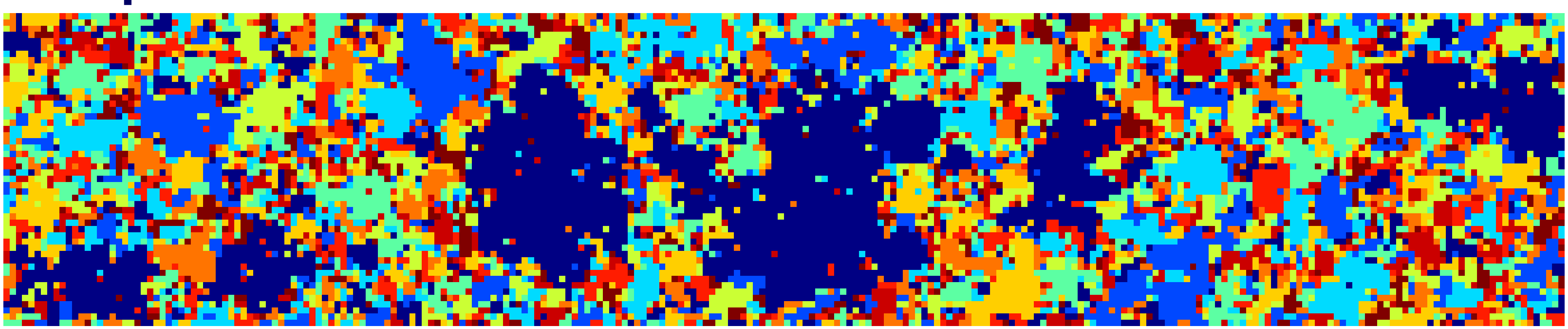




Samples from PY Spatial Prior



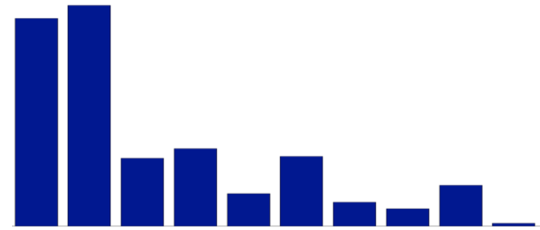
Comparison: Potts Markov Random Field



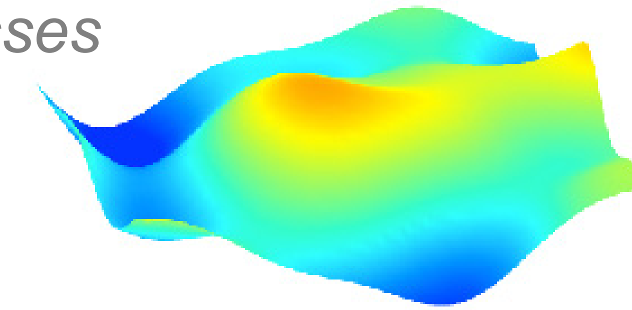
Outline

Model

- Dependent *Pitman-Yor processes*

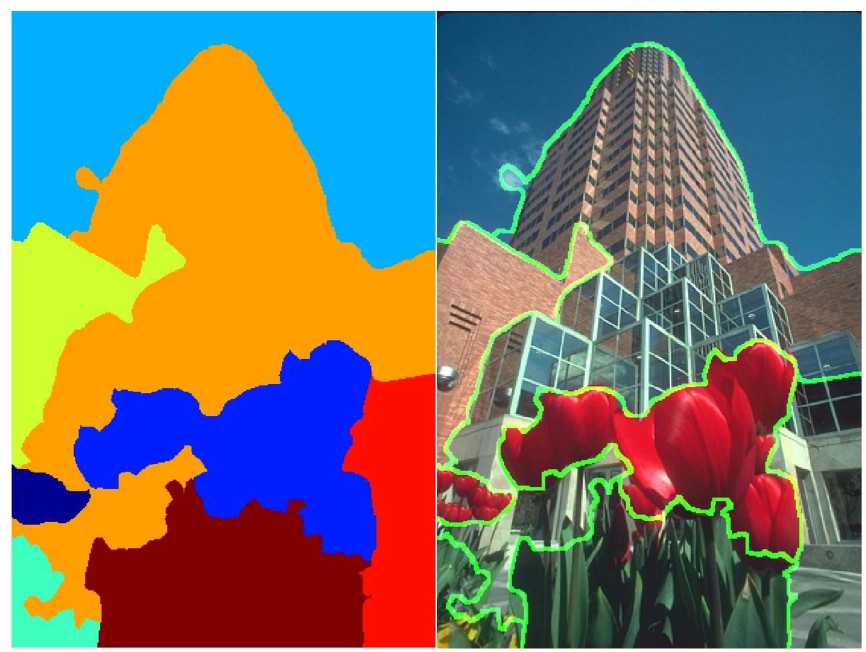


- Spatial coupling via *Gaussian processes*



Inference

- Stochastic search & *expectation propagation*



Learning

- Conditional covariance calibration

Results

- Multiple segmentations of natural images

Mean Field for Dependent PY

Factorized Gaussian Posteriors

$$q(\mathbf{u}) = \prod_{k=1}^{K} \prod_{i=1}^{N} \mathcal{N}(u_{ki} \mid \mu_{ki}, \lambda_{ki})$$

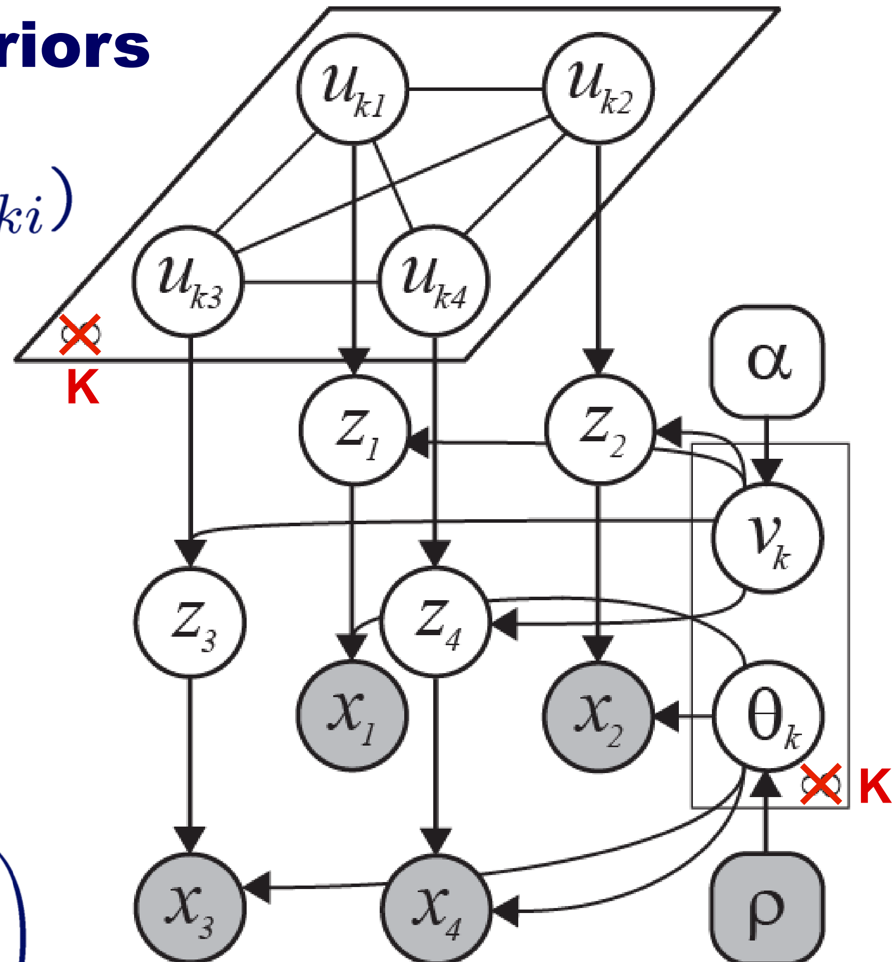
$$q(\bar{\mathbf{v}}) = \prod_{k=1}^{K} \mathcal{N}(\bar{v}_k \mid \nu_k, \delta_k)$$

Sufficient Statistics

$$z_i = \min\{k \mid u_{ik} < \bar{v}_k\}$$

Allows closed form update of $q(\theta_k)$ *via*

$$\mathbb{P}_q[u_{ki} < \bar{v}_k] = \Phi\left(\frac{\nu_k - \mu_{ki}}{\sqrt{\delta_k + \lambda_{ki}}}\right)$$



$$\log p(\mathbf{x} \mid \alpha, \rho) \geq H(q) + \mathbb{E}_q[\log p(\mathbf{u}, \bar{\mathbf{v}}, \boldsymbol{\theta} \mid \alpha, \rho)]$$

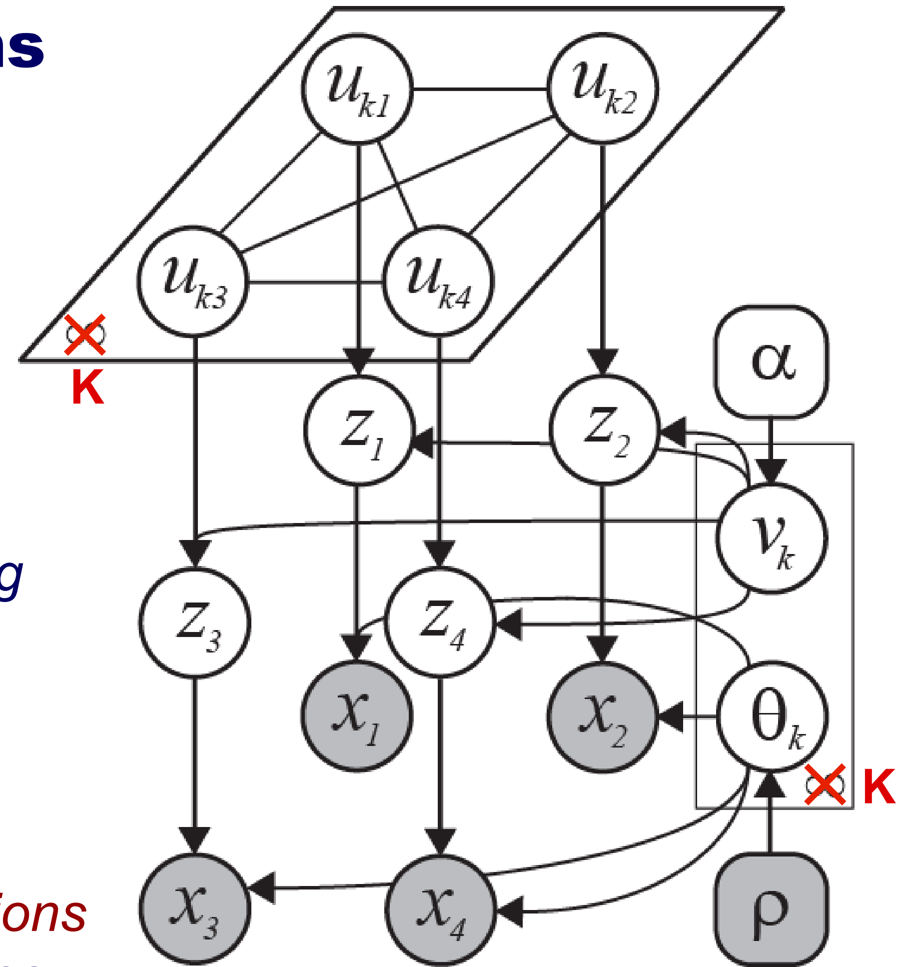
Mean Field for Dependent PY

Updating Layered Partitions

Evaluation of beta normalization constants:

$$\mathbb{E}_q[\log \Phi(\bar{v}_k)] \le \log \mathbb{E}_q[\Phi(\bar{v}_k)]$$
$$= \log \Phi\left(\frac{\nu_k}{\sqrt{1+\delta_k}}\right)$$

Jointly optimize each layer's threshold and Gaussian assignment surface, fixing all other layers, via backtracking conjugate gradient with line search

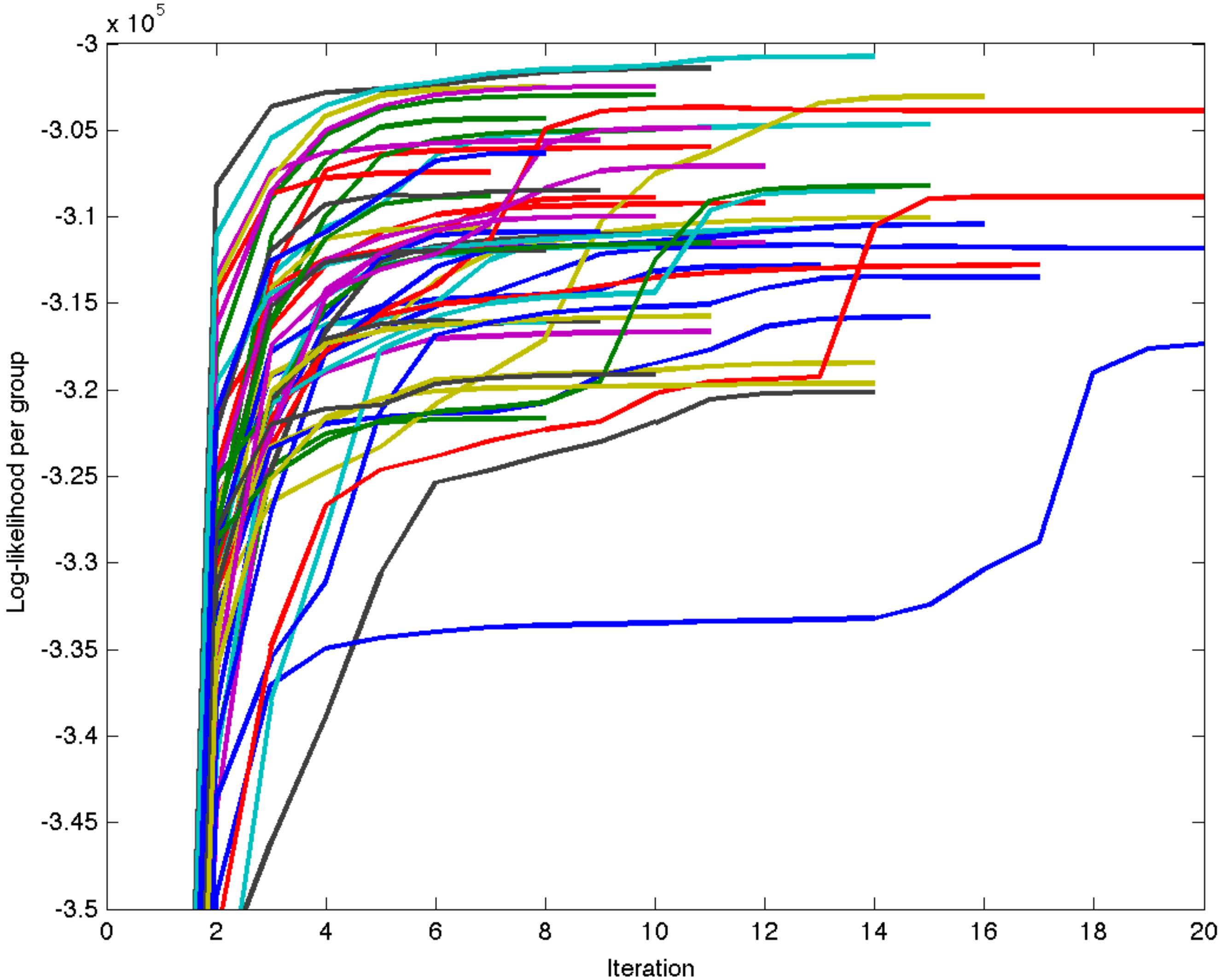


Reducing Local Optima

Place factorized posterior on eigenfunctions of Gaussian process, not single features

$$\log p(\mathbf{x} \mid \alpha, \rho) \ge H(q) + \mathbb{E}_q[\log p(\mathbf{u}, \bar{\mathbf{v}}, \boldsymbol{\theta} \mid \alpha, \rho)]$$

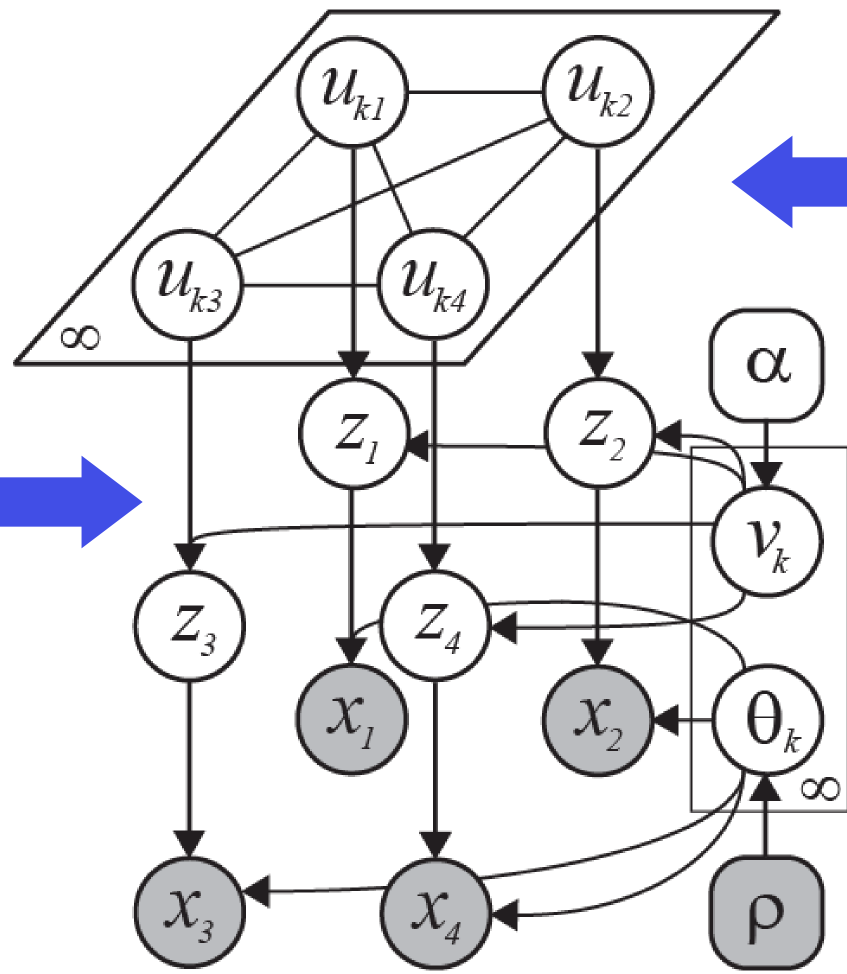
Robustness and Initialization



Log-likelihood bounds versus iteration, for many random initializations of mean field variational inference on a single image.

Alternative: Inference by Search

Consider hard assignments of superpixels to layers (partitions)



Marginalize layer support functions via expectation propagation (EP): approximate but very accurate

Integrate likelihood parameters analytically (conjugacy)

No need for a finite, conservative model truncation!

Maximization Expectation

EM Algorithm

- E-step: Marginalize latent variables (approximate)
- M-step: Maximize likelihood bound given model parameters

ME Algorithm

Kurihara & Welling, 2009

- M-step: Maximize likelihood given latent assignments
- E-step: Marginalize random parameters (exact)

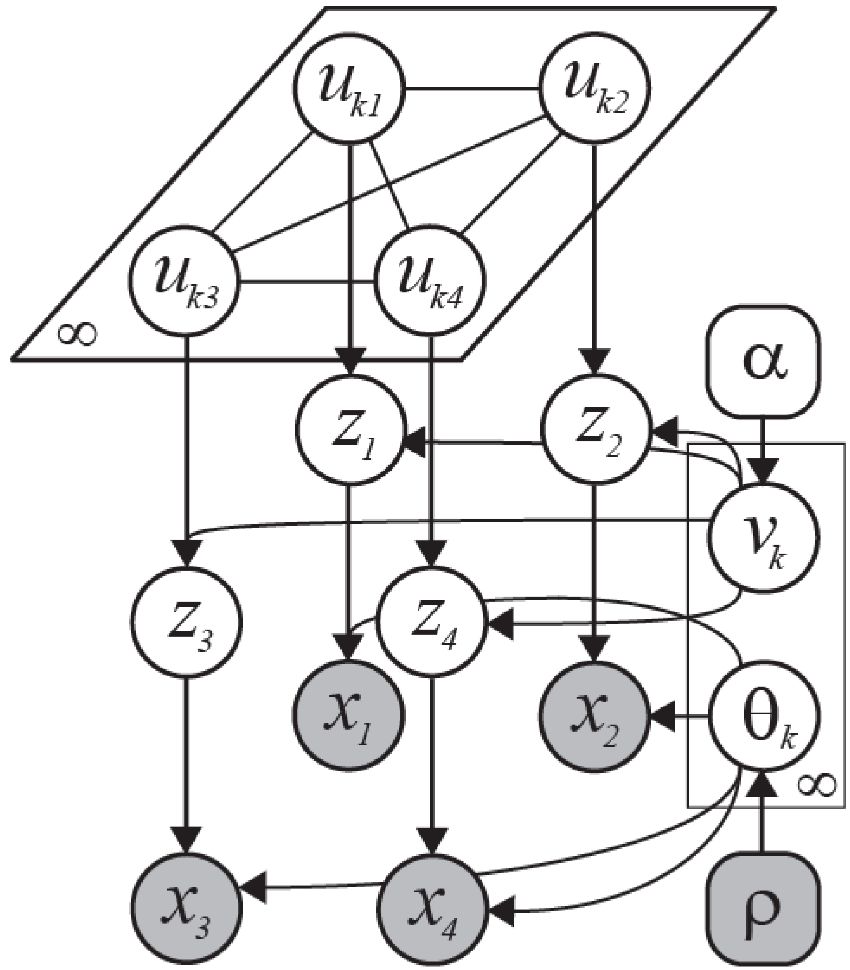
Why Maximization-Expectation?

- Parameter marginalization allows Bayesian "model selection"
- Hard assignments allow efficient algorithms, data structures
- Hard assignments consistent with clustering objectives
- *No need for finite truncation of nonparametric models*

Discrete Search Moves

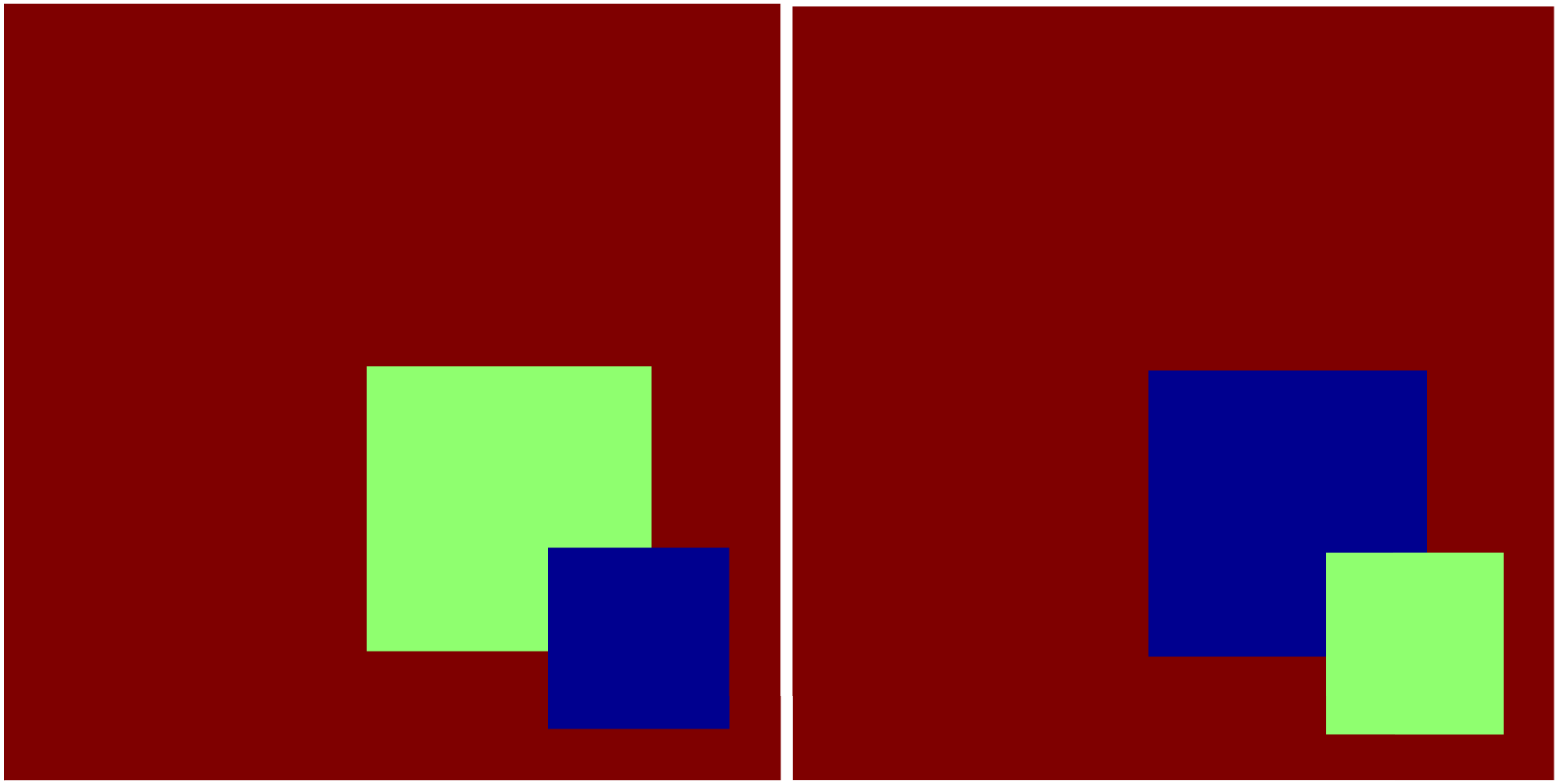
Stochastic proposals, accepted if and only if they improve our EP estimate of marginal likelihood:

- **Merge:** Combine a pair of regions into a single region
- **Split:** Break a single region into a pair of regions (for diversity, a few proposals)
- **Shift:** Sequentially move single superpixels to the most probable region
- **Permute:** Swap the position of two layers in the order



Marginalization of continuous variables simplifies these moves…

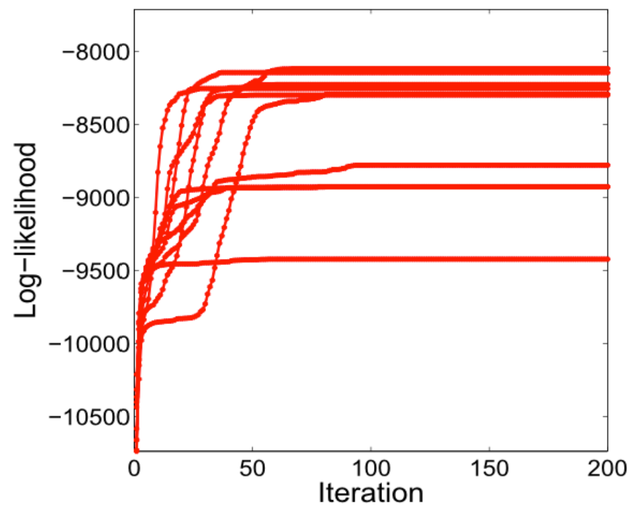
Inferring Ordered Layers

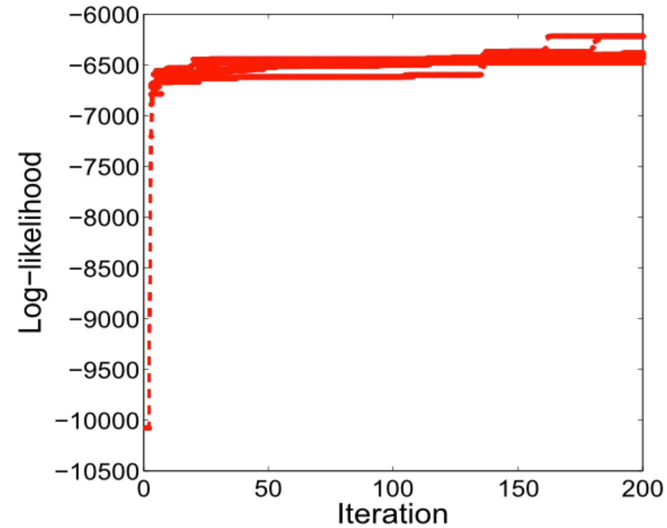


Order A: Front, Middle, Back **Order B: Front, Middle, Back**

- Which is preferred by a diagonal covariance? Order B
- Which is preferred by a spatial covariance? Order A

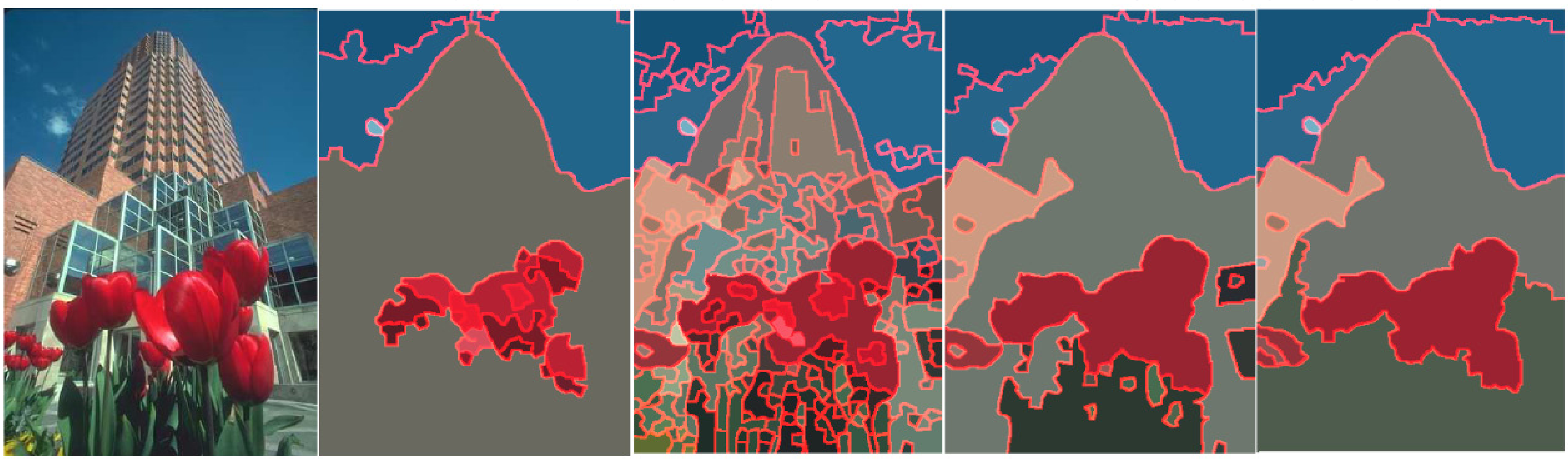
Inference Across Initializations





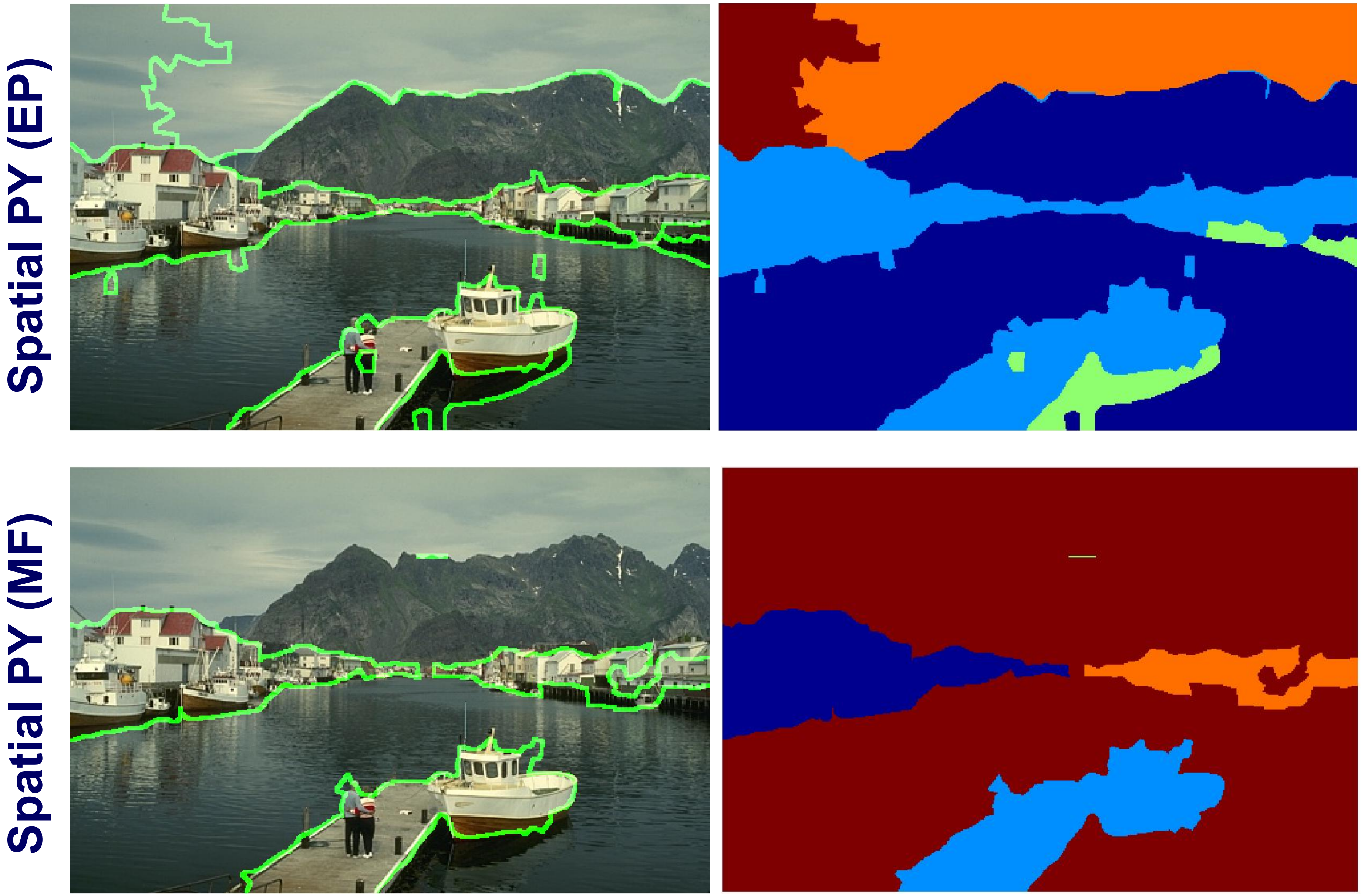
Mean Field Variational

EP Stochastic Search



Best *Worst* *Best* *Worst*

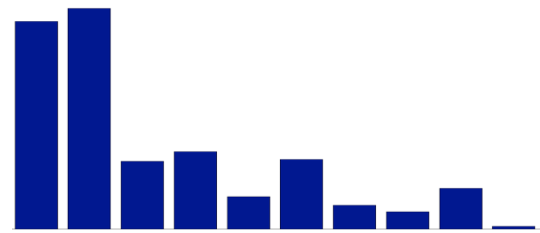
BSDS: Spatial PY Inference

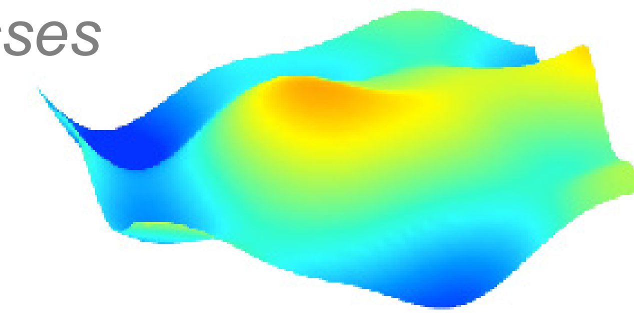


Outline

Model

- Dependent *Pitman-Yor processes*
- Spatial coupling via *Gaussian processes*





Inference

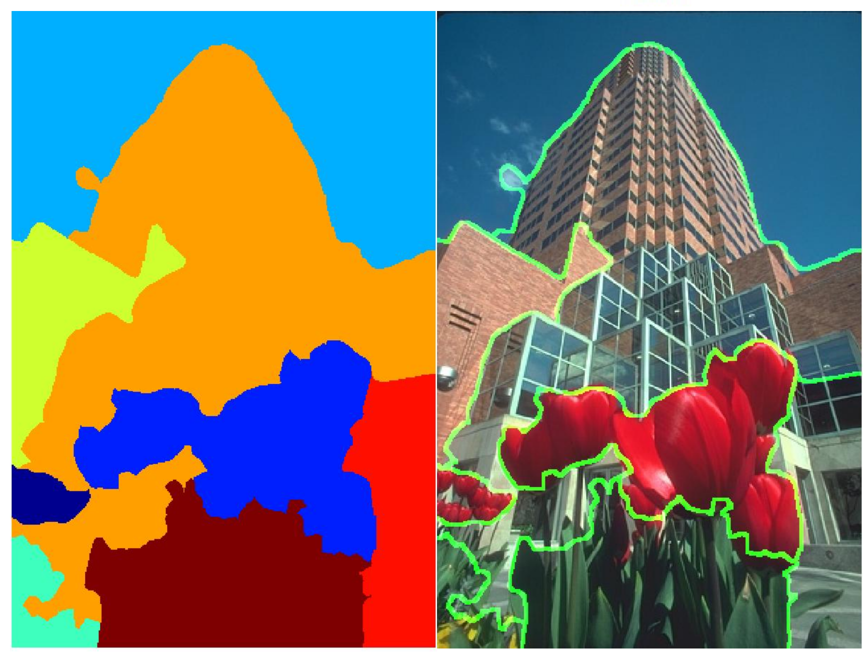
- Stochastic search & *expectation propagation*

Learning

- Conditional covariance calibration

Results

- Multiple segmentations of natural images



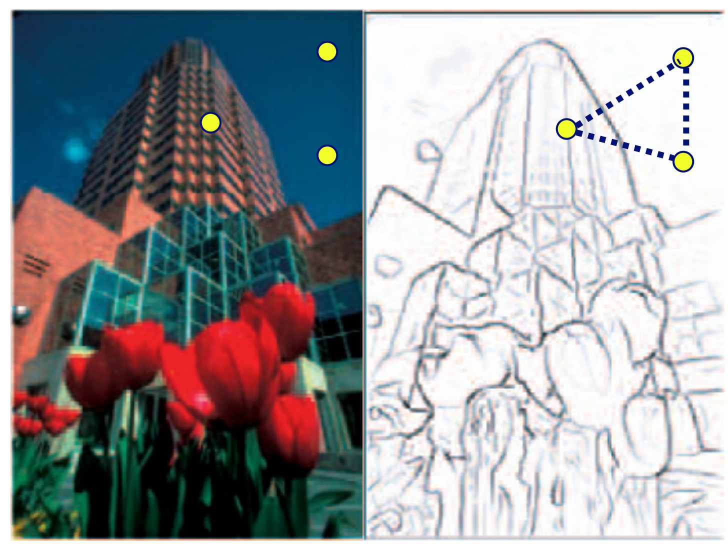
Covariance Kernels

- Thresholds determine segment *size*: Pitman-Yor
- Covariance determines segment *shape*:

$C(y_i, y_j)$ ⟺ *probability that features at locations* (y_i, y_j) *are in the same segment*

Roughly Independent Image Cues:

- Color and texture histograms within each region: Model generatively via multinomial likelihood (Dirichlet prior)
- Pixel locations and *intervening contour* cues: Model conditionally via GP covariance function



Berkeley Pb (probability of boundary) detector

Learning from Human Segments

- Data unavailable to learn models of all the categories we're interested in: We want to discover new categories!
- Use logistic regression, and basis expansion of image cues, to learn binary "are we in the same segment" predictors:
  - *Generative: Distance only*
  - *Conditional: Distance, intervening contours, …*

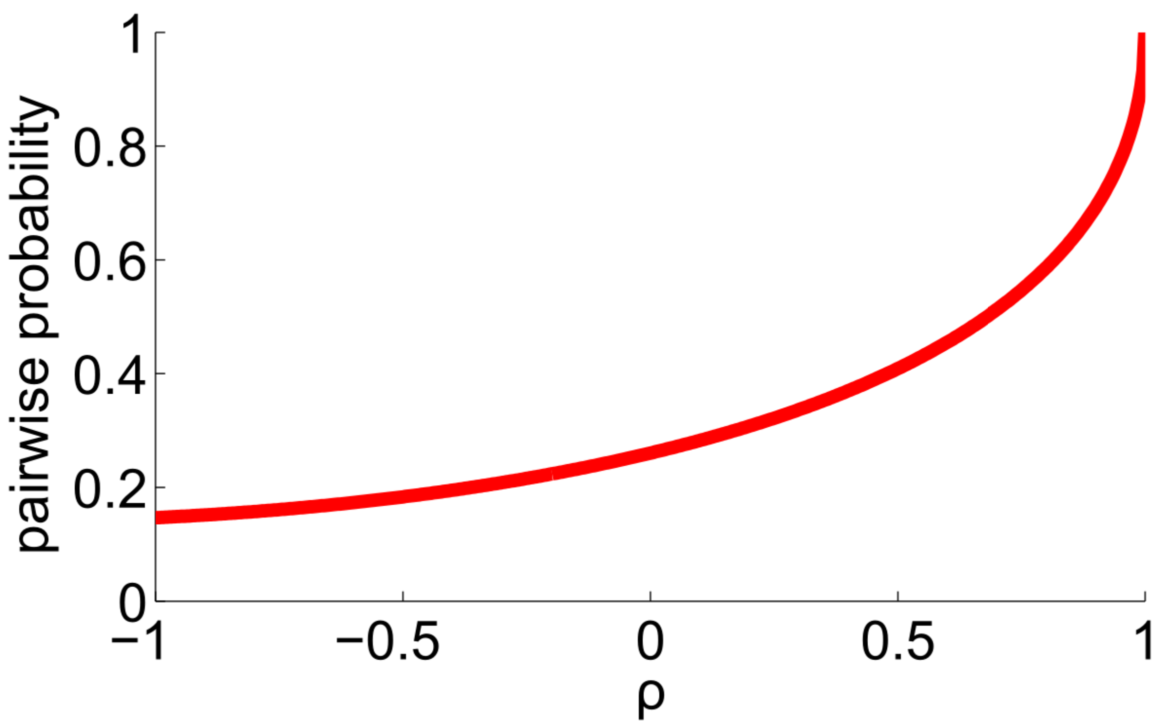
From Probability to Correlation

$$q_-^k(\alpha,\rho) = \int_{-\infty}^{\infty}\int_{-\infty}^{\delta_k}\int_{-\infty}^{\delta_k} \mathcal{N}\left(\begin{bmatrix} u_i \\ u_j \end{bmatrix} \middle| \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right) p(\delta_k|\alpha)du_i du_j d\delta_k$$

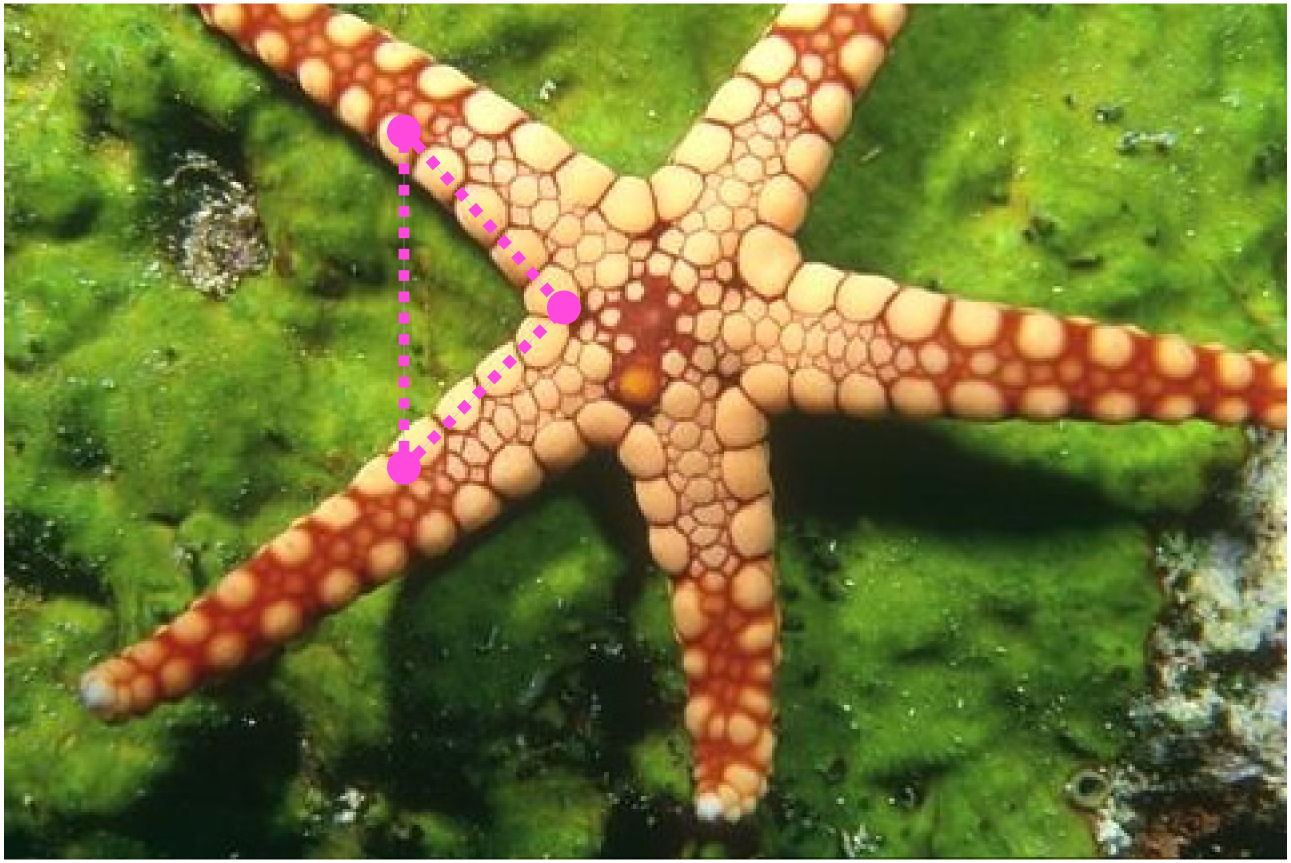
$$q_+^k(\alpha,\rho) = \int_{-\infty}^{\infty}\int_{\delta_k}^{\infty}\int_{\delta_k}^{\infty} \mathcal{N}\left(\begin{bmatrix} u_i \\ u_j \end{bmatrix} \middle| \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right) p(\delta_k|\alpha)du_i du_j d\delta_k$$

$$p_{ij} = q_-^1(\alpha,\rho) + q_-^2(\alpha,\rho)q_+^1(\alpha,\rho) + q_-^3(\alpha,\rho)q_+^1(\alpha,\rho)q_+^2(\alpha,\rho) + \ldots$$

There is an injective mapping between covariance and the probability that two superpixels are in the same segment.

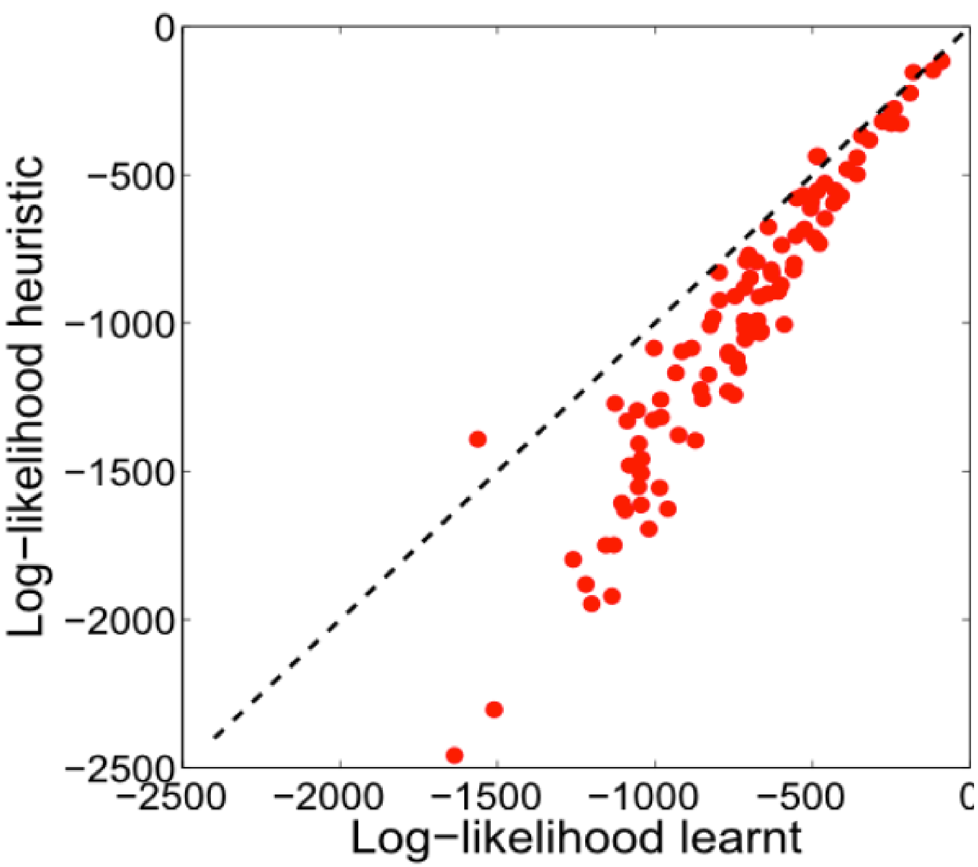


Low-Rank Covariance Projection

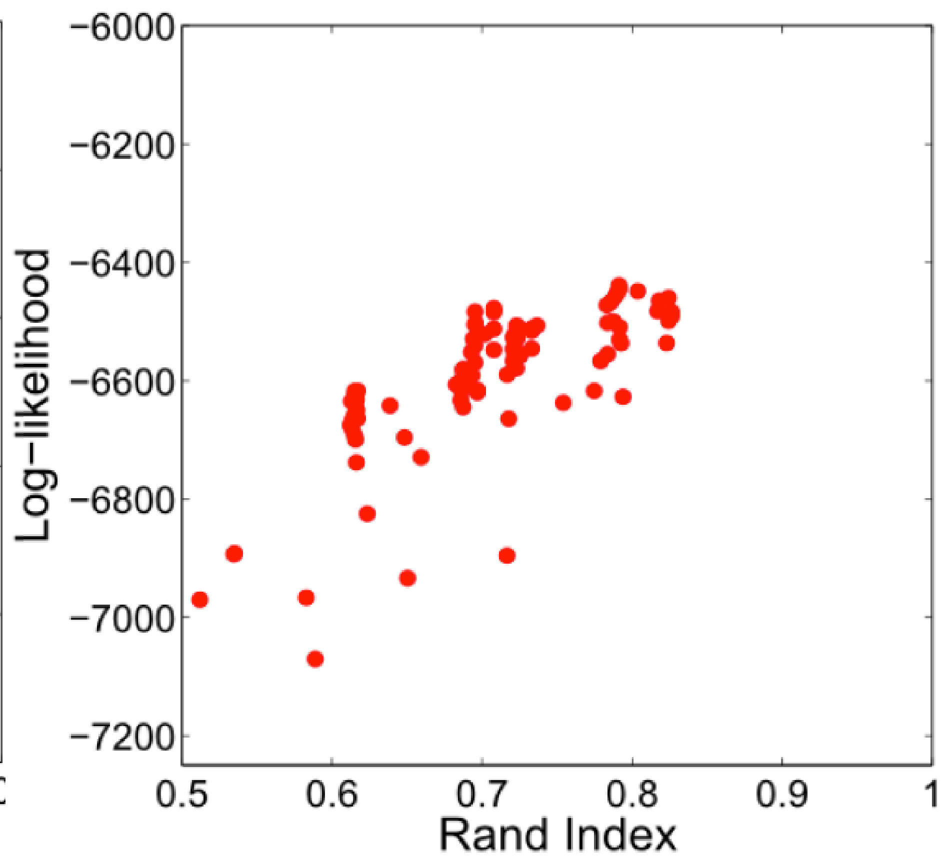


- The pseudo-covariance constructed by considering each superpixel pair independently may not be positive definite
- Projected gradient method finds *low rank* (factor analysis), unit diagonal covariance close to target estimates

Prediction of Test Partitions



Heuristic versus Learned Image Partition Probabilities



Learned Probability versus Rand index measure of partition overlap

Comparing Spatial PY Models

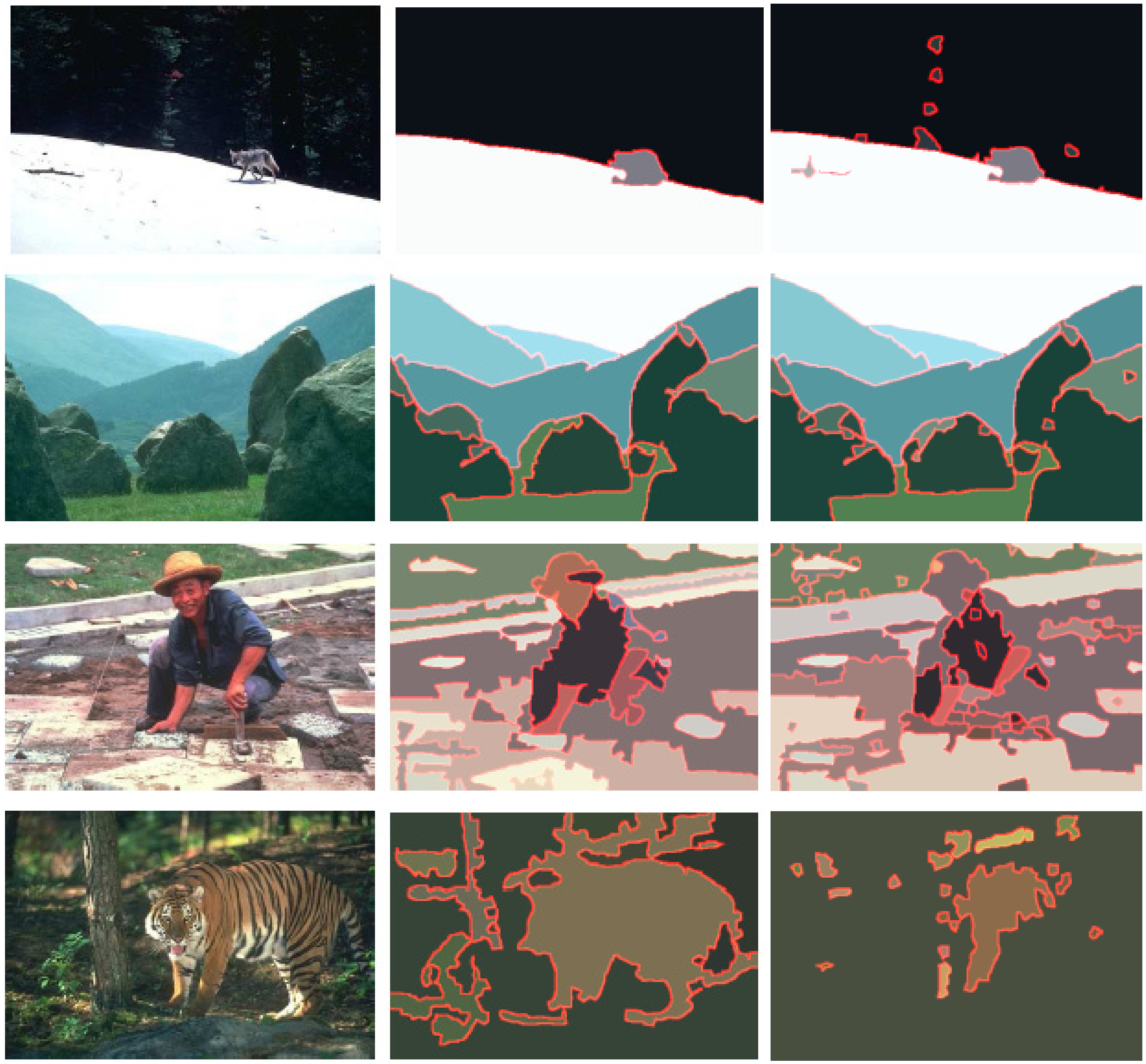
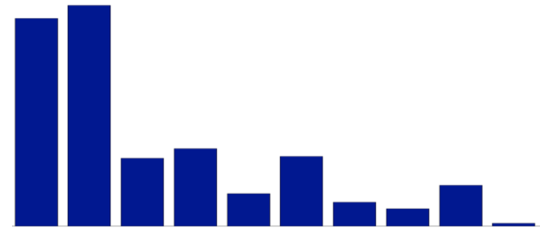


Image PY Learned PY Heuristic

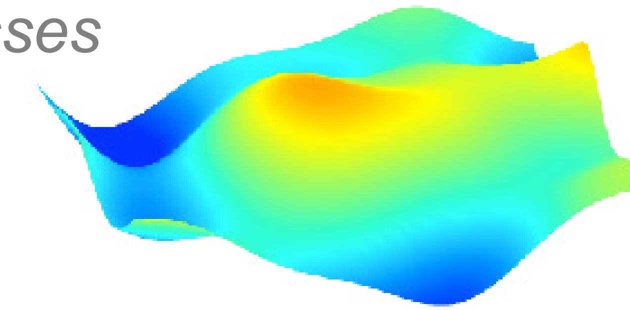
Outline

Model

- Dependent *Pitman-Yor processes*



- Spatial coupling via *Gaussian processes*



Inference

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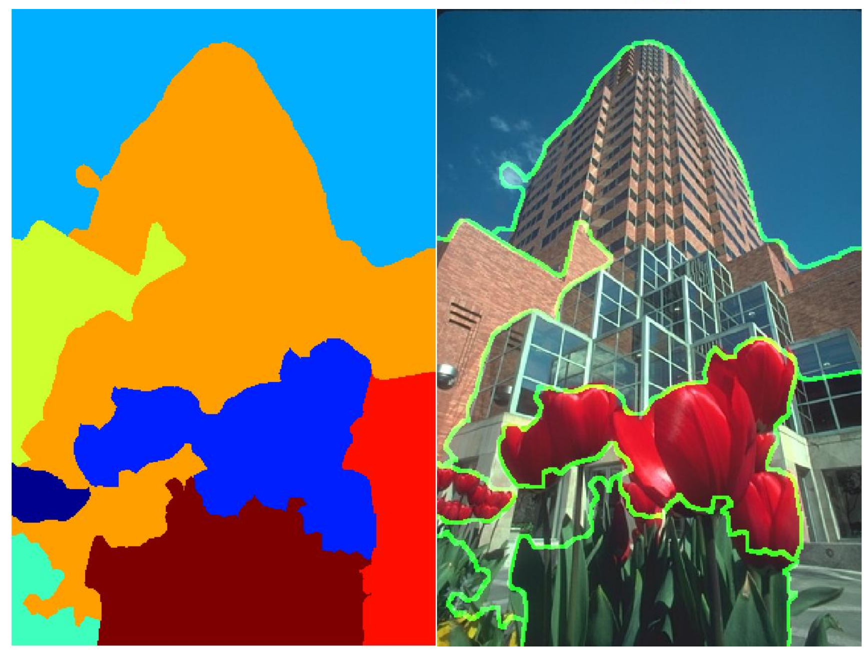
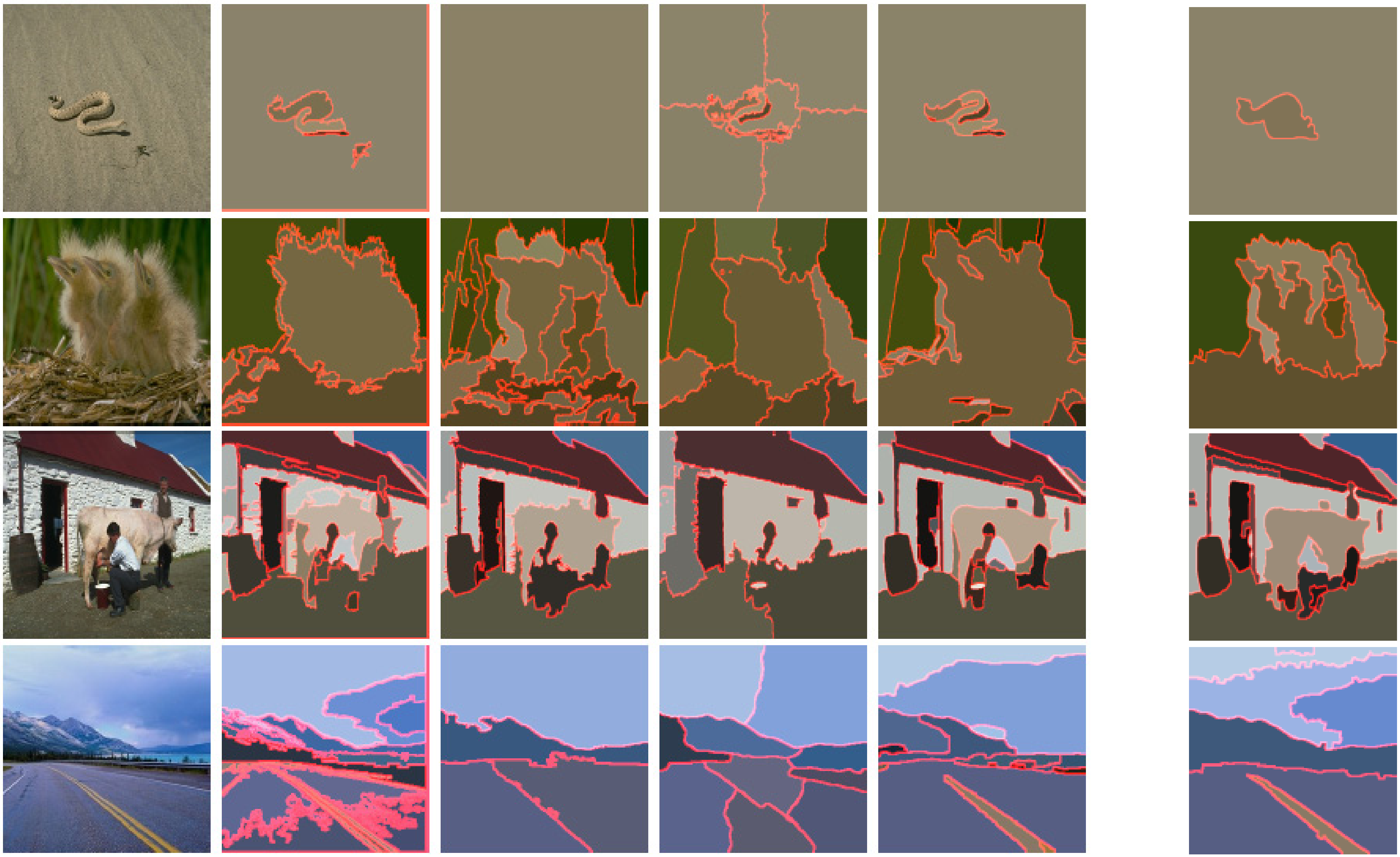
- Multiple segmentations of natural images



Other Segmentation Methods



FH Graph *Mean Shift* *NCuts* *gPb+UCM* *Spatial PY*

Quantitative Comparisons

Algorithms	PRI	VI	SegCover
Ncuts	0.74	2.5	0.38
MS	0.77	2.5	0.44
FH	0.77	2.1	0.52
gPb	0.81	2.0	0.58
PYdist	0.72	2.1	0.51
PYall	0.76	2.1	0.52

Berkeley Segmentation

gPb	0.74	2.1	0.53
PYall	0.73	1.9	0.55

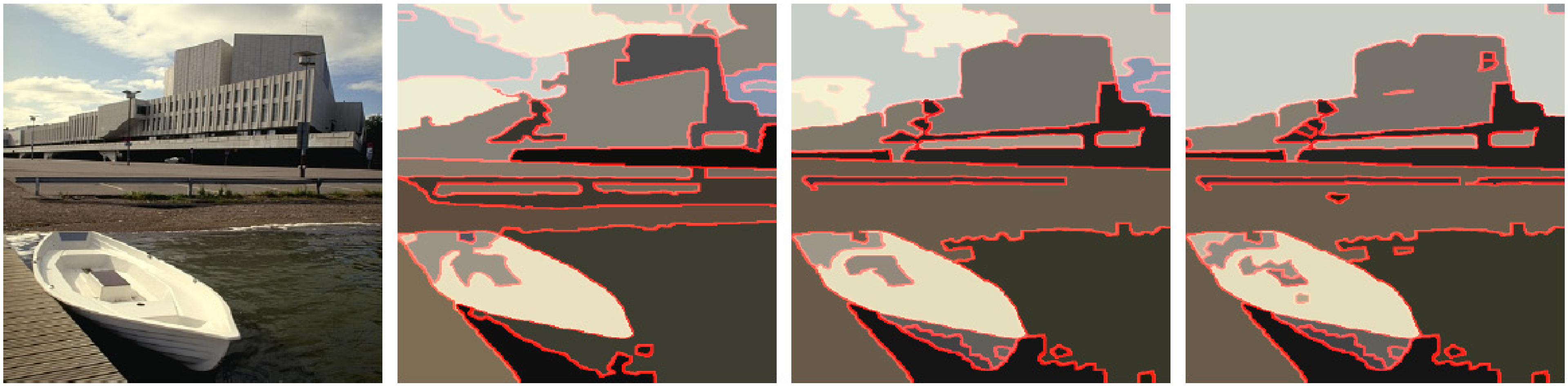
LabelMe Scenes

- On BSDS, similar or better than all methods except gPb
- On LabelMe, performance of Spatial PY is better than gPb

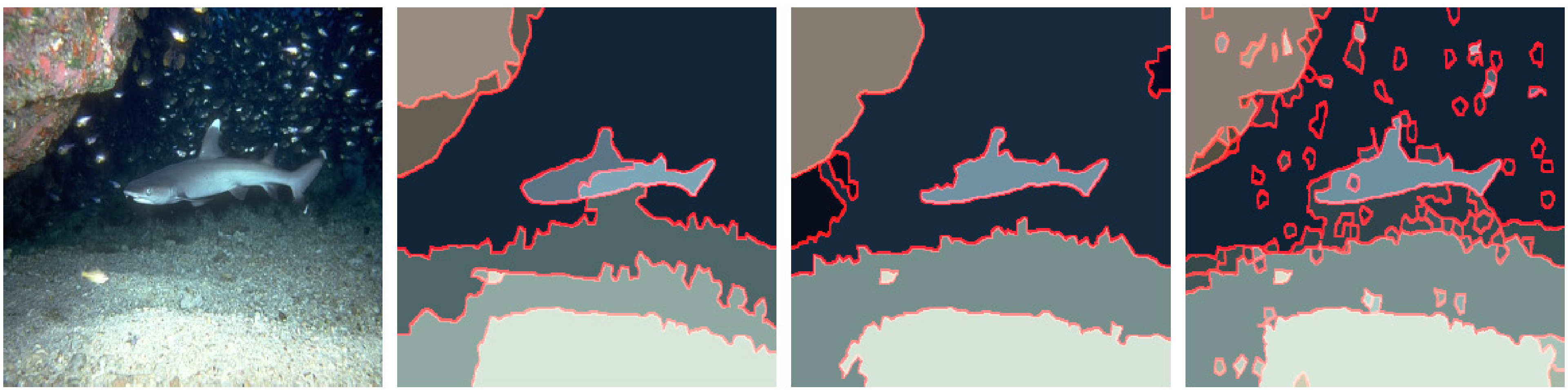
Room for Improvement:

- Implementation efficiency and search run-time
- Histogram likelihoods discard too much information
- Most probable segmentation does not minimize Bayes risk

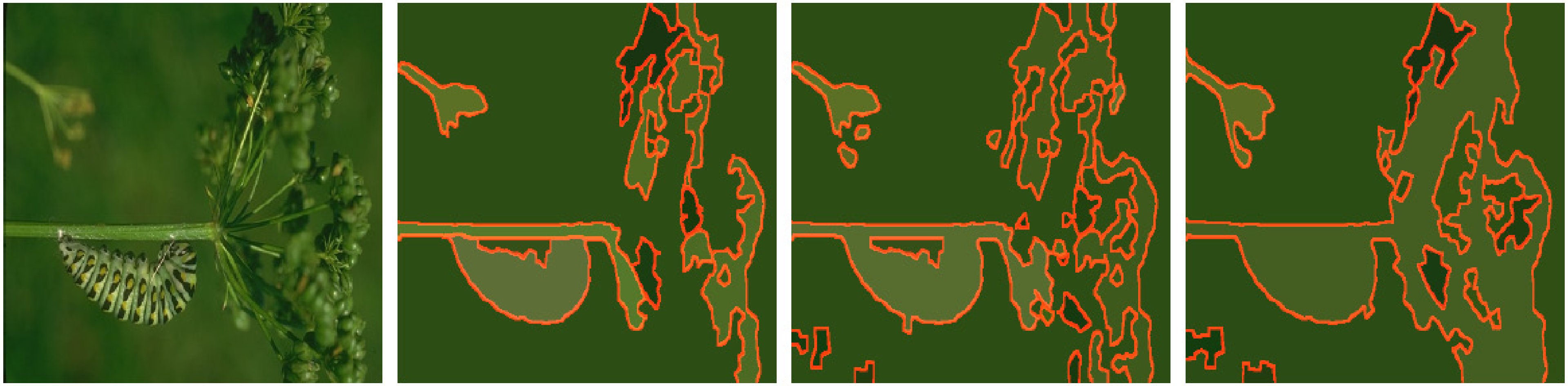
Multiple Spatial PY Modes



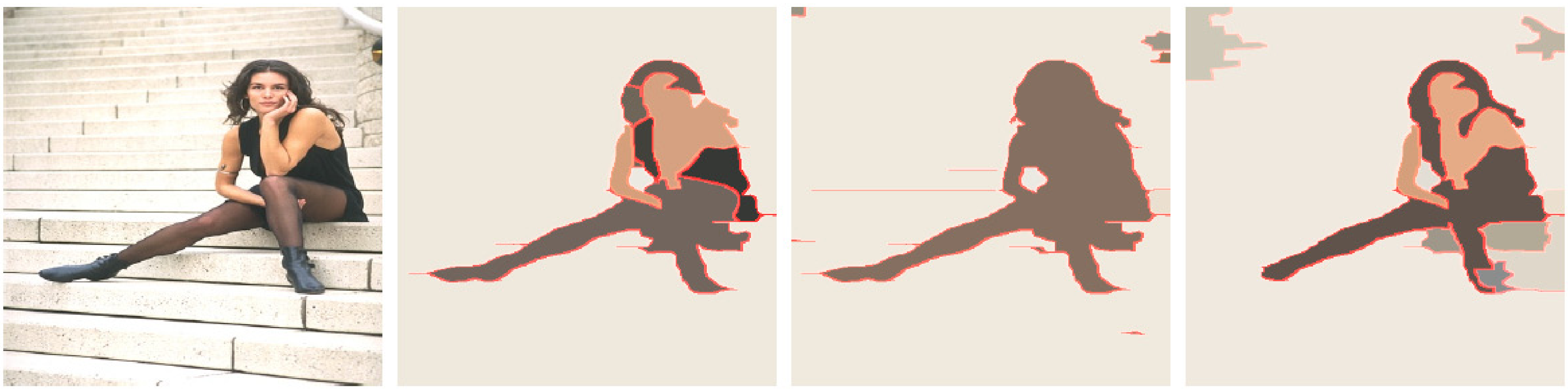
Most Probable



Multiple Spatial PY Modes

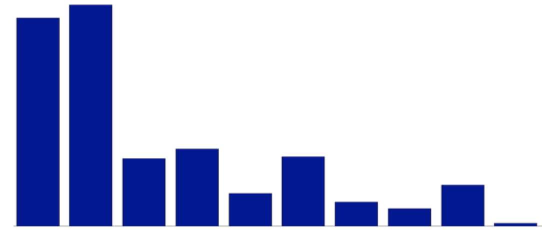


Most Probable



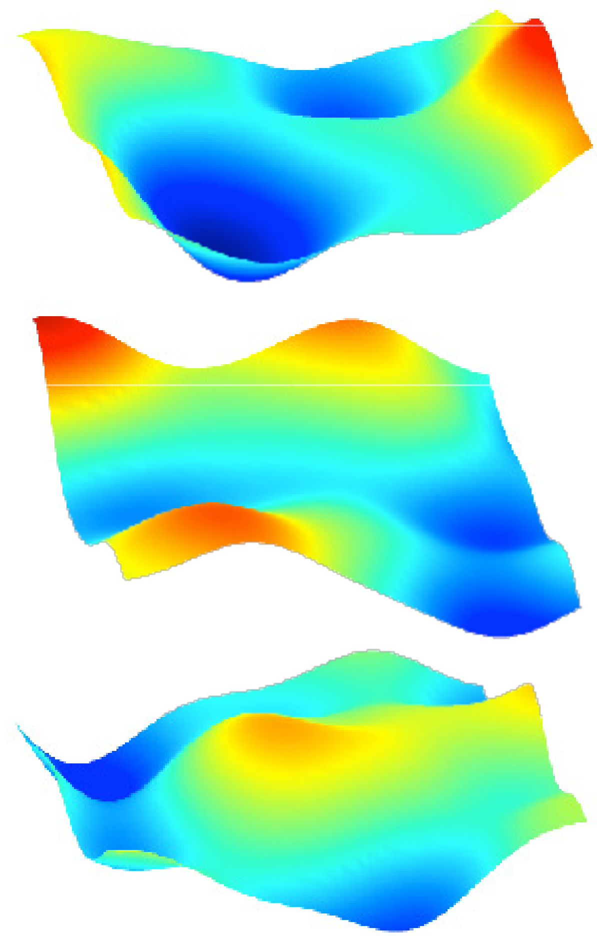
Spatial PY Segmentations

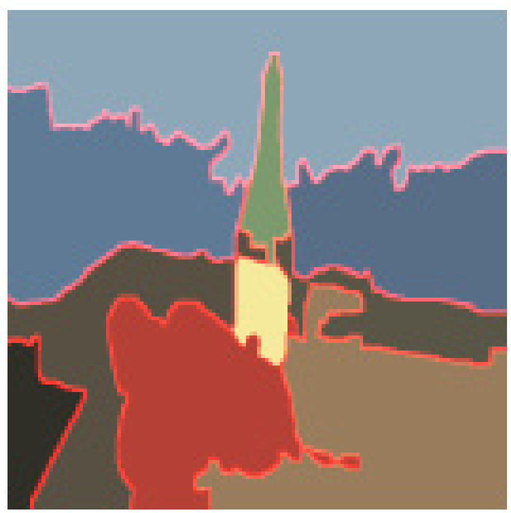
Conclusions



Spatial Pitman-Yor Processes allow…

- efficient variational *parsing* of scenes into unknown numbers of segments
- empirically justified *power law* priors
- accurate learning of non-local spatial statistics of natural scenes
- promise in other application domains…





Conclusions

…but bravery is required

- Conventional MCMC & variational learning prone to local optima, hard to scale to large datasets.
  But better methods on the way!
- Literature remains fairly technical.
  But growing number of tutorials!