Möbius Transformations, Power Diagrams, Lombardi Drawings, and Soap Bubbles

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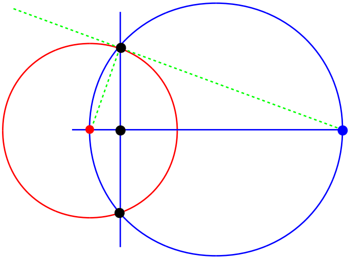
Outline

We use three-dimensional hyperbolic geometry to define a novel type of *power diagram* for disks in the plane that is invariant under Möbius transformations

Using this diagram and the circle packing theorem, we show that every planar graph of degree at most three (and some of degree four) has a planar *Lombardi drawing*

We also use it to characterize the graphs of *planar soap bubble clusters*: they are exactly the bridgeless 3-regular planar graphs

Inversion through a circle



Transformation of the plane (extended by one point at infinity) defined by a circle (e.g. the red circle in the image)

Each point moves to another point on same ray from circle center product of old and new distances from center = squared radius

E.g. exterior blue point moves to interior black point, vice versa

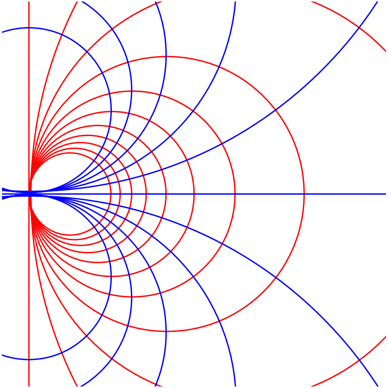
Möbius transformations

Products of inversions with multiple (different) circles

Conformal (preserve angles between curves that meet at a point)

Preserve circularity (counting lines as infinite-radius circles)

Include all translations, rotations, congruences, and similarities



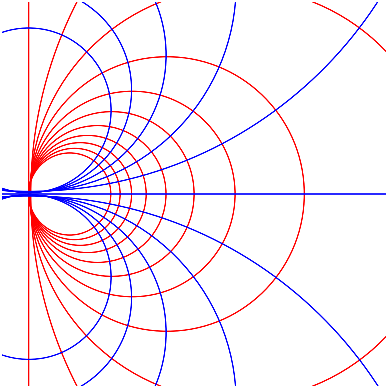
Möbius transformations

If we represent each point in the plane by a complex number, the Möbius transformations are exactly the fractional linear transformations

$$z \mapsto \frac{az+b}{cz+d}$$

and their complex conjugates,

where a, b, c, and d are complex numbers with $ad - bc \neq 0$

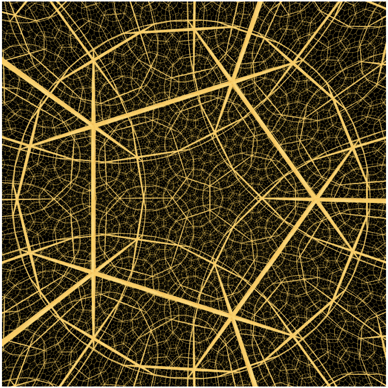


Shadows of higher dimensional transformations

3d hyperbolic geometry can be modeled as a Euclidean halfspace

Hyperbolic lines and planes are modeled as semicircles and hemispheres perpendicular to the boundary plane of the halfspace

In this model, congruences of hyperbolic space correspond one-for-one with Möbius transformations of the boundary plane



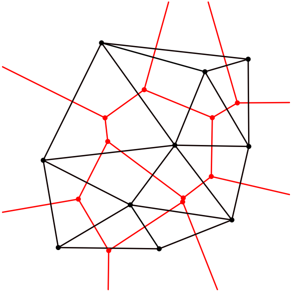
PD image "Hyperbolic orthogonal dodecahedral honeycomb.png" by Tomruen from Wikimedia commons

Things defined by circle incidences are Möbius invariant

The Delaunay triangulation has an edge for circle containing two points

DT of transformed point set is (almost) same as transformation of original DT

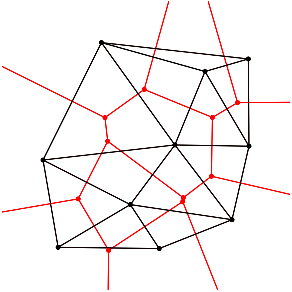
(Caveats: invariant as a topological triangulation, not as straight line segments, and must include farthest point DT together with the usual DT)



Is there a Möbius invariant Voronoi diagram?

Despite the close duality relation between the Delaunay triangulation and the Voronoi diagram, Voronoi is not invariant as a geometric object

For instance, the Voronoi diagram has straight line segment boundaries, whereas its transformed images generally have circular arcs.



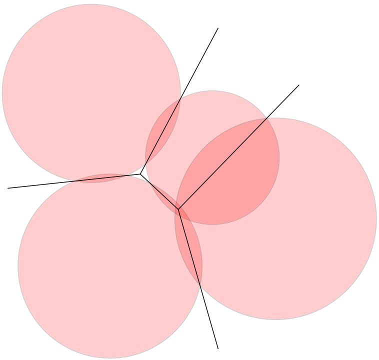
The (classical) power of a circle

For points outside circle, power = (positive) length of tangent segment

For points inside circle, power $= -\frac{1}{2} \times$ length of chord bisected by point

In either case, squared power, multiplied by sign of power, equals $d^2 - r^2$ where d is distance to circle center and r is circle radius

The (classical) power diagram



Minimization diagram of power distance

Bisectors between regions are radical axis lines of circles

Voronoi diagram is special case when all radii are equal

Variant diagrams from hyperbolic geometry

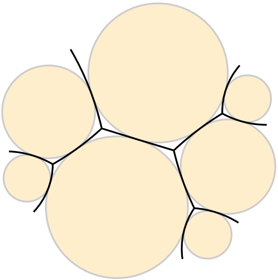
Given non-crossing circles in the plane

View the Euclidean plane as the boundary of a model of 3d hyperbolic geometry

Each circle bounds a hemisphere (modeling a hyperbolic plane)

Construct the 3d hyperbolic Voronoi diagram of these hyperbolic planes

Restrict the Voronoi diagram to the boundary plane of the model

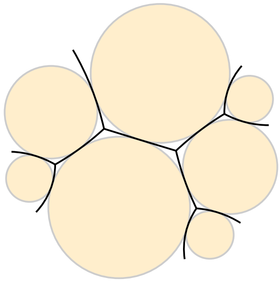


Some properties of the variant diagram

Bisector of disjoint 3d hyperbolic planes is a plane $\Rightarrow$ bisector of disjoint circles is a circle

Voronoi diagram is invariant under hyperbolic congruences $\Rightarrow$ planar diagram is invariant under Möbius transformations

Three tangent circles can be transformed to equal radii $\Rightarrow$ their diagram is a *double bubble* (three circular arcs meeting at angles of $2\pi/3$ at the two isodynamic points of the triangle of tangent points)



What about circles that cross each other?

Möbius transformations can exchange inside and outside
So we need input to be *disks* and *disk complements*, not circles

Each disk is the boundary of a hyperbolic halfspace
(one side of a hyperbolic plane)

Define *signed distance* to a halfspace (in hyperbolic space) as positive distance to plane for points outside the halfspace, negative distance for points inside

Bisectors of signed distance are still planes
$\Rightarrow$ planar diagram still has same properties

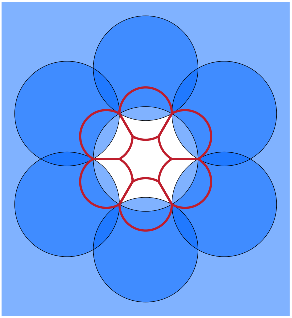
Connectivity of regions in the diagram

In 3d hyperbolic geometry, regions are convex (intersections of halfspaces)

But their restriction to the Euclidean plane can be disconnected

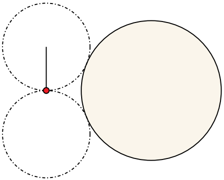
e.g. figure shows six disks and one disk complement forming eight regions

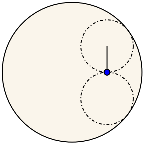
Same phenomenon can happen for non-crossing disks
(e.g. one large disk ringed by many small ones)



Is this a Voronoi diagram? For what distance?

Radial power distance:



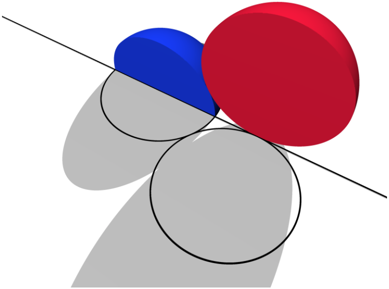


For points outside circle, power = (positive) radius of equal circles tangent to each other at point and tangent to circle

For points inside circle, power = negative radius of equal circles tangent to each other at point and tangent to circle

In either case, it has the formula $\frac{d^2 - r^2}{2r}$

Radial power is not Möbius-invariant; why does it work?



For points in (Euclidean or hyperbolic) 3d space, nearest neighbor
= point that touches smallest concentric sphere

For boundary points of hyperbolic space, replace concentric spheres
by *horospheres* (Euclidean spheres tangent to boundary plane)

Tangent circles for radial power = cross-sections of horospheres

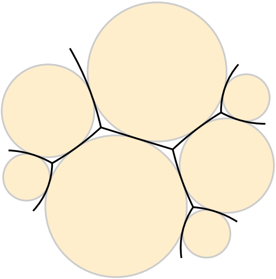
Some local art

Hugo Demartini
Untitled, undated
Kampa Museum

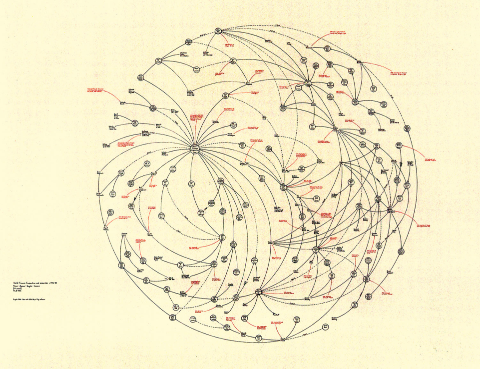
Conclusions (so far):

Minimization diagram of radial power provides a Möbius-invariant decomposition of the plane analogous to the classical power diagram

Its boundary curves are circular arcs and for triples of tangent circles they meet at equal angles



Mark Lombardi



World Finance Corporation and Associates, ca 1970–84: Miami, Ajman, and Bogota–Caracas (Brigada 2506: Cuban Anti-Castro Bay of Pigs Veteran), 7th version, Mark Lombardi, 1999, from *Mark Lombardi: Global Networks*, Independent Curators, 2003, p. 71

American neo-conceptual fine artist (1951–2000)

"Narrative structures", drawings of social networks relating to international conspiracies, based on newspapers and legal documents

Unlike much graph drawing research, used curved arcs instead of polylines

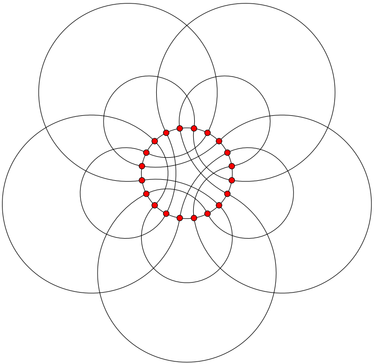
Lombardi Drawing

A style of graph drawing inspired by Lombardi's art

[Duncan, E, Goodrich, Kobourov, & Nöllenburg, *Graph Drawing* 2010]

Edges drawn as circular arcs

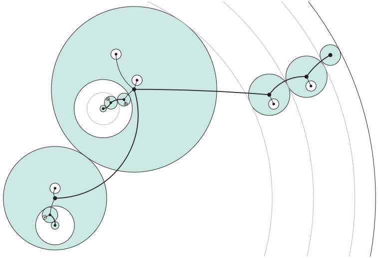
Edges must be equally spaced around each vertex



The Folkman Graph

Smallest edge-transitive but not vertex-transitive graph

Past results from Lombardi drawing



All plane trees (with ordered children) may be drawn with perfect angular resolution and polynomial area [Duncan et al, *GD* 2010]

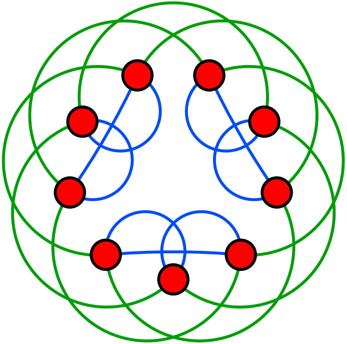
(Straight line drawings may require exponential area)

Past results from Lombardi drawing

k-Regular graphs have drawings with circular vertex placement if and only if

- $k = 0 \pmod 4$,
- k is odd and the graph has a perfect matching,
- the graph has a bipartite 2-regular subgraph, or
- there is a Hamiltonian cycle

[Duncan et al, *GD* 2010]

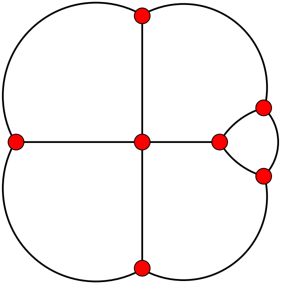


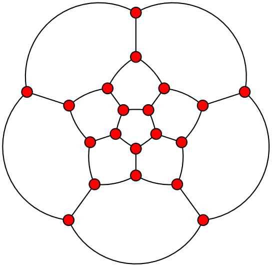
The 9-vertex Paley graph

Past results from Lombardi drawing

Halin graphs and the graphs of symmetric polyhedra
have planar Lombardi drawings

[Duncan et al, *GD* 2010]

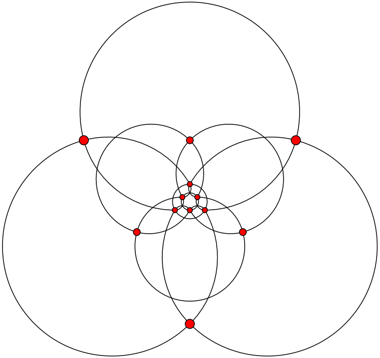


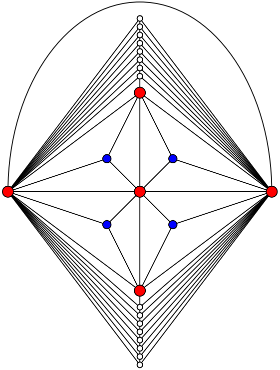


Past results from Lombardi drawing

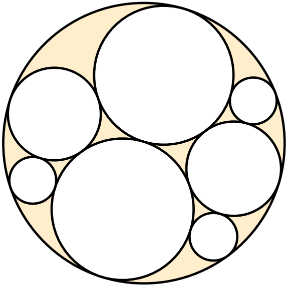
Not every planar graph has a planar Lombardi drawing

[Duncan et al, *GD* 2010; Duncan, E, Goodrich, Kobourov, Löffler, *GD* 2011]

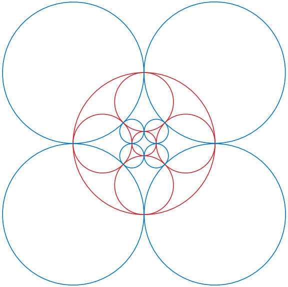




Koebe–Andreev–Thurston circle packing theorems



The vertices of every maximal planar graph may be represented by interior-disjoint circles such that vertices are adjacent iff circles are tangent



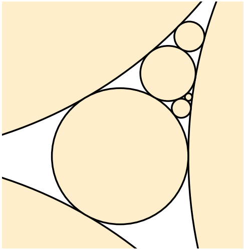
The vertices of every 3-connected planar graph and its dual may be represented by circles that are perpendicular for incident vertex-face pairs

Both representations are unique up to Möbius transformations

More local art

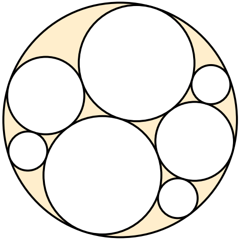
Čestmír Suška
Chmýří / Fuzz, 2007
Kampa Museum

Lombardi drawing for 3-connected 3-regular planar graphs



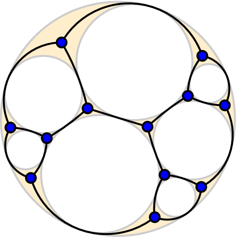
Find a circle packing for the dual (a maximal planar graph)

[Mohar, *Disc. Math.* 1993; Collins, Stephenson, *CGTA* 2003]



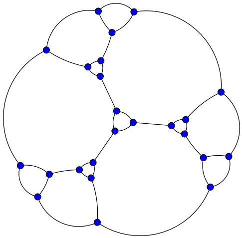
Use a Möbius transformation to make one circle exterior, maximize smallest radius

[Bern, E, *WADS* 2001]



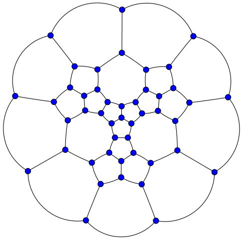
The Möbius-invariant power diagram is a Lombardi drawing of the original graph

Examples of 3-connected planar Lombardi drawings



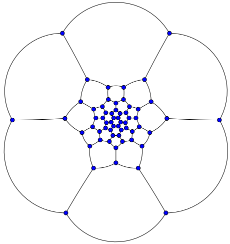
Smallest power-of-two cycle has length 16

[Markström, *Cong. Num.* 2004]



Non-Hamiltonian cyclically 5-connected graph

[Grinberg, *Latvian Math. Yearbook* 1968]



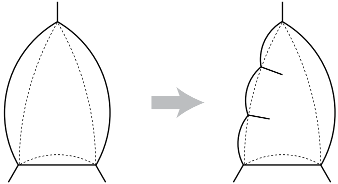
Buckyball (truncated icosahedron)

Lombardi drawing for arbitrary planar graphs of degree ≤ 3

For 2-connected graphs, decompose using an SPQR tree, and use Möbius transformations to glue together the pieces

For graphs with bridges:

- Split into 2-connected subgraphs by cutting each bridge
- Use SPQR trees to decompose into 3-connected components
- Modify 3-connected drawings to make attachments for bridges



- Möbius transform and glue back together

Lombardi drawing for (some) 4-regular planar graphs

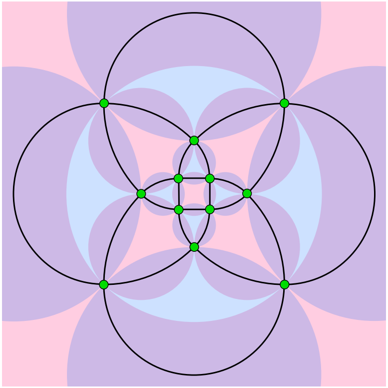
Two-color the faces of the graph G

Construct the incidence graph H of one color class

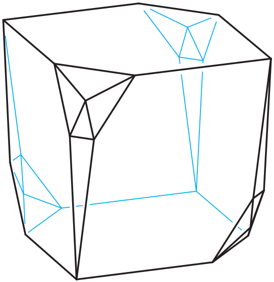
If H is 3-connected, then:

Find an orthogonal circle packing of H and its dual

The Möbius-invariant power diagram is a Lombardi drawing of G

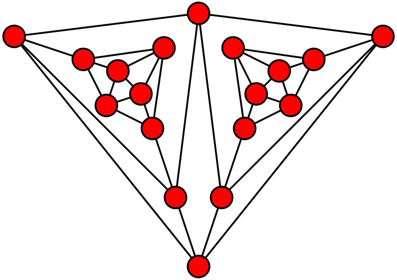


But it doesn't work for all 4-regular graphs



A 3-connected 4-regular graph for which H is not 3-connected

[Dillencourt, E, *Elect. Geom. Models* 2003]



A 2-connected 4-regular planar graph with no planar Lombardi drawing

Summary of new Lombardi drawing results

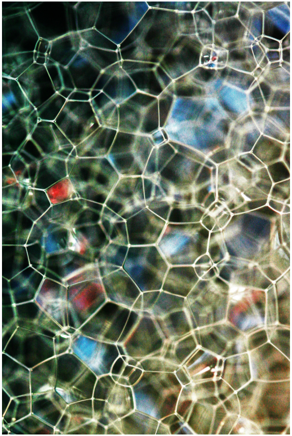
Every planar graph of maximum degree ≤ 3
has a planar Lombardi drawing

Runtime depends on numerics of circle packing
but implemented for the 3-connected case

4-regular medial graphs of 3-connected planar graphs
have planar Lombardi drawings

But other 4-regular planar graphs
may not have a planar Lombardi drawing

Soap bubbles and soap bubble foams



Soap molecules form double layers separating thin films of water from pockets of air

A familiar physical system that produces complicated arrangements of curved surfaces, edges, and vertices

What can we say about the combinatorics of these structures?

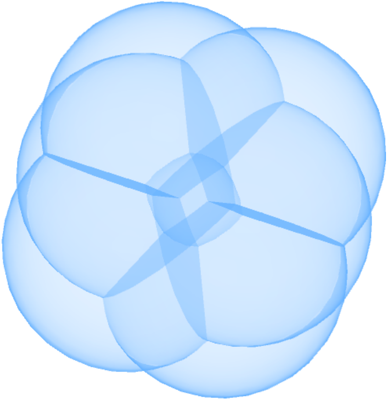
Plateau's laws

In any soap bubble cluster:

- Each surface has constant mean curvature
- Each curve bounds three surfaces with $2\pi/3$ dihedral angles
- Each vertex is the endpoint of four curves with angles of $\cos^{-1}(-1/3)$

Observed in 19th c. by Joseph Plateau

Proved in 1976 by Jean Taylor



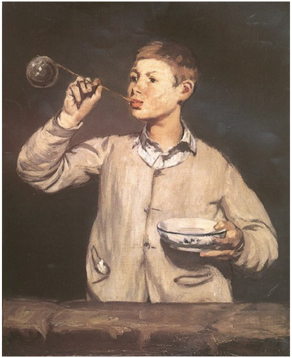
Stereographic projection of a 4-dimensional hypercube, visualized using Jenn

Young–Laplace equation

For any surface in a soap bubble cluster:

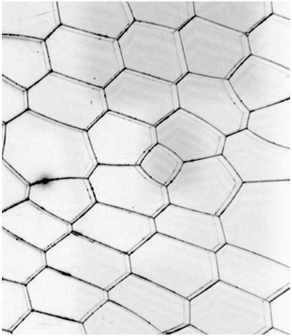
Its mean curvature (the average of the two principal curvatures) is proportional to the pressure difference between the volumes it separates

Formulated in 19th c., by Thomas Young and Pierre-Simon Laplace



Édouard Manet, *Les Bulles de savon*, 1867

Planar soap bubbles



PD image "2-dimensional foam (colors inverted).jpg" by Klaus-Dieter Keller from Wikimedia commons

3d is too complicated, let's restrict to two dimensions

Equivalently, form 3d bubbles between parallel glass plates

Bubble surfaces are perpendicular to plates, so principal curvature in that direction is zero

The soap bubble computer

Soap film connecting pins between the plates forms a minimal Steiner tree (approximate solution to an NP-hard problem)

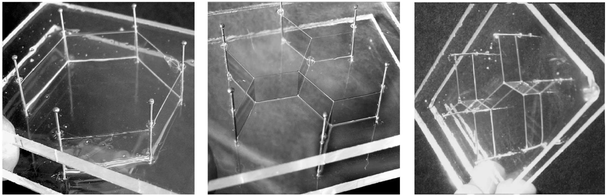


Image from Dutta, Khastgir, and Roy, *Amer. J. Phys.* 2010; see also Isenberg, *Amer. Scientist* 1976; Hoffman, *Math. Teacher* 1979; Aaronson, *SIGACT News* 2005; etc.

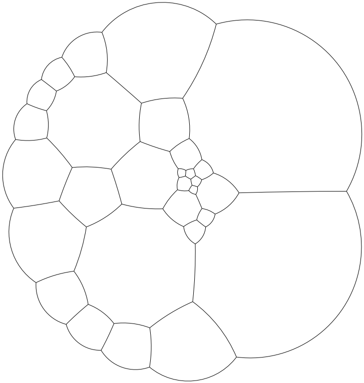
However, we are interested in free-standing bubbles without pins

They can be described by planar graphs, but which graphs?

Plateau and Young–Laplace for planar bubbles

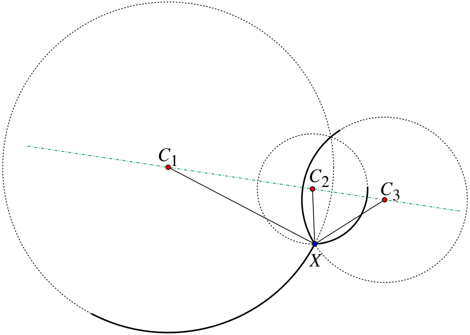
In any planar soap bubble cluster:

- Each curve is an arc of a circle or a line segment
- Each vertex is the endpoint of three curves with angles of $2\pi/3$
- It is possible to assign pressures to the bubbles so that curvature is inversely proportional to pressure difference



Local-global principle for planar bubbles

Existence of a global pressure assignment is equivalent to requiring that at each vertex the (signed) curvatures sum to zero



Zero-curvature condition is equivalent to requiring three incident arcs to be part of a *standard double bubble* (three collinear centers forming two circles with three triple crossings)

Möbius transformation of bubbles

CC-NC photograph "Funhouse Anna at the Museum of Science & Industry" by The Shifted Librarian from Flickr

Möbius transformation is not physically meaningful: it does not preserve volume, surface area, inside-outside relations, etc.

Nevertheless, Möbius transformation of a planar soap bubble cluster is another planar soap bubble cluster

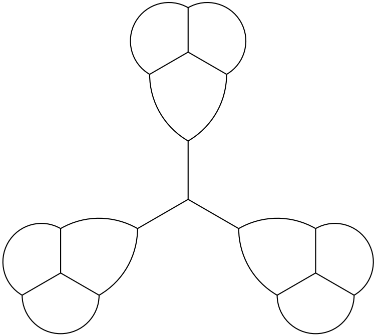
Why? Because it preserves circularity of arcs, $2\pi/3$ angles between arcs, and the triple crossing property in the local characterization

Planar soap bubbles are bridgeless

Proof idea:

If the bridge is a circular arc, it would violate Young–Laplace

If the bridge is a straight line segment, it can be made into an arc by a Möbius transformation, after which it would violate Young–Laplace



This is not a soap bubble

Bridgeless planar graphs are soap bubbles

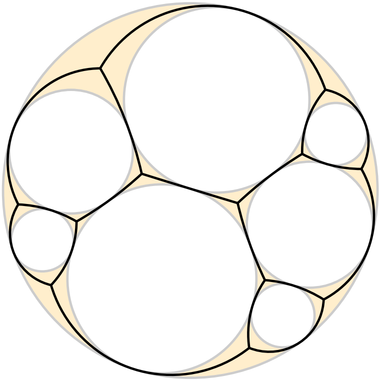
Proof idea:

Use power diagrams and SPQR trees to find a Lombardi drawing

Each three tangent circles can be transformed to equilateral position

So the edges at each junction can be transformed to straight line segments

Therefore, it meets the local curvature conditions on soap bubbles



Conclusions

We have defined a new Möbius-invariant power diagram of disks

Using it, we show that every planar graph of degree at most three has a planar Lombardi drawing

and that planar soap bubble graphs are exactly bridgeless 3-regular planar graphs

Future work?

Complexity and algorithms for invariant power diagrams?

(Circle packings are an easy linear special case; disconnected regions make general case messier.)

Future work?

For which other graphs does circle packing + power diagram give Lombardi drawings?

(Can check numerically by constructing a drawing and checking whether it works, using uniqueness of circle packings, but can we do it combinatorially?)

CC-SA image "world of soap" by Martin Fisch on Flickr

Future work?

How stable are our soap bubbles? Can they ever be in an unstable equilibrium?

Can this be extended to any interesting classes of 3d bubbles?