Discrete Mathematics for Computer Science

Lecture 11

Permutations and Combinations (4.3)

Introduction

- Counting problems:
- à Find the number of elements of a set given a description of this set
- Counting techniques
  - Two lectures ago:
    - Product rule
    - Sum rule
    - Inclusion-Exclusion principle
  - Today:
    - Permutations
    - Combinations

Outline

- Permutations
- Combinations
- Properties of combinations

Permutations

- Given a set A, a permutation of A is an ordered arrangement of its elements
- Example
  - A={a,b,c}
  - Permutations: (a,b,c) , (a,c,b) , (b,a,c) , (b,c,a) , (c,a,b) , (c,b,a)

Permutations

- How many permutations for a finite set A?
- We have to choose
  - n=|A| elements one after each other
  - Each element is different

à Use the product rule:
  - First choice n possibilities
  - Second choice, (n-1) possibilities
  - …
- Total possibilities: nx(n-1)x…x2x1 = n!

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r-permutations

- Given a set A, a r-permutation of A is an ordered arrangement of size r of its elements
- Example
  - A={a,b,c,d}
  - 4-permutations are permutations: (a,b,c,d) , (a,b,d,c) , …
    - 4x3x2x1=24 4-permutations
  - 3-permutations: (a,b,c) , (a,c,b) , (d,b,c) , …
    - 4x3x2=24 3-permutations
  - 2-permutations: (a,b) , (a,d) , (d,a) , …
    - 4x3=12 2-permutations
  - 1-permutations: (a) , (b) , (c) and (d)
    - 4 1-permutations

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r-permutations

- How many r-permutations for a finite set A with |A|=n?

choice	1st	2nd	3rd	…	rth
Nb possibilities	n	n-1	n-2		n-(r-1) =n-r+1

- Total number of possibilities à apply the product rule

$$P(n,r) = n\times(n-1)\times(n-2)\times\ldots\times(n-r+1)$$

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r-permutations

- Compact expression:

$$P(n,r) = n\times(n-1)\times(n-2)\times\ldots\times(n-r+1)$$

$$= \frac{n\times(n-1)\times(n-2)\times\ldots\times(n-r+1)\times(n-r)\times(n-r-1)\times\ldots\times2\times1}{(n-r)\times(n-r-1)\times\ldots\times2\times1}$$

$$P(n,r) = \frac{n!}{(n-r)!}$$

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r-permutations

- When use r-permutations?
- Choose r elements in A
  - The order of the choice is relevant
  - Every choice has to be different
- Number of possible choices:

  P(|A| , r)

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Example (2 page 321)

- How many ways are there to select 1st 2nd and 3rd price winners among 100 people?

à We have to choose 3 among 100, the order of the choice being important
  - (Eva,John,Bob) is different from (Bob,Eva,John)

à Total number of possibilities

P(100,3) = 100x99x98 = 970,200

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Outline

- Permutations
- <u>Combinations</u>
- Properties of combinations

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r-combinations

- r-combination of a set A à subset of A with r elements
- Example
  - Deck of cards: A={1,2,3,…,Q,K}
  - A poker hand: subset of A with 5 elements

    {1,2,3,4,5} = {5,4,3,2,1}

  à How many different poker hands?
- Difference with r-permutations: the order is irrelevant for r-combinations

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Counting r-combinations

- How to count r-combinations?
- Idea:
  - Count r-permutations
  - Put together all r-permutations that contain the same elements
- Example: how many 2-combinations of {a,b,c,d}?
  - P(4,2)=4x3=12 2-permutations
  - Each 2-combination correspond to two 2-permutations (e.g. {a,b} à (a,b) and (b,a))
  - P(4,2)/2=6 2-combinations

Counting r-combinations

- General case: r-combinations of a set containing n elements
- Count r-permutations à P(n,r)
- How many r-permutations for one r-combination?
  - For each r-combination $\{a_1,a_2,\ldots,a_r\}$

  à $(a_1,a_2,\ldots,a_r)$, $(a_1,a_3,\ldots,a_r)$, …
  - Ordered selection of r object among r à P(r,r) = r!
- Total number of r-combinations:

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!}{(n-r)!r!}$$

Example (8 page 324)

- How many possibilities to select 5 players from a 10-member team?
- Select 5 different objects among 10, the order being irrelevant

à 5-combinations

$$C(10,5) = \frac{10!}{(10-5)!5!} = 252$$

Example (10 page 324)

- How many bit strings of length 10 contain exactly 4 1s
- One string correspond to 4 different numbers between 1 and 10
  - 1110000001 corresponds to {1,2,3,10}
  - The order is not relevant
- Finally, $C(10,4) = \frac{10!}{(10-4)!4!} = 210$ possibilities

Outline

- Permutations
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Property of r-combination

- Theorem:
  Let r and n be such that r≤n. Then
  C(n,r)=C(n,n-r)
- Proof:

$$C(n,r) = \frac{n!}{(n-r)!r!}$$

$$C(n,n-r) = \frac{n!}{(n-(n-r))!(n-r)!} = \frac{n!}{r!(n-r)!}$$

Combinatorial proof

- Proof that consist of counting the same number of objects in two different ways
- C(n,r) = the number of subset with r elements of a set A, with |A|=n
- To each subset E of A, we can associate the complement of E

à There are the same number of subsets and complements of these subsets

- Complements have n-r elements
- Number of complements: C(n,n-r)
- Finally, C(n,r)=C(n,n-r)

Summary

- Select r different objects in a set that contain n objects
- The order is relevant
  - r-permutation

$$P(n,r) = \frac{n!}{(n-r)!}$$

- The order is not relevant
  - r-combination

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!}{(n-r)!r!}$$

Conclusion

- We have seen new techniques to solve counting problems:
  - r-permutations
  - r-combinations
- We have pointed out a simple property of r-combinations
- Next lecture: several properties and applications of r-combinations

Extra

- Page 324-325 exercises 11,15,19,23