How I entered Constraints: Some of the Early Milestones

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Mechanical Heuristic generation

**Observation:** People generate heuristics by consulting simplified/relaxed models.

**Context:** Heuristic search (A*) of state-space graph (Nilsson, 1980)

**Context:** Weak methods vs. strong methods

**Domain knowledge:** Heuristic function

\[ h(n) \text{: Heuristic underestimate the best cost from } n \text{ to the solution} \]
THE OPTIMALITY OF A* REVISITED

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ABSTRACT

This paper examines the optimality of A*, in the sense of expanding the least number of distinct nodes, over three classes of algorithms which return solutions of comparable costs to that found by A*. We first show that A* is optimal over those algorithms guaranteed to find a solution at least as good as A*’s for every heuristic assignment A. Second, we consider a wider class of algorithms which, like A*, are guaranteed to find an optimal solution (i.e., admissible) if all cost estimates are optimistic (i.e., A < h). On this class we show that A* is not optimal and that no optimal algorithm exists unless h is also consistent, in which case A* is optimal. Finally we show that A* is optimal over the subclass of best-first algorithms which are admissible whenever h < A.

I. INTRODUCTION AND PRELIMINARIES

1.1 A* and Informed Best-First Strategies

Of all search strategies used in problem solving, one of the most popular methods of exploiting heuristic information to cut down search time is the informed best-first strategy. The general philosophy of this strategy is to use the heuristic information to assess the “merit” of each candidate solution, as a way to order the direction of the search. Formal descriptions of this strategy are usually given in the context of path searching problems, a formulation which represents many combinatorial problems which have heuristic guidance (e.g., search, scheduling, speech recognition, scene analysis, and others). Given a weighted directed graph G with a distinguished start node s and a set of goal nodes t, the optimal path problem is to find a path from s to t with the minimum cost. The path may, in general, be an arbitrary function of the weights assigned to the nodes and branches along the path.

By far, the most studied version of informed best-first strategies is the algorithm A* [Hart, Nilsson and Raphael, 1968] which was developed for additive cost measures, i.e., where the cost of a path is the sum of the costs of the arcs along the path. A* employs a special additive form of the evaluation function f which makes up the current cost of the solution. A heuristic estimate of the cost of the path remaining between a and some goal node. A* constructs a tree T of selected paths of G using the elementary operation of node expansion, i.e., generating all successors of a given node. The algorithm selects for expansion the leaf node of T which has the lowest f value, and only maintains the lowest-k path to any given node. The search halts as soon as the path to a node selected for expansion is found to satisfy the goal conditions. It is known that if h(n) is a lower bound, then A* is guaranteed to find the optimal path.

1.2 Previous Works

The optimality of A* in the sense of expanding the least number of distinct nodes, has been a subject of some confusion. The well-known property of A* which predicts that decreasing errors h < A can only improve its performance [Nilsson, 1980, rework A] has often been interpreted to reflect some supremacy of A* over other search algorithms which rely on the heuristic information. Consequently, several authors have assumed that A*’s optimality is an established fact (e.g., Nilsson, 1971, Martelli, 1978, Mero, 1981, Barr and Feigenbaum, 1982, 1985). In fact, all this property says is that some A* algorithms are better than other A* algorithms depending on the heuristics which guide them. It does not indicate whether the heuristic h is better than other ways of combining h and g (e.g., h + g + h², h + g²), neither does it assure us that expansion policies based only on h can do as well as more sophisticated best-first policies using the entire information gathered along the path (e.g., h(n) = max_{p(i|n)} f(i|n), where p(i|n) is the path to n).

Instead of using the entire information gathered along the path, A* only uses the current cost of the path. This makes the algorithm very simple and efficient, but also forbids it from processing common information in a better way than A does.
The Simplified models Paradigm

Pearl 1983 (On the discovery and generation of certain Heuristics, 1983, AI Magazine, 22-23) : “knowledge about easy problems could serve as a heuristic in the solution of difficult problems, i.e., that it should be possible to manipulate the representation of a difficult problem until it is approximated by an easy one, solve the easy problem, and then use the solution to guide the search process in the original problem.”

The implementation of this scheme requires three major steps:
  a) simplification,
  b) solution, and
  c) advice generation.

Simplified = relaxed is appealing because:
  1. implies admissibility, monotonicity,
  2. explains many human-generated heuristics (15-puzzle, traveling salesperson)

“We must have a simple a-priori criterion for deciding when a problem lends itself to easy solution.”
Systematic relaxation of STRIPS

STRIPS (Stanford Research Institute Problem Solver, Nillson and Fikes 1971) action representation:

Move(x,c1,c2)
Precond list: on(x1,c1), clear(c2), adj(c1,c2)
    Add-list: on(x1,c2), clear(c1)
    Delete-list: on(x1,c1), clear(c2)

Relaxation (Sacerdoti, 1974): Remove literals from the precondition-list:
1. clear(c2), adj(c2,c3) → #misplaced tiles
2. Remove clear(c2) → manhattan distance
3. Remove adj(c2,c3) → h3, a new procedure that transfer to the empty location a tile appearing there in the goal

But the main question remained:
“Can a program tell an easy problem from a hard one without actually solving?” (Pearl 1984, Heuristics)
Easy = Greedily solved?

Pearl, 84: Most easy problems we encounter are solved by “greedy” hill-climbing methods without backtracking” and that the features that make them amenable to such methods is their “decomposability”

The question now:
Can we recognize a greedily solved STRIPS problem?”
On the greedy solution of ordering/scheduling problems

**Job-shop:** minimizing weighted average flow-time on a single processor

\[ C(1,2,\ldots,n) = \sum_{i=1}^{n} q_i \sum_{j=1}^{i} p_j \]

**Spanish treasure problem:** An unknown number of chests of Spanish treasure have been buried on a random basis in \( n \) sites. For each site there is a probability \( p_i \) that the chest is there and the cost of excavating a site is \( q_i \). Find a sequence of excavations that will minimize the average cost of finding the first chest.

**Greedy strategy:**

\[ \frac{q_i}{p_i} \]

So, the question now: Can we characterize when does an ordering problem has a **ranking function**, or a **greedy rule**, (not necessarily using the cost function) that yields an optimal solution?

On the greedy solution…
(continued)

**Theorem:** If $P$ is any greedily optimized problem then an optimizing ranking function $f$ has to agree with the ordering dictated by the cost function on pairs of elements. Namely for every two elements $a$ and $b$

$C(a,b) > C(b,a)$ iff $f(a) > f(b)$.

\[
C(\sigma) - C(\sigma') = (u_{i+1} p_i - u_i p_{i+1})
= u_{i+1} u_i \left( \frac{p_i}{u_i} - \frac{p_{i+1}}{u_{i+1}} \right).
\] (8)

The theory explained all known greedy rules for ordering problems.

**Conclusion:** “The paper provides necessary and sufficient conditions for a problem to be greedily optimized by a uniform ranking function. The virtue of these conditions is that they are easy to test and thus may be useful in mechanizing the process of generating greedy strategies by computers.”
On the Greedy Solution of Ordering Problems

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The greedy method is a well-known approach for problem solving directed mainly at the solution of optimization problems. Leading theoretical frameworks dealing with the optimality of greedy solutions (e.g., the matroid and greedoid theories) tacitly assume that the greedy algorithm is always guided by the cost function to be optimized. Namely, it builds a solution by adding, in each step, an element that contributes the most to the value of the cost function. This paper studies a class of problems for which this type of a greedy algorithm does not optimize the given cost function, but for which there exists a secondary objective function, called a greedy rule, such that applying the greedy algorithm to the secondary objective function yields a solution which is optimal with respect to the original cost function.

This paper is concerned exclusively with ordering problems, involving a set of elements and a cost function defined on all permutations of the elements, where the task is to order the elements so as to maximize (or minimize) the value of the cost function. Job sequencing on a single machine and the traveling salesman problems are two examples of this class of problems.

A theoretical framework, called greedoid theory, which characterizes a class of ordering problems that can be solved optimally by greedy algorithms, is due to Korte and Lovasz. The greedoid structure is a generalization of the matroid structure which provides a theoretical foundation for the optimality of the greedy algorithm on selection problems. (In contrast with ordering problems, selection problems involve a set of elements and a cost function defined on all unordered subsets of elements, where the task is to select a subset of elements which satisfies some property, so as to maximize (or minimize) the value of the cost function. The minimum weight spanning tree problem is a well-known example of this class of problems. For further details on matroids refer, for example, to Lawler or Welsh.)

The greedoid theory (as well as the matroid theory) considers only greedy algorithms that use the cost function to be optimized as their selection criterion, namely, which build the solution by adding, at each step, that element which results in maximum improvement in the value of that cost function. The appendix to this paper lists a number of known ordering problems for which this greedy algorithm does not optimize the cost function, but for which there exists a secondary objective function, which we call a greedy rule, such that.

Subject classification: Analysis of algorithms; optimality of greedy algorithms.
Freuder, JACM 1982: “A sufficient condition for backtrack-free search”

Whow! Backtrack-free is greedy!
Got absorbed…

**Sufficient condition (Freuder 82):**
1. Trees (width-1) and arc-consistency implies backtrack-free
2. Width=i and (i+1)-consistency implies backtrack-free search

If 3-consistent
no deadends

Else, impose consistency, but it add arcs except for trees. So trees are easy.
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Dechter and Pearl, 1985:
if a problem has \textit{induced-width} \(i\) it can be solved by \textit{directional} \(i\)-consistent

\textbf{Led to:} 
\begin{itemize}
  \item Directional-consistency,
  \item Adaptive-consistency,
  \item Join-tree clustering and treewidth
  \item Optimization (dynamic programming),
  \item Counting, all solved by a single Algorithm, with induced-width complexity.
  \item Later, generalized to bucket-elimination for probabilistic reasoning, Later to cycle-cutset.
\end{itemize}

It is all based on generalizing \textbf{trees}:
(Macworth, Freuder, 85, Pearl 83)
Back to Automatic generation of heuristics... for CSPs

I did not abandon the general goal of heuristic generation. Just shifted to CSP where heuristic indicate existence of a solution. Or, alternatively, how many solutions are below a given node. Since trees are easy for counting, I relaxed the problem into a tree and... count (Dechter, 1985).

Results in 1985
On random 15 vars, 5 vals:

We revisited this idea with Kask, Gogate and Dechter (CP, 2004) estimated counts using GBP/IJGP.
From Then On…
(just tried to understand what was going on around me)

Backjumping and no-good learning (1987-88)
(Wanted to understand TMS and Logic programming)

Sat-based Nonmon-reasoning (with Ben-Eliyahu, 1990)
(answer-set programming)
(Wanted to understand default logic, logic programming)

Temporal constraint networks (with Meiri and Pearl, 1988-90)
(Understanding what Dean and Macdermoth and James Allen were doing)

Distributed constraints (with Collin and Katz, 1990)
Neural networks hyped up again.

On the expressiveness of networks with hidden variables (Dechter 1990),
from local to global consistency (Dechter 1992)
Neural networks (will explain)

Identifiability of structures (trees) from relations (with Meiri and Pearl, 1990)
Learnability / PAC learning.

Bucket-elimination (Dechter, 96) (bringing treewidth/induced-width
to Bayesian networks)
Understanding probabilistic reasoning through VE

Mini-buckets, (with Rish 1997) finally Generating heuristics for real
(with Kask, Marinescu, 2001, 2004)

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ON THE EXPRESSIVENESS OF NETWORKS WITH HIDDEN VARIABLES

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Abstract

This paper investigates design issues associated with representing relations in binary networks augmented with hidden variables. The trade-off between the number of variables required and the size of their domains is discussed. We show that if the number of values available to each variable is just two, then hidden variables cannot improve the expressional power of the network, regardless of their number. However, for $k \geq 3$, we can always find a layered network using $k$-valued hidden variables that represent an arbitrary relation. We then provide a scheme for decomposing an arbitrary relation, $p$, using $\frac{1}{k-2}$ hidden variables, each having $k$ values ($k > 2$).

1. Introduction

Hidden units play a central role in connectionist models, without which the model would not represent many useful functions and relations. In the early days of the Perceptrons [Minsky 1969] it was noted that even simple functions like the XOR were not expressible in a single layer perceptron; a realization that slowed research in the area until the notion of hidden units had emerged [Rumelhart 1988a, Hinton 1988]. Nevertheless, a formal treatment of the expressiveness gained by hidden units, and systematic schemes for designing systems with hidden units within the neural network paradigm are still not available.

Our intention is to investigate formally the role of hidden units and devise systematic schemes for designing systems incorporating hidden units. Specifically, we address the following task: given a relation on $n$ variables, called visible, we wish to design a network having $n+k$ units whose stable patterns, (relative to the visible units) coincide with the original relation. This task is central to most applications of connectionist networks, in particular to its role as associative memory. The task will be investigated for a connectionist architecture which is different from classic connectionist networks in that it is based on constraint networks. The sequential constraint network model is defined next.

A Network of binary constraints involves a set of $n$ variables $X_1,\ldots,X_n$, each represented by its domain values, $D_1,\ldots,D_n$, and a set of constraints. A binary constraint $R_{ij}$ between two variables $X_i$ and $X_j$ is a subset of the cartesian product $D_i \times D_j$ that specifies which values of the variables are compatible with each other. A solution is an assignment of values to all the variables which satisfy all the constraints, and the constraint satisfaction problem (CSP) associated with these networks is to find one or all solutions. A binary CSP can be associated with a constraint-graph in which nodes represent variables and arcs connect pairs of variables which are constrained explicitly. Figure 1a presents a constraint network where each node represents a variable having values $(a,b,c)$ and each link is associated with a strict lexicographic order (where $X_i < X_j$ iff $i < j$). (The domains and the constraints explicitly indicated on some of the links.)

![Figure 1: An example of a binary CN](image)

Our constraint-based connectionist architecture assumes that each unit plays the role of a variable having $k$ states, and that the links, representing the constraints, are quantified by compatibility relations between states of adjacent units. Each unit asynchronously updates its state.

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On the expressiveness of networks with hidden variables

Can a relation be expressed by a binary constraint networks with hidden variables?

Yes. If no limit on number of values

And, with limit?

\[ U_5 = \begin{pmatrix}
X_1 & X_2 & X_3 & X_4 & X_5 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \]

With 2 values?
With 3 values?
How many hidden variables?
On the expressiveness of networks with hidden variables

Theorem: Relations which are not binary network decomposable cannot be binary network decomposable by adding any number of bi-valued Hidden variables.
Proof: We want to exclude \((x_1,x_2,x_3)=(0,0,0)\) using a variable \(Y=\{0,1\}\). …

Reason: 3-consistent bi-valued binary networks are globally consistent

Theorem (Dechter, 1992): \(k\)-valued binary networks which are strong \((k+1)\)-consistent are globally consistent.

However, No simple criterion for tractability emerged;
Semantic based tractability: row-convex constraints (van Beek 1995)
A whole major line of work by Jeavons and Cohen (1995-2007)
(Constraint book, chapter 10, 2003)
On the expressiveness of networks with hidden variables

Figure 6: A layered decomposition of $U^*_{12}$
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Structure Identification in Relational Data *

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Another Example

\[ S = \]
\[(\text{relation})\]

\[ T = \{ A \rightarrow B, B \rightarrow C \rightarrow D \} \]
\[(\text{Horn theory})\]
Identifying Tree Structures in Categorical Relations
(Dechter, 1990, Meiri, Dechter, Pearl, 1991)

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- A tree $T$ represents a relation $\rho$ iff

$$\rho = \bigwedge_{(X_i, X_j) \in T} \rho X_i X_j$$

- Convenience: storage, query processing, parallelism.
3.2 Identifying Constraint Trees

Problem

- Given a relation \( \rho \) (e.g., as a table)
- If \( \rho \) has a tree decomposition, **find** one
- If it does not, **acknowledge** and find a **best** approximation.

**Theorem 8 (Dechter, 1987):** Let

\[
n(x_i) = \text{the number of } n\text{-tuples in } \rho \text{ for which } X_i = x_i,
\]

\[
n(x_i, x_j) = \text{number of } n\text{-tuples in } \rho \text{ for which } X_i = x_i \text{ and } X_j = x_j.
\]

The MWST algorithm using the arc-weights:

\[
m(X_i, X_j) = \frac{1}{|\rho|} \sum_{(x_i, x_j) \in \rho} n(x_i, x_j) \log \frac{n(x_i, x_j)}{n(x_i) n(x_j)}
\]

is guaranteed to produce a tree-decomposition to \( \rho \) if such a decomposition exists.

The decomposition is exact iff \( |\rho| = |\rho_T| \)

**Complexity:** \( O[(|\rho| + \log n)n^2] \)

**Best approximation? Open problem**

**Qualitative:** Heiri, Dechter, Pearl (AAAI-80)
Example

n(A=0)=8, n(B=1)=6, n(B=0)=2
n(B=0,C=1)=2, n(B=1,C=1)=3...

m(B,C)=-13.97
m(B,D)=-15.95
m(B,E)=-16.55
m(C,D)=-16.55
m(C,E)=-17.13
m(D,E)=-15.50
m(A,B)=m(A,C)=
m(A,E)=m(A,D)=-16.63
Qualitative tree decomposition
(Meiri, Dechter, Pearl, AAAI-90)

Method: Given minimal network \( M \)

1. Consider all triangles \( t = \{e_1, e_2, e_3\} \)
   - generate labeling on arcs such that if \( e_1 \) is redundant in \( t \)
     \[ w(e_1) \leq w(e_2), w(e_3) \leq w(e_1) \]
   - if \( e_1 \) is redundant and \( e_2 \)
     is not \( w(e_2) < w(e_3) \)
     (called triangle labeling)

2. Find a maximal weight spanning tree \( T \) w.r.t. \( w \).

Definition: \( e_1 \) is redundant in \( t = \{e_1, e_2, e_3\} \)

\[
\begin{align*}
\text{iff} & \quad \text{if } M_{e_1} = M_{e_2} \text{ or } M_{e_3} \\
\end{align*}
\]

3. Test if \( T \) decompose \( M \) \text{ rel}(\text{N})
   if not there is no tree decomposition
Identifying Structure from Relational Data (Dechter, Meiri and Pearl 1988-1991)


A preliminary version PODS pp. 185-189.
In Summary

• Uncovering structure from data and how to exploit hidden variables are still central scientific questions.…

• As to heuristic generation nowadays…
  – Simplification and solution steps combine (e.g., mini-bucket, heuristics for planning, using MDPs…)

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