Problem 1

$R_{xy} = \{(a,b),(c,d),(d,e)\}$, $R_{yz} = \{(b,c),(e,a),(b,d)\}$.

Since the two relations have different scopes, the operations of union and difference can only be applied to them if the relations are viewed as ordinary sets, and not as relations between variables. In that sense, then, we find:

(a) $R_{xy} \cup R_{yz} = \{(a,b),(b,c),(b,d),(c,d),(d,e),(e,a)\}$.

(b) Since $R_{xy} \cap R_{yz} = \emptyset$, it follows that $R_{xy} - R_{yz} = R_{xy} = \{(a,b),(c,d),(d,e)\}$.

(c) The join operation, however, *is* defined for relations on different scopes, and we find that $R_{xy} \bowtie R_{yz} = \{(a,b,c),(a,b,d),(d,e,a)\}$.

(d) Since no pair of tuples in R_{xy} have the same value for their first variable, we obtain $\prod_x R_{xy} = \{(a),(c),(d)\}$.

(e) Since no tuple in the join $R_{xy} \bowtie R_{yz}$ has the value c for its first variable, it follows that $\sigma_{x=c}(R_{xy} \bowtie R_{yz}) = \emptyset$. ■

Problem 2

Since any pair of values present in a binary constraint minimal network must necessarily appear in at least one solution to the network and since the crossword puzzle of Figure 1.2 has no solution, it is the case that the minimal network for a binary constraint representation of the problem must be the empty set. This is also consistent with the fact that the minimal network is the projection network of the set of solutions (I suppose the projection network of the empty set is defined to be the empty set). ■

Problem 3

$\rho(x,y,z,t) = \{(a,a,a,a),(a,b,b,b),(b,b,a,c)\}$.

(a) The projection network is the set of all pairwise projections, $P(\rho) = \{P_{xy}, P_{xz}, P_{xt}, P_{yz}, P_{yt}, P_{zt}\}$, where:

$$\begin{aligned} P_{xy} &= \{(a,a),(a,b),(b,b)\}\,, & P_{xz} &= \{(a,a),(a,b),(b,a)\}\,, \\ P_{xt} &= \{(a,a),(a,b),(b,c)\}\,, & P_{yz} &= \{(a,a),(b,b),(b,a)\}\,, \\ P_{yt} &= \{(a,a),(b,b),(b,c)\}\,, & P_{zt} &= \{(a,a),(b,b),(a,c)\}\,. \end{aligned}$$

The projection network $P(\rho)$ has the same set of solutions as the relation $\rho(x,y,z,t)$; the domain of t is $\{a,b,c\}$ so either $t = a$, or $t = b$, or $t = c$, and each case uniquely resolves the values of the other variables, as follows:

$$\begin{aligned}
t = a \quad &\Rightarrow \quad x = a \text{ (thru } P_{xt}) \text{ and } y = a \text{ (thru } P_{yt}) \text{ and } z = a \text{ (thru } P_{zt}),\\
t = b \quad &\Rightarrow \quad x = a \text{ (thru } P_{xt}) \text{ and } y = b \text{ (thru } P_{yt}) \text{ and } z = b \text{ (thru } P_{zt}),\\
t = c \quad &\Rightarrow \quad x = b \text{ (thru } P_{xt}) \text{ and } y = b \text{ (thru } P_{yt}) \text{ and } z = a \text{ (thru } P_{zt}).
\end{aligned}$$

Since $P(\rho)$ has the same set of solutions as $\rho(x, y, z, t)$, they're equivalent. Hence, $\rho(x, y, z, t)$ does have a binary network representation.

(b) $\prod_{xyz} \rho = \{(a, a, a), (a, b, b), (b, b, a)\}$. Again we need to find the projection network, only this time it is $P(\prod_{xyz} \rho) = \Big\{\{(a, a), (a, b), (b, b)\}, \{(a, a), (a, b), (b, a)\}, \{(a, a), (b, b), (b, a)\}\Big\}$, where the order inside the main brackets is again P_{xy}, P_{xz}, P_{yz}. Since $P(\prod_{xyz} \rho)$ admits the solution $\{(a, b, a)\}$, which is not in the set of solutions of $\prod_{xyz} \rho$, the networks $P(\prod_{xyz} \rho)$ and $\prod_{xyz} \rho$ are *not* equivalent and the answer this time is no; the projection $\prod_{xyz} \rho$ is *not* representable by a network of binary constraints.

(c) (t, z, x, y) is an ordering for which $P(\rho)$ is backtrack–free. ■

Problem 4

Clearly, if $R^{(1)}$ and $R^{(2)}$ are equivalent networks, the result is trivial since equivalent networks necessarily have the same set of solutions, regardless of their relative tightness. So suppose that $R^{(1)}$ and $R^{(2)}$ are not equivalent. If $R^{(1)}$ is tighter than $R^{(2)}$, then to obtain $R^{(1)}$ from $R^{(2)}$ one must add a new constraint to $R^{(2)}$, or tighten one or more of its existing constraints. The result is either irrelevant as far as the solution set of $R^{(2)}$ is concerned, or it's not. If it is irrelevant, that means the solution set of $R^{(2)}$ hasn't changed with the addition/tightening, from which we conclude that $\text{sol}(R^{(1)}) = \text{sol}(R^{(2)})$. If, however, the addition/tightening is relevant to the solution set of $R^{(2)}$, then it can only restrict that solution set further, from which we conclude that $\text{sol}(R^{(1)}) \subset \text{sol}(R^{(2)})$. In either case, then, it follows that $\text{sol}(R^{(1)}) \subseteq \text{sol}(R^{(2)})$. A formal proof goes as follows:

$$\begin{aligned}
\text{sol}(R^{(1)}) \quad &= \quad \bowtie\Big|_{i \neq j} R^{(1)}_{ij} && \text{(definition of solution for } R^{(1)}),\\
&\subseteq \quad \bowtie\Big|_{i \neq j} R^{(2)}_{ij} && \text{(since } R^{(1)}_{ij} \subseteq R^{(2)}_{ij}\text{, i.e., } R^{(1)} \text{ is tighter than } R^{(2)}),\\
&\subseteq \quad \text{sol}(R^{(2)}) && \text{(definition of solution for } R^{(2)}).
\end{aligned}$$

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