Advanced consistency methods
Chapter 8

ICS-275

Spring 14

Relational consistency (Chapter 8)

- Relational arc-consistency
- Relational path-consistency
- Relational m-consistency
- Relational consistency for Boolean and linear constraints:
  - Unit-resolution is relational-arc-consistency
  - Pair-wise resolution is relational path-consistency

Example

- Consider a constraint network over five integer domains, where the constraints take the form of linear equations and the domains are integers bounded by
  - D_x in [-2,3]
  - D_y in [-5,7]
  - R_{xyz} : x + y = z
  - R_{zlt} : z + t = l
  - From D_x and R_{xyz} infer z-y in [-2,3] from this and D_y we can infer z in [-7,10]

Relational arc-consistency

- Let *R* be a constraint network , *X*= $\{x_1,\ldots, x_n\}$, $D_1,\ldots, D_n$, R_S a relation.

R_S in *R* is *relational-arc-consistent* relative to x in S, iff any *consistent* instantiation of the variables in *S*- *{x}* has an extension to a value in D_x that satisfies R_S. Namely,

$$\rho(S - x) \subseteq \pi_{S-x} R_S \boxed{\bowtie} D_x$$

Enforcing relational arc-consistency

- If arc-consistency is not satisfied add:

$$R_{S-x} \leftarrow R_{S-x} \cap \pi_{S-x} R_S \otimes D_S$$

Example

- R_{xyz} = {(a,a,a),(a,b,c),(b,b,c)}.
- This relation is not relational arc-consistent, but if we add the projection:

  R_{xy}= {(a,a),(a,b),(b,b)}, then R_{xyz} will be relational arc-consistent relative to {z}.
- To make this network relational-arc-consistent, we would have to add all the projections of R_{xyz} with respect to all subsets of its variables.

Relational path-cosistency

- Let R_S and R_T be two constraints in a network.
- R_S and R_T are relational-path-consistent relative to a variable x in S U T iff any consistent instantiation of variables in S T - {x} has an extension to in the domain D_x, s.t. R_S and R_T simultaneously;
-

$$\rho(A) \subseteq \pi_A R_S \otimes R_T \otimes D_x$$

$$A = S \cup T - x$$

- A pair of relations R_S and R_T is relational-path-consistent iff it is relational-path-consistent relative to every variable in S ∩ T. A network is relational-path-consistent iff every pair of its relations is relational-path-consistent.

D_x in [-2,3]

D_y in [-5,7]

Example:

R_{xyz}:= x + y = z

R_{ztl}:= z + t = l

- We can assign to *x, y, l* and *t* values that are consistent relative to the relational-arc-consistent network generated in earlier. For example, the assignment

- *(x=2, y= -5, t=3, l=15)* is consistent, since only domain restrictions are applicable, but no value of z that satisfies $x+y = z$ and $z+t = l$.
- To make the two constraints relational path-consistent relative to *z* add : $x+y+t = l$.

Enforcing relational arc, path and m-consistency

- If arc-consistency is not satisfied add:

r.a.c

$$R_{S-x} \leftarrow R_{S-x} \cap \pi_{S-x} R_S \otimes D_S$$

r.p.c

$$\rho(A) \subseteq \pi_A R_S \otimes R_T \otimes D_x$$

$$A = S \cup T - x$$

r.m.c

$$\rho(A) \subseteq \pi_A \otimes_{i=1,m} R_{S_i} \otimes D_x$$

$$A = S_1 \cup \ldots S_m - x$$

Extended composition

- The extended composition of relation $R_{S_1} \ldots R_{S_m}$ relative to A is defined by

$$EC_A(R_{1,} \ldots, R_m) = \pi_A(R_1 \otimes R_2 \otimes, \ldots, \otimes R_m)$$

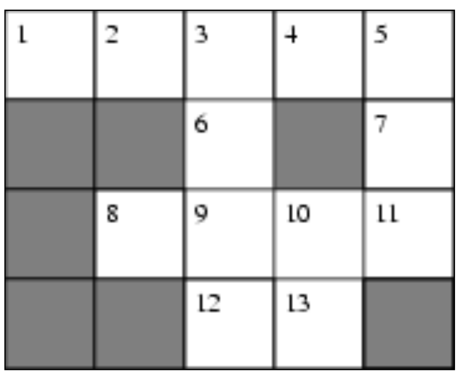
- If the projection operation is restricted to subsets of size i, it is called extended (*i,m*)-composition.
- Special cases: domain propagation and relational arc-consistency

$$D_x \leftarrow D_x \cap \pi_x(R_S \otimes D_S)$$

$$R_{S-x} \leftarrow R_{S-x} \cap \pi_{S-x}(R_S \otimes D_S)$$

Example: crossword puzzle, DRC_2

$R_{1,2,3,4,5} = \{(H,O,S,E,S),(L,A,S,E,R),(S,H,E,E,T),$
$(S,N,A,I,L),(S,T,E,E,R)\}$
$R_{3,6,9,12} = \{(H,I,K,E),(A,R,O,N),(K,E,E,T),(E,A,R,N),$
$(S,A,M,E)\}$
$R_{5,7,11} = \{(R,U,N),(S,U,N),(L,E,T),(Y,E,S),(E,A,T),(T,E,N)\}$
$R_{8,9,10,11} = R_{3,6,9,12}$
$R_{10,13} = \{(N,O),(B,E),(U,S),(I,T)\}$
$R_{12,13} = R_{10,13}$



$bucket(x_1)$

$bucket(x_2)$

$bucket(x_3)$

$bucket(x_4)$

$bucket(x_5)$

$bucket(x_6)$

$bucket(x_7)$

$bucket(x_8)$

$bucket(x_9)$

$bucket(x_{10})$

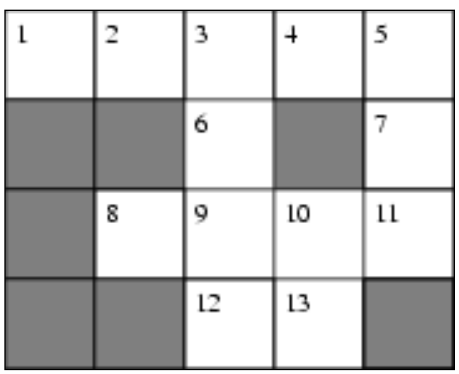
$bucket(x_{11})$

$bucket(x_{12})$

$bucket(x_{13})$

Example: crossword puzzle, Directional-relational-2

$R_{1,2,3,4,5} = \{(H,O,S,E,S),(L,A,S,E,R),(S,H,E,E,T),$
$(S,N,A,I,L),(S,T,E,E,R)\}$
$R_{3,6,9,12} = \{(H,I,K,E),(A,R,O,N),(K,E,E,T),(E,A,R,N),$
$(S,A,M,E)\}$
$R_{5,7,11} = \{(R,U,N),(S,U,N),(L,E,T),(Y,E,S),(E,A,T),(T,E,N)\}$
$R_{8,9,10,11} = R_{3,6,9,12}$
$R_{10,13} = \{(N,O),(B,E),(U,S),(I,T)\}$
$R_{12,13} = R_{10,13}$



$bucket(x_1)$	$R_{1,2,3,4,5}$		
$bucket(x_2)$		$H_{2,3,4,5}$	
$bucket(x_3)$	$R_{3,6,9,12}$	$H_{3,4,5}$	
$bucket(x_4)$		$H_{4,5,6,9,12}$	
$bucket(x_5)$	$R_{5,7,11}$	$H_{5,6,9,12}$	
$bucket(x_6)$		$H_{6,7,9,11,12}$	
$bucket(x_7)$		$H_{7,9,11,12}$	
$bucket(x_8)$	$R_{8,9,10,11}$		
$bucket(x_9)$	$H_{9,10,11}$	$H_{9,11,12}$	
$bucket(x_{10})$	$R_{10,13}$	$H_{10,11,12}$	
$bucket(x_{11})$			*Empty relation . . . exit.*
$bucket(x_{12})$	$R_{12,13}$		
$bucket(x_{13})$			

Complexity

- Theorem: DRC_2 is exponential in the induced-width.
- (because size of the recorded relations are exp in w).

- Crossword puzzles can be made directional backtrack-free by DRC_2

Domain tightness

- **Theorem:** a strong relational 2-consistent constraint network over bi-valued domains is globally consistent.

- **Theorem**: A strong relational k-consistent constraint network with at most k values is globally consistent.

Inference for Boolean theories

- Resolution is identical to *extended 2 decomposition*
- check: $\{(f \vee x \vee y \vee \backsim z), (x \vee y \vee f)\}$
- Boolean theories have domain size 2
- Therefore DRC_2 makes a cnf globally consistent.
- DRC_2 expressed on cnfs is directional resolution

Directional resolution

DIRECTIONAL-RESOLUTION

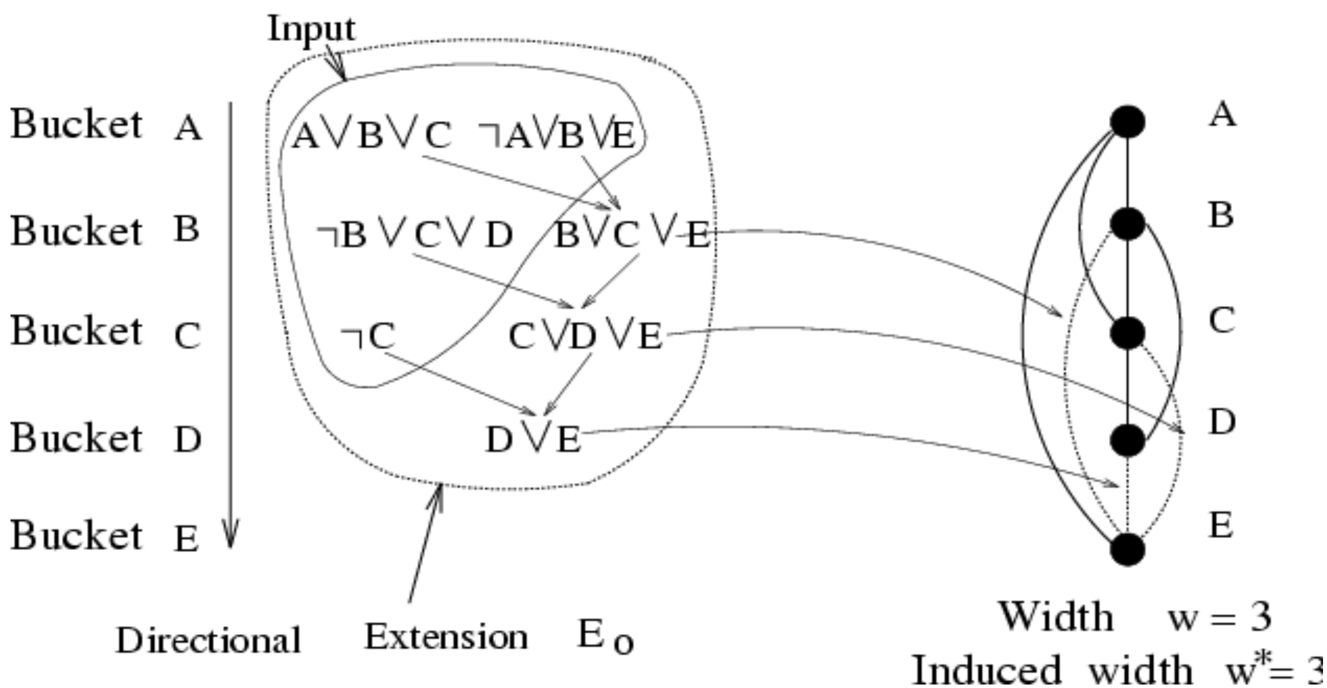
Input: A *CNF* theory φ, an ordering $d = Q_1, \ldots, Q_n$ of its variables.

Output: A decision of whether φ is satisfiable. If it is, a theory $E_d(\varphi)$, equivalent to φ, else an empty directional extension.

1. **Initialize:** generate an ordered partition of clauses into buckets. $bucket_1, \ldots, bucket_n$, where $bucket_i$ contains all clauses whose highest literal is Q_i.
2. **for** $i \leftarrow n$ **downto** 1 process $bucket_i$:
3. **if** there is a unit clause **then** (the instantiation step)
   apply unit-resolution in $bucket_i$ and place the resolvents in their right buckets.
   if the empty clause was generated, theory is not satisfiable.
4. **else** resolve each pair $\{(\alpha \vee Q_i), (\beta \vee \neg Q_i)\} \subseteq bucket_i$.
   if $\gamma = \alpha \vee \beta$ is empty, return $E_d(\varphi) = \{\}$, theory is not satisfiable
   else determine the index of γ and add it to the appropriate bucket.
5. **return** $E_d(\varphi) \leftarrow \bigcup_i bucket_i$

Figure 4.20: Directional-resolution

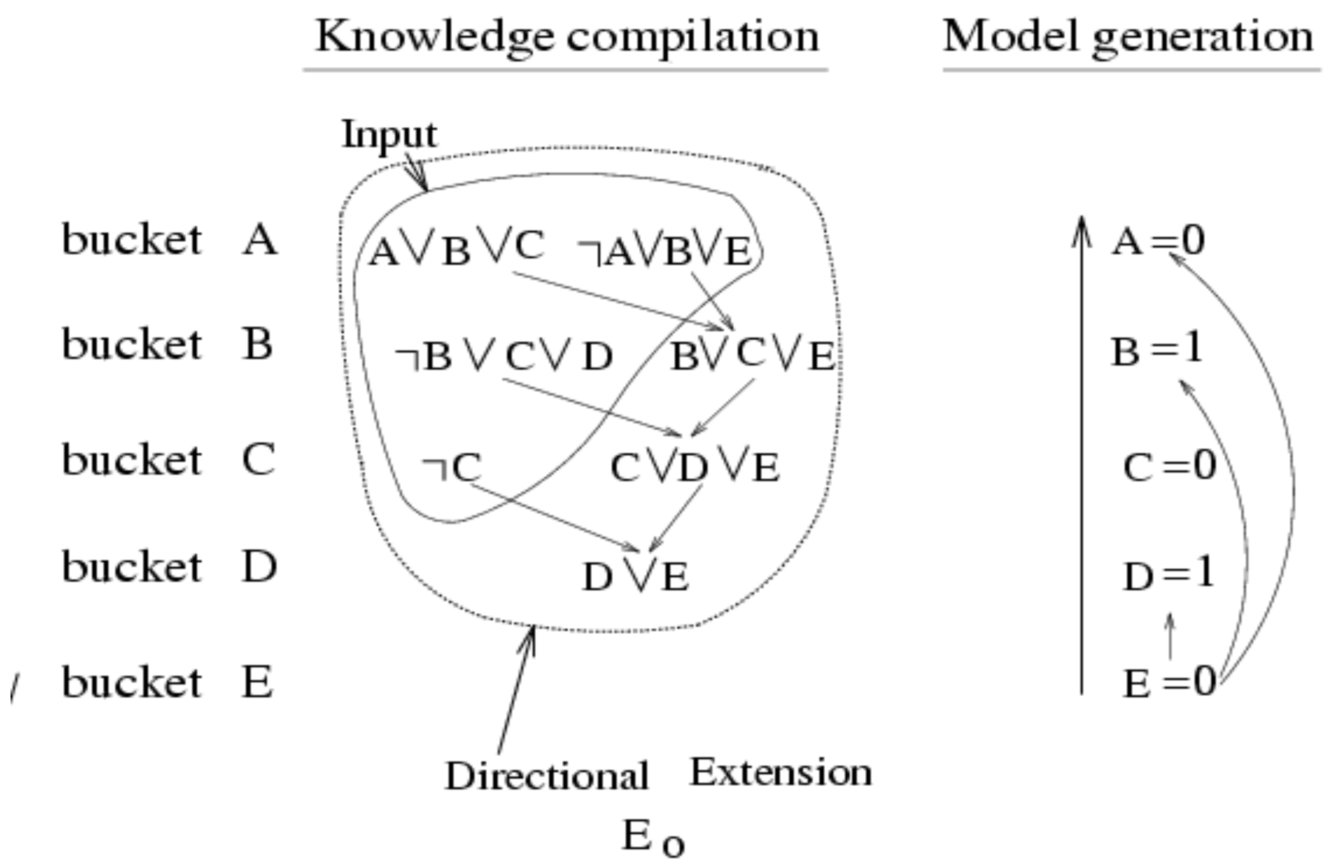
DR resolution = adaptive-consistency=directional relational path-consistency



$$|\, bucket_i \,| = O(\exp(w^*))$$

$$\text{DR time and space}: O(n \exp(w^*))$$

Directional Resolution ⇔ Adaptive Consistency



Resolution – An Example

$$(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3) \wedge (x_3 \vee x_4) \wedge (x_3 \vee \neg x_4) \quad \vdash$$

$$(\neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3) \wedge (x_3 \vee x_4) \wedge (x_3 \vee \neg x_4) \quad \vdash$$

History

- 1960 – resolution-based Davis-Putnam algorithm
- 1962 – resolution step replaced by conditioning (Davis, Logemann and Loveland, 1962) to avoid memory explosion, resulting into a backtracking search algorithm known as Davis-Putnam (DP), or DPLL procedure.
- The dependency on induced width was not known in 1960.
- 1994 – Directional Resolution (DR), a rediscovery of the original Davis-Putnam, identification of tractable classes (Dechter and Rish, 1994).

Complexity of DR

Theorem 4.7.6 *(complexity of DR)*
Given a theory φ and an ordering of its variables o, the time complexity of algorithm DR along o is $O(n \cdot 9^{w_o^*})$, and $E_o(\varphi)$ contains at most $n \cdot 3^{w_o^*+1}$ clauses, where w_o^* is the induced width of φ's interaction graph along o. □

- 2-cnfs and Horn theories

Theorem 4.7.7 *Given a 2-cnf theory φ, its directional extension $E_o(\varphi)$ along any ordering o is of size $O(n \cdot {w_o^*}^2)$, and can be generated in $O(n \cdot {w_o^*}^2)$ time.*

Theorem 4.7.8 *The consistency of Horn theories can be determined by unit propagation. If the empty clause is not generated, the theory is satisfiable.* □

Linear inequalities

- Consider r-ary constraints over a subset of variables $,x_1 .. \, x_r$ of the form
- $a_r x_r$ + … + $a_r x_r$ <= c, a_i are rational constants. The r-ary inequalities define corresponding r-ary relations that are *row convex*.
- Since r-ary linear inequalities that are closed under relational path-consistency are row-convex, relative to any set of integer domains (using the natural ordering).
- **Proposition:** A set of linear inequalities that is closed under RC_2 is globally consistent.

Linear inequalities

- Gausian elimination with domain constraint is relational-arc-consistency
- Gausian elimination of 2 inequalities is relational path-consistency
- Theorem: directional relational path-consistency is complete for CNFs and for linear inequalities

DIRECTIONAL-LINEAR-ELIMINATION (φ, d)

Input: A *set of linear inequalities* φ, an ordering $d = x_1, \ldots, x_n$.

Output:A decision of whether φ is satisfiable. If it is, a backtrack-free theory $E_d(\varphi)$.

1. **Initialize:** Partition inequalities into ordered buckets.
2. **for** $i \leftarrow n$ **downto** 1 **do**
3. **if** x_i has one value in its domain **then**

. substitute the value into each inequality in the bucket and put the resulting inequality in the right bucket.

4. **else,for each** pair $\{\alpha, \beta\} \subseteq bucket_i$, compute $\gamma = elim_i(\alpha, \beta)$ **if** γ has no solutions, return $E_d(\varphi) = \{\}$, "inconsistency" **else** add γ to the appropriate lower bucket.
5. **return** $E_d(\varphi) \leftarrow \bigcup_i bucket_i$

Figure 4.22: Fourier Elimination; DLE

Directional linear elimination, DLE : generates a backtrack-free representation

Theorem 4.8.3 *Given a set of linear inequalities φ, algorithm DLE (Fourier elimination) decides the consistency of φ over the Rationals and the Reals, and it generates an equivalent backtrack-free representation.* $\square$

Example

$$
\begin{aligned}
&bucket_4: \quad 5x_4 + 3x_2 - x_1 \le 5,\ x_4 + x_1 \le 2,\ \ -x_4 \le 0,\\
&bucket_3: \quad x_3 \le 5,\ x_1 + x_2 - x_3 \le -10\\
&bucket_2: \quad x_1 + 2x_2 \le 0.\\
&bucket_1: \quad
\end{aligned}
$$

Figure 4.23: initial buckets

$$
\begin{aligned}
&bucket_4: \quad 5x_4 + 3x_2 - x_1 \le 5,\ x_4 + x_1 \le 2,\ \ -x_4 \le 0,\\
&bucket_3: \quad x_3 \le 5,\ x_1 + x_2 - x_3 \le -10\\
&bucket_2: \quad x_1 + 2x_2 \le 0\ ||\ 3x_2 - x_1 \le 5, x_1 + x_2 \le -5\\
&bucket_1: \quad ||\ x_1 \le 2.
\end{aligned}
$$

Figure 4.24: final buckets