Class Information

• Instructor: Rina Dechter

• Lectures: Monay & Wednesday
• Time: 11:00 - 12:20 pm
• Discussion (optional): Wednesdays 12:30-1:20

• Class page:
  http://www.ics.uci.edu/~dechter/courses/ics-275a/spring-2014/
Text book (required)

Rina Dechter,

**Constraint Processing**, Morgan Kaufmann
Outline

✓ Motivation, applications, history
✓ CSP: Definition, and simple modeling examples
✓ Mathematical concepts (relations, graphs)
✓ Representing constraints
✓ Constraint graphs
✓ The binary Constraint Networks properties
Outline

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Graphical Models

Those problems that can be expressed as:

- A set of variables
- Each variable takes its values from a finite set of domain values
- A set of local functions

Main advantage:
- They provide unifying algorithms:
  - Search
  - Complete Inference
  - Incomplete Inference
Combinatorial Problems

Many Examples

Graph Coloring

Timetabling

EOS Scheduling

Bayesian Networks

... and many others.
Example: student course selection

- **Context**: You are a senior in college
- **Problem**: You need to register in 4 courses for the Spring semester
- **Possibilities**: Many courses offered in Math, CSE, EE, CBA, etc.
- **Constraints**: restrict the choices you can make
  - Courses have prerequisites you have/don't have
  - Courses/instructors you like/dislike
  - Courses are scheduled at the same time
  - In CE: 4 courses from 5 tracks such as at least 3 tracks are covered

- **You have choices, but are restricted by constraints**
  - Make the right decisions!!
  - **ICS Graduate program**
Student course selection (continued)

• **Given**
  – A set of variables: 4 courses at your college
  – For each variable, a set of choices (values): the available classes.
  – A set of constraints that restrict the combinations of values the variables can take at the same time

• **Questions**
  – Does a solution exist? (classical decision problem)
  – How many solutions exists? (counting)
  – How two or more solutions differ?
  – Which solution is preferable?
  – etc.
The field of Constraint Programming

• How did it started:
  – Artificial Intelligence (vision)
  – Programming Languages (Logic Programming),
  – Databases (deductive, relational)
  – Logic-based languages (propositional logic)
  – SATisfiability

• Related areas:
  – Hardware and software verification
  – Operation Research (Integer Programming)
  – Answer set programming

• Graphical Models; deterministic
Scene labeling constraint network
Scene labeling constraint network

Fork: + + - - - + +

Arrow: + + + + + + +

Ell: + + + + + + +

Tee: + + + + + + +
3-dimensional interpretation of 2-dimensional drawings

Fork: 

Arrow: 

Ell: 

Tee: 

Spring 2014
The field of Constraint Programming

• **How did it started:**
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• **Graphical Models; deterministic**
Applications

• Radio resource management (RRM)
• Databases (computing joins, view updates)
• Temporal and spatial reasoning
• Planning, scheduling, resource allocation
• Design and configuration
• Graphics, visualization, interfaces
• Hardware verification and software engineering
• HC Interaction and decision support
• Molecular biology
• Robotics, machine vision and computational linguistics
• Transportation
• Qualitative and diagnostic reasoning
Outline

✓ Motivation, applications, history
✓ **CSP: Definitions and simple modeling examples**
✓ Mathematical concepts (relations, graphs)
✓ Representing constraints
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Constraint Networks

Example: map coloring

Variables - countries (A, B, C, etc.)
Values - colors (red, green, blue)
Constraints:  \( A \neq B, \ A \neq D, \ D \neq E, \ etc. \)
### Example: map coloring

- **Variables**: countries (A, B, C, etc.)
- **Values**: colors (e.g., red, green, yellow)
- **Constraints**: $A \neq B$, $A \neq D$, $D \neq E$, etc.

#### Are the constraints consistent?

#### Find a solution, find all solutions

#### Count all solutions

#### Find a good solution
Information as Constraints

• I have to finish my class in 50 minutes
• 180 degrees in a triangle
• Memory in our computer is limited
• The four nucleotides that makes up a DNA only combine in a particular sequence
• Sentences in English must obey the rules of syntax
• Susan cannot be married to both John and Bill
• Alexander the Great died in 333 B.C.
Constraint Network; Definition

- A constraint network is: $R=(X,D,C)$
  - $X$ variables
    $$X = \{X_1, \ldots, X_n\}$$
  - $D$ domain
    $$D = \{D_1, \ldots, D_n\}$$, $D_i = \{v_1, \ldots, v_k\}$
  - $C$ constraints
    $$C = \{C_1, \ldots, C_i\}$$, $C_i = (S_i, R_i)$
  - $R$ expresses allowed tuples over scopes

- A solution is an assignment to all variables that satisfies all constraints (join of all relations).

- **Tasks:** consistency?, one or all solutions, counting, optimization
The N-queens problem

The network has four variables, all with domains $\mathcal{D} = \{1, 2, 3, 4\}$.
(a) The labeled chess board. (b) The constraints between variables.

\[
\begin{array}{cccc}
  x_1 & x_2 & x_3 & x_4 \\
  1 & & & \\
  2 & & & \\
  3 & & & \\
  4 & & & \\
\end{array}
\]

\[
R_{12} = \{(1,3), (1,4), (2,4), (3,1), (4,1), (4,2)\}
\]
\[
R_{13} = \{(1,2), (1,4), (2,1), (2,3), (3,2), (3,4), (4,1), (4,3)\}
\]
\[
R_{14} = \{(1,2), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (3,4), (4,2), (4,3)\}
\]
\[
R_{23} = \{(1,3), (1,4), (2,4), (3,1), (4,1), (4,2)\}
\]
\[
R_{24} = \{(1,2), (1,4), (2,1), (2,3), (3,2), (3,4), (4,1), (4,3)\}
\]
\[
R_{34} = \{(1,3), (1,4), (2,4), (3,1), (4,1), (4,2)\}
\]
A solution and a partial consistent tuple

Not all consistent instantiations are part of a solution:
(a) A consistent instantiation that is not part of a solution.
(b) The placement of the queens corresponding to the solution (2, 4, 1, 3).
c) The placement of the queens corresponding to the solution (3, 1, 4, 2).

\[ \begin{array}{|c|c|c|c|}
\hline
Q & & & \\
\hline
& Q & & \\
\hline
& & & Q \\
\hline
\end{array} \]

\[ \begin{array}{|c|c|c|c|}
\hline
& & Q & \\
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Q & & & \\
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\end{array} \]

\[ \begin{array}{|c|c|c|c|}
\hline
& & & Q \\
\hline
Q & & & \\
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& & & \\
\hline
\end{array} \]
Example: Crossword puzzle

- Variables: $x_1, \ldots, x_{13}$
- Domains: letters
- Constraints: words from

{HOSES, LASER, SHEET, SNAIL, STEER, ALSO, EARN, HIKE, IRON, SAME, EAT, LET, RUN, SUN, TEN, YES, BE, IT, NO, US}
Configuration and Design

- Want to build: recreation area, apartment complex, a cluster of 50 single-family houses, cemetery, and a dump
  - Recreation area near lake
  - Steep slopes avoided except for recreation area
  - Poor soil avoided for developments
  - Highway far from apartments, houses and recreation
  - Dump not visible from apartments, houses and lake
  - Lots 3 and 4 have poor soil
  - Lots 3, 4, 7, 8 are on steep slopes
  - Lots 2, 3, 4 are near lake
  - Lots 1, 2 are near highway
Example: Sudoku (constraint propagation)

Each row, column and major block must be all different.

“Well posed” if it has unique solution: 27 constraints.
Sudoku (inference)

Each row, column and major block must be alldifferent

“Well posed” if it has unique solution
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Mathematical background

- Sets, domains, tuples
- Relations
- Operations on relations
- Graphs
- Complexity
Two Representations of a relation: 
\( R = \{(\text{black, coffee}), (\text{black, tea}), (\text{green, tea})\} \).

Variables: Drink, color

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
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<tbody>
<tr>
<td>black</td>
<td>coffee</td>
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<tr>
<td>black</td>
<td>tea</td>
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<tr>
<td>green</td>
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(a) table
Two Representations of a relation: \( R = \{(\text{black, coffee}), (\text{black, tea}), (\text{green, tea})\} \).

Variables: Drink, color

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<td>green</td>
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</tbody>
</table>

(a) table

\[
\begin{array}{lll}
\text{apple juice} \\
\text{coffee} \\
\text{tea}
\end{array}
\]

\[
\begin{array}{lll}
x_2 \\
\text{coffee} \\
\text{tea}
\end{array}
\]

(b) \((0,1)\)-matrix

\[
\begin{bmatrix}
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\]
### Three Relations

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<tr>
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(a) Relation $R$  
(b) Relation $R'$  
(c) Relation $R''$
Operations with relations

- Intersection
- Union
- Difference
- Selection
- Projection
- Join
- Composition
Relations are special case of a Local function

\[ f : \prod_{x_i \in Y} D_i \rightarrow A \]

where

\[ \text{var}(f) = Y \subseteq X: \text{ scope of function } f \]

\[ A: \text{ is a set of valuations} \]

In constraint networks: functions are boolean

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<tr>
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<td>a</td>
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Example of Set Operations: intersection, union, and difference applied to relations.

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(a) Relation $R$  
(b) Relation $R'$  
(c) Relation $R''$

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(a) $R \cap R'$  
(b) $R \cup R'$  
(b) $R - R'$
## Selection, Projection, and Join

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(a) Relation $R$

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(b) Relation $R'$

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(c) Relation $R''$

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(a) $\sigma_{x_3=c}(R')$

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(b) $\pi_{x_2,x_3}(R')$

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</table>

(c) $R' \bowtie R''$
Local Functions

Combination

- **Join:** \( f \Join g \)

\[
\begin{array}{cccc}
\hline
x_1 & x_2 & f & x_1 & x_2 & x_3 & g \\
\hline
a & a & true & a & a & true & a & a & true \\
a & b & false & a & b & true & a & b & true \\
b & a & false & b & a & true & b & a & true \\
b & b & true & b & b & false & b & b & false \\
\hline
\end{array}
\]

- **Logical AND:** \( f \land g \)

\[
\begin{array}{cccc}
\hline
x_1 & x_2 & x_3 & h \\
\hline
a & a & a & true \\
a & a & b & true \\
a & b & a & false \\
a & b & b & false \\
b & a & a & false \\
b & a & b & true \\
b & b & a & true \\
b & b & b & false \\
\hline
\end{array}
\]

Spring 2014
Outline

✓ Motivation, applications, history
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✓ Mathematical concepts (relations, graphs)
✓ Representing constraints/ Languages
✓ Constraint graphs
✓ The binary Constraint Networks properties
Modeling; Representing a problems

• If a CSP $M = \langle X, D, C \rangle$ represents a real problem $P$, then every solution of $M$ corresponds to a solution of $P$ and every solution of $P$ can be derived from at least one solution of $M$

• The variables and values of $M$ represent entities in $P$

• The constraints of $M$ ensure the correspondence between solutions

• The aim is to find a model $M$ that can be solved as quickly as possible

• **goal of modeling:** choose a set of variables and values that allows the constraints to be expressed easily and concisely
Given a proposition theory $\varphi = \{(A \lor B), (C \lor \neg B)\}$ does it have a model?

**Can it be encoded as a constraint network?**

**Variables:** \{A, B, C\}

**Domains:** $D_A = D_B = D_C = \{0, 1\}$

**Relations:**

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If this constraint network has a solution, then the propositional theory has a model.
Constraint’s representations

- Relation: allowed tuples
  
  \[
  \begin{array}{ccc}
  X & Y & Z \\
  1 & 3 & 2 \\
  2 & 1 & 3 \\
  \end{array}
  \]

- Algebraic expression:
  \[X + Y^2 \leq 10, X \neq Y\]

- Propositional formula:
  \[(a \lor b) \rightarrow \neg c\]

- A decision tree, a procedure

- Semantics: by a relation
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Constraint Graphs:
Primal, Dual and Hypergraphs

• A (primal) constraint graph: a node per variable, arcs connect constrained variables.
• A dual constraint graph: a node per constraint’s scope, an arc connect nodes sharing variables = hypergraph

{HOSES, LASER, SHEET, SNAIL, STEER, ALSO, EARN, HIKE, IRON, SAME, EAT, LET, RUN, SUN, TEN, YES, BE, IT, NO, US}
Graph Concepts Reviews: Hyper Graphs and Dual Graphs

A hypergraph

Primal graphs

Dual graph

Factor graphs
Example: Cryptarithmetic

Variables: $F, T, U, W, R, O, X_1, X_2, X_3$

Domains: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Constraints: Alldiff $(F, T, U, W, R, O)$

- $O + O = R + 10 \cdot X_1$
- $X_1 + W + W = U + 10 \cdot X_2$
- $X_2 + T + T = O + 10 \cdot X_3$
- $X_3 = F, T \neq 0, F \neq 0$

What is the primal graph?
What is the dual graph?
Propositional Satisfiability

\[ \varphi = \{ (\neg C), (A \lor B \lor C), (\neg A \lor B \lor E), (\neg B \lor C \lor D) \}. \]
Given a telecommunication network (where each communication link has various antennas), assign a frequency to each antenna in such a way that all antennas may operate together without noticeable interference.

**Encoding?**

Variables: one for each antenna

Domains: the set of available frequencies

Constraints: the ones referring to the antennas in the same communication link
Constraint graphs of 3 instances of the Radio frequency assignment problem in CELAR’s benchmark
Examples

Scheduling problem

Five tasks: T1, T2, T3, T4, T5
Each one takes one hour to complete
The tasks may start at 1:00, 2:00 or 3:00
Requirements:
- T1 must start after T3
- T3 must start before T4 and after T5
- T2 cannot execute at the same time as T1 or T4
- T4 cannot start at 2:00

Encoding?

Variables: one for each task
Domains: \( D_{T1} = D_{T2} = D_{T3} = D_{T4} = \{1:00, 2:00, 3:00\} \)
Constraints:

\[
\begin{array}{c}
\text{T4} \\
1:00 \\
3:00
\end{array}
\]
The constraint graph and relations of scheduling problem

**Unary constraint**
\[ D_{T4} = \{1:00, 3:00\} \]

**Binary constraints**
\[ R_{\{T1,T2\}}: \{(1:00,2:00), (1:00,3:00), (2:00,1:00), (2:00,3:00), (3:00,1:00), (3:00,2:00)\} \]
\[ R_{\{T1,T3\}}: \{(2:00,1:00), (3:00,1:00), (3:00,2:00)\} \]
\[ R_{\{T2,T4\}}: \{(1:00,2:00), (1:00,3:00), (2:00,1:00), (2:00,3:00), (3:00,1:00), (3:00,2:00)\} \]
\[ R_{\{T3,T4\}}: \{(1:00,2:00), (1:00,3:00), (2:00,3:00)\} \]
\[ R_{\{T3,T5\}}: \{(2:00,1:00), (3:00,1:00), (3:00,2:00)\} \]
A combinatorial circuit: $M$ is a multiplier, $A$ is an adder
More examples

• Given \( P = (V, D, C) \), where

\[
V = \{V_1, V_2, \ldots, V_n\} \\
D = \{D_{V_1}, D_{V_2}, \ldots, D_{V_n}\} \\
C = \{C_1, C_2, \ldots, C_l\}
\]

Example I:

• Define \( C \) ?
Example: Temporal Reasoning

- Give one solution: .......
- Satisfaction, yes/no: decision problem
Outline

✓ Motivation, applications, history
✓ CSP: Definition, representation and simple modeling examples
✓ Mathematical concepts (relations, graphs)
✓ Representing constraints
✓ Constraint graphs
✓ The binary Constraint Networks properties
Properties of Binary Constraint Networks

A graph $\mathcal{R}$ to be colored by two colors, an equivalent representation $\mathcal{R}'$ having a newly inferred constraint between $x_1$ and $x_3$.

Equivalence and deduction with constraints (composition)
Composition of relations (Montanari'74)

Input: two binary relations $R_{ab}$ and $R_{bc}$ with 1 variable in common.

Output: a new induced relation $R_{ac}$ (to be combined by intersection to a pre-existing relation between them, if any).

Bit-matrix operation: matrix multiplication

$$R_{ac} = R_{ab} \cdot R_{bc}$$

$$R_{ab} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad R_{bc} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R_{ac} = ?$$
Equivalence, Redundancy, Composition

• Equivalence: Two constraint networks are equivalent if they have the same set of solutions.

• Composition in matrix notation

$R_{xz} = R_{xy} \times R_{yz}$

• Composition in relational operation

$R_{xz} = \pi_{xz}(R_{xy} \otimes R_{yz})$
Relations vs networks

• Can we represent by binary constraint networks the relations
  • \( R(x_1, x_2, x_3) = \{(0,0,0)(0,1,1)(1,0,1)(1,1,0)\} \)
  • \( R(X_1, x_2, x_3, x_4) = \{(1,0,0,0)(0,1,0,0) (0,0,1,0)(0,0,0,1)\} \)
• Number of relations \( 2^k \times n \)
• Number of networks: \( 2^{(k^2)(n^2)} \)
• Most relations cannot be represented by binary networks
The minimal and projection networks

• The projection network of a relation is obtained by projecting it onto each pair of its variables (yielding a binary network).

• \( Relation = \{(1,1,2)(1,2,2)(1,2,1)\} \)
  – What is the projection network?

• What is the relationship between a relation and its projection network?

• \( \{(1,1,2)(1,2,2)(2,1,3)(2,2,2)\} \), solve its projection network?
The N-queens constraint network.

The network has four variables, all with domains \( D_i = \{1, 2, 3, 4\} \). (a) The labeled chess board. (b) The constraints between variables.

\[
R_{12} = \{(1,3), (1,4), (2,4), (3,1), (4,1), (4,2)\}
\]
\[
R_{13} = \{(1,2), (1,4), (2,1), (2,3), (3,2), (3,4), (4,1), (4,3)\}
\]
\[
R_{14} = \{(1,2), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (3,4), (4,2), (4,3)\}
\]
\[
R_{23} = \{(1,3), (1,4), (2,4), (3,1), (4,1), (4,2)\}
\]
\[
R_{24} = \{(1,2), (1,4), (2,1), (2,3), (3,2), (3,4), (4,1), (4,3)\}
\]
\[
R_{34} = \{(1,3), (1,4), (2,4), (3,1), (4,1), (4,2)\}
\]
The 4-queens constraint network:
(a) The constraint graph. (b) The minimal binary constraints. (c) The minimal unary constraints (the domains).

\[ M_{12} = \{ (2,4), (3,1) \} \]
\[ M_{13} = \{ (2,1), (3,4) \} \]
\[ M_{14} = \{ (2,3), (3,2) \} \]
\[ M_{23} = \{ (1,4), (4,1) \} \]
\[ M_{24} = \{ (1,2), (4,3) \} \]
\[ M_{34} = \{ (1,3), (4,2) \} \]
\[ D_1 = \{ 1,3 \} \]
\[ D_2 = \{ 1,4 \} \]
\[ D_3 = \{ 1,4 \} \]
\[ D_4 = \{ 1,3 \} \]

Solutions are: (2,4,1,3) (3,1,4,2)
Projection network (continued)

• **Theorem:** Every relation is included in the set of solutions of its projection network.

• **Theorem:** The projection network is the tightest upper bound binary networks representation of the relation.

Therefore, If a network cannot be represented by its projection network it has no binary network representation.
Partial Order between networks,
The Minimal Network

Definition 2.3.10 Given two binary networks, $\mathcal{R}'$ and $\mathcal{R}$, on the same set of variables $x_1, ..., x_n$, $\mathcal{R}'$ is at least as tight as $\mathcal{R}$ iff for every $i$ and $j$, $R'_{ij} \subseteq R_{ij}$.

• An intersection of two networks is tighter (as tight) than both
• An intersection of two equivalent networks is equivalent to both

Definition 2.3.14 Let $\{\mathcal{R}_1, ... \mathcal{R}_l\}$ be the set of all networks equivalent to $\mathcal{R}_0$ and let $\rho = \text{sol}(\mathcal{R}_0)$. Then the minimal network $M$ of $\mathcal{R}_0$ is defined by $M(\mathcal{R}_0) = \cap_{i=1}^l \mathcal{R}_i$.

Theorem 2.3.15 For every binary network $\mathcal{R}$ s.t. $\rho = \text{sol}(\mathcal{R})$, $M(\rho) = P(\rho)$. 
The N-queens constraint network.

The network has four variables, all with domains $D_i = \{1, 2, 3, 4\}$. (a) The labeled chess board. (b) The constraints between variables.

\[
\begin{align*}
R_{12} &= \{(1,3), (1,4), (2,4), (3,1), (4,1), (4,2)\} \\
R_{13} &= \{(1,2), (1,4), (2,1), (2,3), (3,2), (3,4), (4,1), (4,3)\} \\
R_{14} &= \{(1,2), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (3,4) \\
&\quad (4,2), (4,3)\} \\
R_{23} &= \{(1,3), (1,4), (2,4), (3,1), (4,1), (4,2)\} \\
R_{24} &= \{(1,2), (1,4), (2,1), (2,3), (3,2), (3,4), (4,1), (4,3)\} \\
R_{34} &= \{(1,3), (1,4), (2,4), (3,1), (4,1), (4,2)\}
\end{align*}
\]
The 4-queens constraint network:
(a) The constraint graph. (b) The minimal binary constraints. (c) The minimal unary constraints (the domains).

\[ M_{12} = \{(2,4), (3,1)\} \]
\[ M_{13} = \{(2,1), (3,4)\} \]
\[ M_{14} = \{(2,3), (3,2)\} \]
\[ M_{23} = \{(1,4), (4,1)\} \]
\[ M_{24} = \{(1,2), (4,3)\} \]
\[ M_{34} = \{(1,3), (4,2)\} \]

\[ D_1 = \{1,3\} \]
\[ D_2 = \{1,4\} \]
\[ D_3 = \{1,4\} \]
\[ D_4 = \{1,3\} \]

Solutions are: (2,4,1,3) (3,1,4,2)
The Minimal vs Binary decomposable networks

• The minimal network is perfectly explicit for binary and unary constraints:
  – Every pair of values permitted by the minimal constraint is in a solution.

• Binary-decomposable networks:
  – A network whose all projections are binary decomposable
  – Ex: \((x,y,x,t) = \{(a,a,a,a)(a,b,b,b)(b,b,a,c)\}:
    is binary representable? and what about its projection on \(x,y,z\)?

  – Proposition: The minimal network represents fully binary-decomposable networks.