Directional consistency Chapter 4

CS-275

Winter 2016

Tractable classes

Theorem 3.7.1 *1. The consistency binary constraint networks having no cycles can be decided by arc-consistent*

2. The consistency of binary constraint networks with bi-valued domains can be decided by path-consistency,

3. The consistency of Horn cnf theories can be decided by unit propagation.

Backtrack-free search: or

What level of consistency will guarantee global-consistency

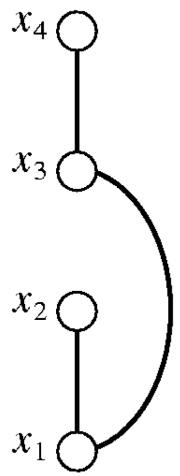
Definition 4.1.1 (backtrack-free search) *A constraint network is backtrack-free relative to a given ordering $d = (x_1, ..., x_n)$ if for every $i \leq n$, every partial solution of $(x_1, ..., x_i)$ can be consistently extended to include x_{i+1}.*

Backtrack free and queries:
Consistency,
All solutions
Counting
optimization

Directional arc-consistency:
another restriction on propagation

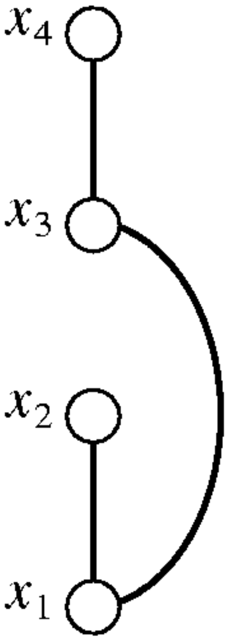
Definition 4.3.1 (directional arc-consistency) *A network is* directional-arc-consistent *relative to order* $\mathbf{d} = (x_1, ..., x_n)$ *iff every variable* x_i *is arc-consistent relative to every variable* x_j *such that* $i \leq j$.

D4={white,blue,black}
D3={red,white,blue}
D2={green,white,black}
D1={red,white,black}
X1=x2, x1=x3,x3=x4



Directional arc-consistency:
another restriction on propagation

- D4={white,blue,black}
- D3={red,white,blue}
- D2={green,white,black}
- D1={red,white,black}
- X1=x2,
- x1=x3,
- x3=x4



After DAC:

- D1= {white},
- D2={green,white,black},
- D3={white,blue},
- D4={white,blue,black}

Algorithm for directional arc-consistency (DAC)

DAC($\mathcal{R}$)

Input:A network $\mathcal{R} = (\mathcal{X}, \mathcal{D}, \mathcal{C})$, its constraint graph G, and an ordering $d = (x_1, ..., x_n)$.

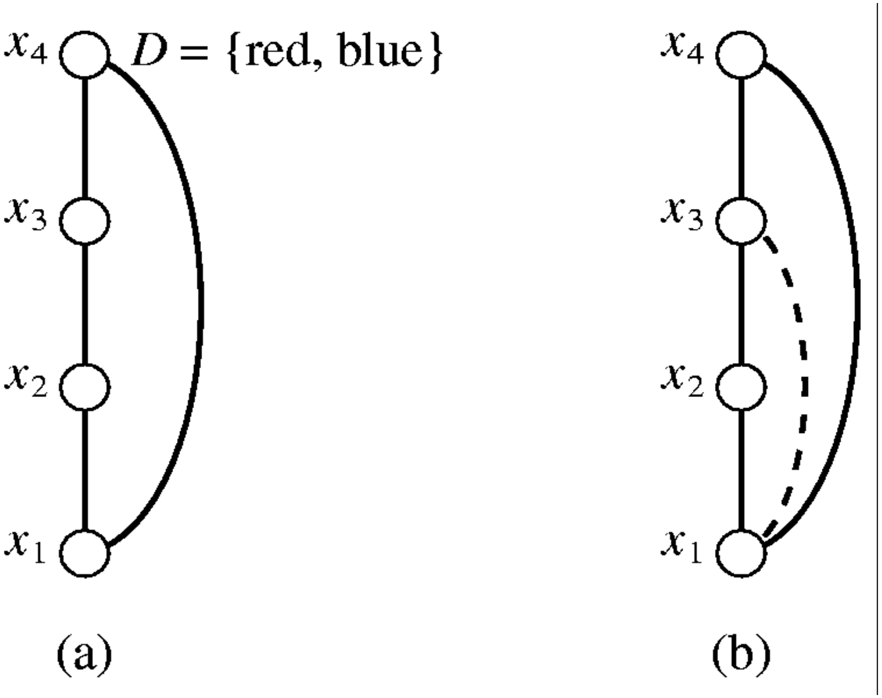
Output: A directional arc-consistent network.

1. **for** $i = n$ to 1 by -1 **do**
2. **for** each $j < i$ s.t. $R_{ji} \in \mathcal{R}$,
3. $D_j \leftarrow D_j \cap \pi_j(R_{ji} \bowtie D_i)$, (this is revise$((x_j), x_i)$).
4. **end-for**

Figure 4.6: Directional arc-consistency (DAC)

- Complexity: $O(ek^2)$

Directional arc-consistency may not be enough → Directional path-consistency



Definition 4.3.5 (directional path-consistency) *A network* $\mathcal{R}$ *is* directional path-consistent *relative to order* $d = (x_1, ..., x_n)$ *iff for every* $k \geq i, j$, *the pair* $\{x_i, x_j\}$ *is path-consistent relative to* x_k.

Algorithm directional path consistency (DPC)

DPC($\mathcal{R}$)

Input:A binary network $\mathcal{R} = (X, D, C)$ and its constraint graph $G = (V, E)$, $d = (x_1, ..., x_n)$.

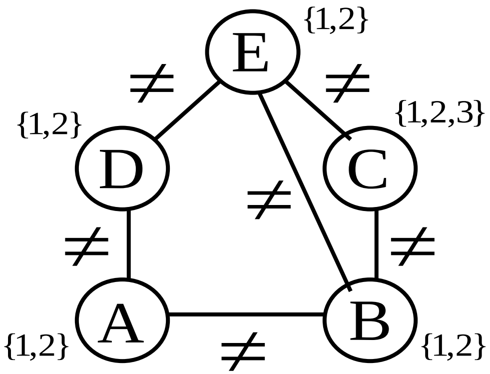
Output:A strong directional path-consistent network and its graph $G' = (V, E')$.

Initialize: $E' \leftarrow E$.

1. **for** $k = n$ to 1 by -1 **do**
2. (a) $\forall\, i \le k$ such that x_i is connected to x_k in the graph, **do**
3. $D_i \leftarrow D_i \cap \pi_i(R_{ik} \bowtie D_k)$ ($Revise((x_i), x_k)$)
4. (b) $\forall\, i, j \le k$ s.t. (x_i, x_k),$(x_j, x_k) \in E'$ **do**
5. $R_{ij} \leftarrow R_{ij} \cap \pi_{ij}(R_{ik} \bowtie D_k \bowtie R_{kj})$ (Revise-3($(x_i, x_j), x_k$))
6. $E' \leftarrow E' \cup (x_i, x_j)$
7. **endfor**
8. **return** The revised constraint network $\mathcal{R}$ and $G' = (V, E')$.

Theorem 4.3.7 *Given a binary network $\mathcal{R}$ and an ordering d, algorithm DPC generates a largest equivalent, strong, directional-path-consistent network relative to d. The time and space complexity of DPC is $O(n^3k^3)$, where n is the number of variables and k bounds the domain sizes.*

Example of DPC



Directional i-consistency

Definition 4.3.8 (directional i-consistency) *A network is* directional i-consistent *relative to order* $d = (x_1, ..., x_n)$ *iff every* $i-1$ *variables are* i*-consistent relative to every variable that succeeds them in the ordering. A network is strong directional* i*-consistent if it is directional* j*-consistent for every* $j < i$.

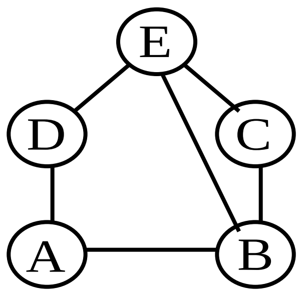
Algorithm directional i-consistency

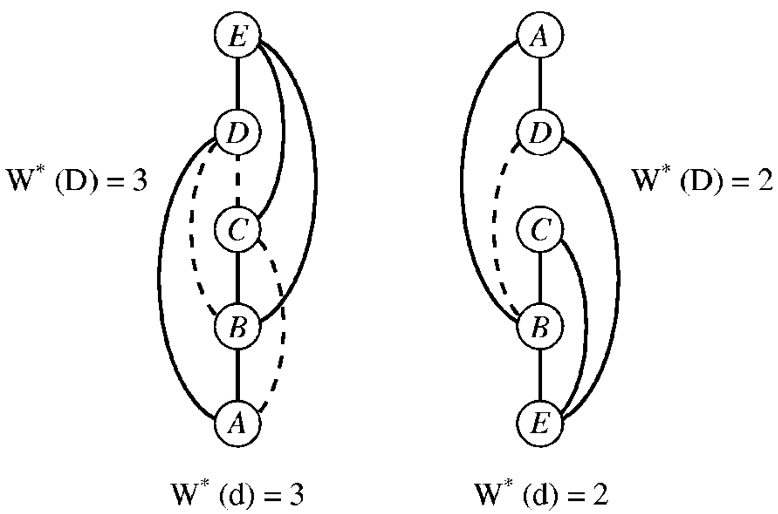
Directional i-consistency $(DIC_i(\mathcal{R}))$
Input: a network $\mathcal{R} = (X, D, C)$, its constraint graph $G = (V, E)$, $d = (x_1, \ldots, x_n)$.
output: A strong directional i-consistent network along d and its graph $G' = (V, E')$.
Initialize: $E' \leftarrow E$, $C' \leftarrow C$.
1. **for** $j = n$ to 1 by -1 **do**
2. **let** $P = parents(x_j)$.
3. **if** $|P| < i - 1$ then
4. $Revise(P, x_j)$
5. **else, for** each subset of $i - 1$ variables S, $S \subseteq P$, **do**
6. Revise(S, x_j)
7. **endfor**
8. $C' \leftarrow C' \cup$ all generated constraints.
8. $E' \leftarrow E' \cup \{(x_k, x_m) | x_k, x_m \in P\}$ (connect all parents of x_j)
9. **endfor.**
10. **return** C' and E'.

Figure 4.9: Algorithm directional i-consistency (DIC_i)

The induced-width

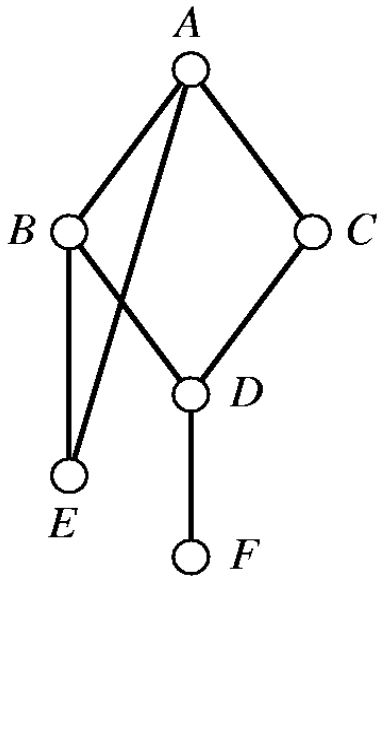
DPC recursively connects parents in the ordered graph, yielding:





- **Width along ordering *d*, w(d):**
  - **max # of previous parents**
- Induced width w*(d):
  - **The width in the ordered *induced graph***
- Induced-width w*:
  - **Smallest induced-width over all orderings**
- Finding w*
  - **NP-complete *(Arnborg, 1985) but greedy heuristics (min-fill).***

Induced-width



(a) (b) (c) (d)

Induced-width and DPC

- The induced graph of (G,d) is denoted (G*,d)
- The induced graph (G*,d) contains the graph generated by DPC along d, and the graph generated by directional i-consistency along d.

Refined complexity using induced-width

Theorem 4.3.11 *Given a binary network $\mathcal{R}$ and an ordering $\mathbf{d}$, the complexity of DPC along $\mathbf{d}$ is $O((w^*(d))^2 \cdot n \cdot k^3)$, where $w^*(\mathbf{d})$ is the induced width of the ordered constraint graph along $\mathbf{d}$.*

Theorem 4.3.13 *Given a general constraint network $\mathcal{R}$ whose constraints' arity is bounded by $\mathbf{i}$, and an ordering $\mathbf{d}$, the complexity of DIC_i along $\mathbf{d}$ is $O(n(w^*(d))^i \cdot (2k)^i)$.* □

- Consequently we wish to have ordering with minimal induced-width
- Induced-width is equal to tree-width to be defined later.
- Finding min induced-width ordering is NP-complete

Greedy algorithms for induced-width

- Min-width ordering
- Min-induced-width ordering
- Max-cardinality ordering
- Min-fill ordering
- Chordal graphs

Min-width ordering

MIN-WIDTH (MW)

input: a graph $G = (V, E)$, $V = \{v_1, ..., v_n\}$

output: A min-width ordering of the nodes $d = (v_1, ..., v_n)$.

1. **for** $j = n$ to 1 by -1 do
2. $\quad r \leftarrow$ a node in G with smallest degree.
3. $\quad$ put r in position j and $G \leftarrow G - r$.
   (Delete from V node r and from E all its adjacent edges)
4. **endfor**

Figure 4.2: The min-width (MW) ordering procedure

Min-induced-width

MIN-INDUCED-WIDTH (MIW)

input: a graph $G = (V, E)$, $V = \{v_1, ..., v_n\}$
output: An ordering of the nodes $d = (v_1, ..., v_n)$.
1. **for** $j = n$ to 1 by -1 do
2. $r \leftarrow$ a node in V with smallest degree.
3. put r in position j.
4. connect r's neighbors: $E \leftarrow E \cup \{(v_i, v_j) | (v_i, r) \in E, (v_j, r) \in E\}$,
5. remove r from the resulting graph: $V \leftarrow V - \{r\}$.

Figure 4.3: The min-induced-width (MIW) procedure

Min-fill algorithm

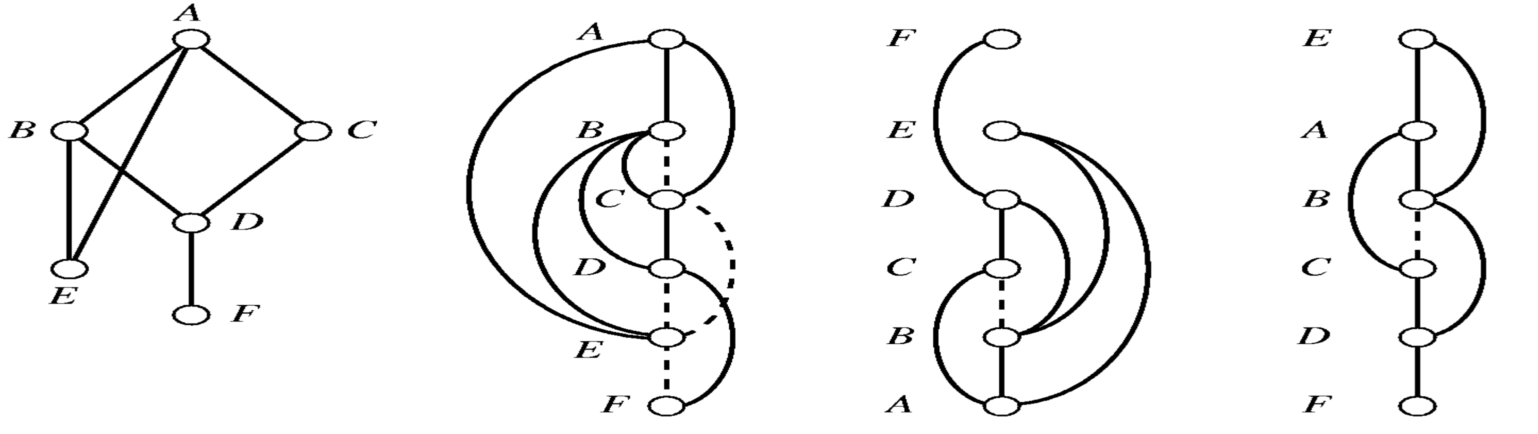
- Prefers a node who adds the least number of fill-in arcs.
- Empirically, fill-in is the best among the greedy algorithms (MW,MIW,MF,MC)

Cordal graphs and max-cardinality ordering

- A graph is cordal if every cycle of length at least 4 has a chord
- Finding w* over chordal graph is easy using the max-cardinality ordering
- If G* is an induced graph it is chordal
- K-trees are special chordal graphs.
- Finding the max-clique in chordal graphs is easy (just enumerate all cliques in a max-cardinality ordering

Example

We see again that *G* in Figure 4.1(a) is not chordal since the parents of *A* are not connected in the max-cardinality ordering in Figure 4.1(d). If we connect *B* and *C*, the resulting induced graph is chordal.



(a) (b) (c) (d)

Max-cardinality ordering

MAX-CARDINALITY (MC)

input: a graph $G = (V, E)$, $V = \{v_1, ..., v_n\}$
output: An ordering of the nodes $d = (v_1, ..., v_n)$.
1. Place an arbitrary node in position 0.
2. **for** $j = 1$ to n do
3. $\quad r \leftarrow$ a node in G that is connected to a largest subset of nodes in positions 1 to $j - 1$, breaking ties arbitrarily.
4. **endfor**

Figure 4.5 The max-cardinality (MC) ordering procedure.

Width vs local consistency: solving trees

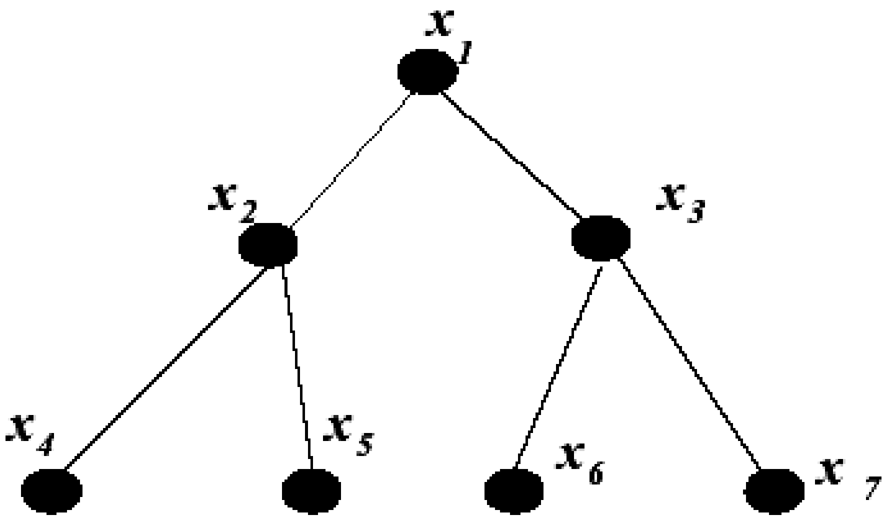


Figure 4.10: A tree network

Theorem 4.4.1 *If a binary constraint network has a width of 1 and if it is arc-consistent, then it is backtrack-free along any width-1 ordering.*

Tree-solving

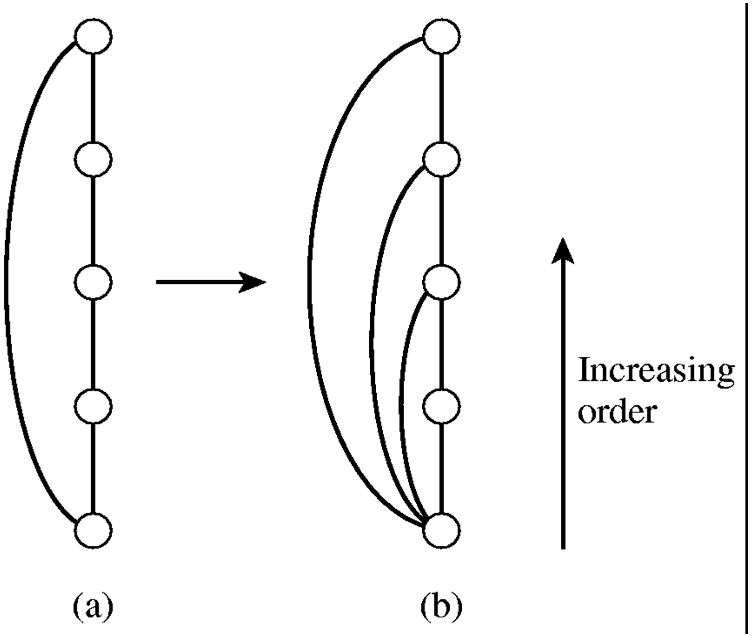
Tree-solving
Input: A tree network $T = (X, D, C)$.
Output: A backtrack-free network along an ordering d.

1. generate a width-1 ordering, $d = x_1, \ldots, x_n$.
2. **let** $x_{p(i)}$ denote the parent of x_i in the rooted ordered tree.
3. **for** $i = n$ to 1 **do**
4. *Revise* $((x_{p(i)}), x_i)$;
5. **if** the domain of $x_{p(i)}$ is empty, exit. (no solution exists).
6. **endfor**

Figure 4.11: Tree-solving algorithm

$$complexity : O(nk^2)$$

Width-2 and DPC



Theorem 4.4.3 (Width-2 and directional path-consistency) *If $\mathcal{R}$ is directional arc and path-consistent along* $\mathbf{d}$, *and if it also has width-2 along* $\mathbf{d}$, *then it is backtrack-free along* $\mathbf{d}$. □

Width vs directional consistency
(Freuder 82)

Theorem 4.4.5 (Width (i-1) and directional i-consistency) *Given a general network $\mathcal{R}$, its ordered constraint graph along $\mathbf{d}$ has a width of $\mathbf{i}-1$ and if it is also strong directional $\mathbf{i}$-consistent, then $\mathcal{R}$ is backtrack-free along $\mathbf{d}$.*

Width vs i-consistency

- DAC and width-1
- DPC and width-2
- DIC_i and width-(i-1)
- → backtrack-free representation

- If a problem has width 2, will DPC make it backtrack-free?
- **Adaptive-consistency**: applies i-consistency when i is adapted to the number of parents

Adaptive-consistency

Adaptive-Consistency (ac1)

Input: a constraint network $\mathcal{R} = (X, D, C)$, its constraint graph $G = (V, E)$, $d = (x_1, \ldots, x_n)$.

output: A backtrack-free network along d

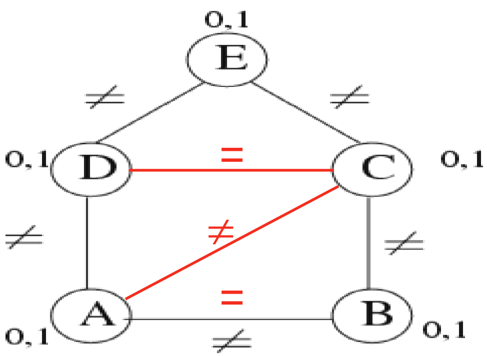
Initialize: $C' \leftarrow C$, $E' \leftarrow E$

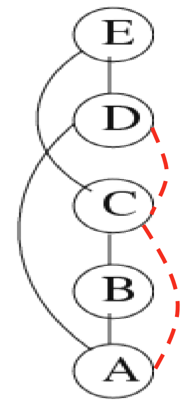
1. **for** $j = n$ to 1 do
2. Let $S \leftarrow parents(x_j)$.
3. $R_S \leftarrow Revise(S, x_j)$ (generate all partial solutions over S that can extend to x_j).
4. $C' \leftarrow C' \cup R_S$
5. $E' \leftarrow E' \cup \{(x_k, x_r) | x_k, x_r \in parents(x_j)\}$ (connect all parents of x_j)
5. **endfor**.

Figure 4.13: Algorithm adaptive-consistency– version 1

Bucket Elimination

Adaptive Consistency (Dechter & Pearl, 1987)





Bucket E: $E \neq D$, $E \neq C$

Bucket D: $D \neq A$ ‖ $D = C$

Bucket C: $C \neq B$ ‖ $A \neq C$

Bucket B: $B \neq A$ ‖ $B = A$

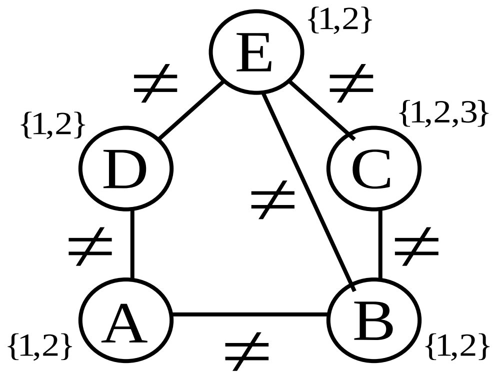
Bucket A: ‖ contradiction

Complexity: nk^{w^*+1}

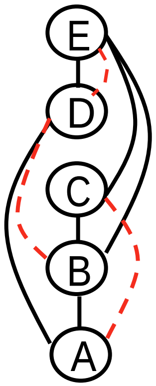
w^* is the induced-width along the ordering

Bucket Elimination

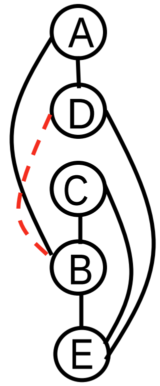
Adaptive Consistency (Dechter & Pearl, 1987)



$$Bucket(E):\ E \neq D,\ E \neq C,\ E \neq B$$
$$Bucket(D):\ D \neq A\ ||\ R_{DCB}$$
$$Bucket(C):\ C \neq B\ ||\ R_{ACB}$$
$$Bucket(B):\ B \neq A\ ||\ R_{AB}$$
$$Bucket(A):\quad R_A$$



$$Bucket(A):\ A \neq D,\ A \neq B$$
$$Bucket(D):\ D \neq E\ ||\ R_{DB}$$
$$Bucket(C):\ C \neq B,\ C \neq E$$
$$Bucket(B):\ B \neq E\ ||\ R^{D}_{BE},\ R^{C}_{BE}$$
$$Bucket(E):\ ||\ R_E$$

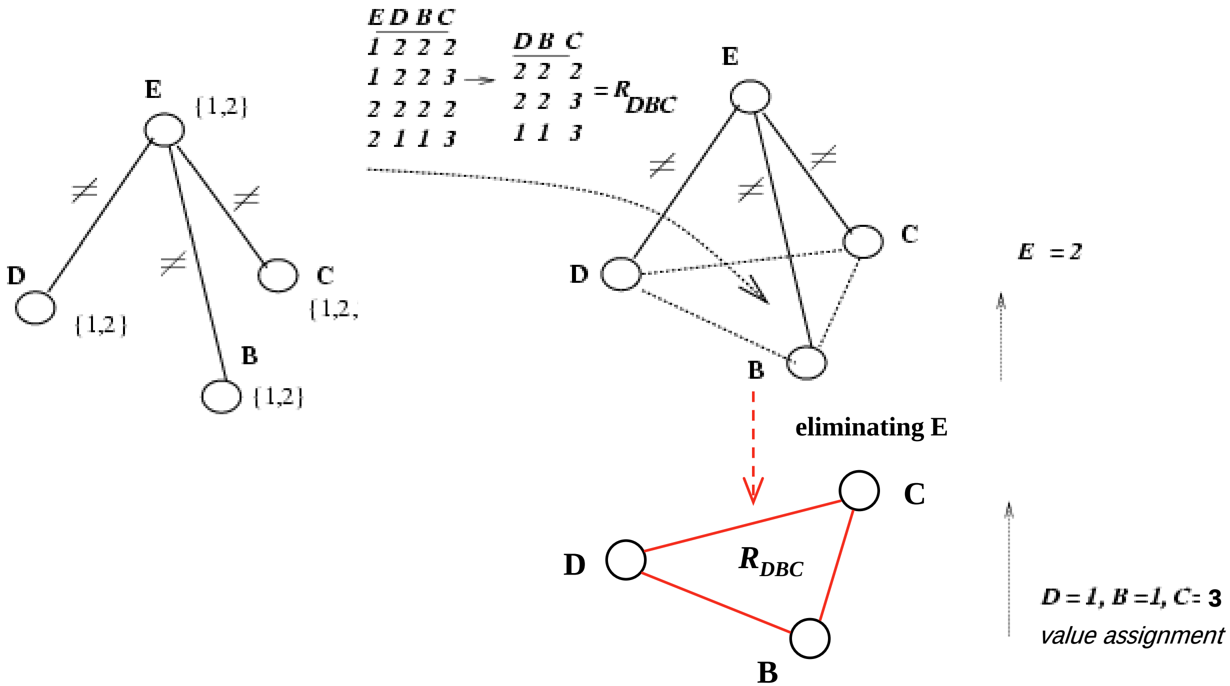


Time : $O(n\,exp(w^*(d)+1))$,

space : $O(n\,exp(w^*(d)))$

w^*(d) - ***induced - width - along - ordering - d***

The Idea of Elimination

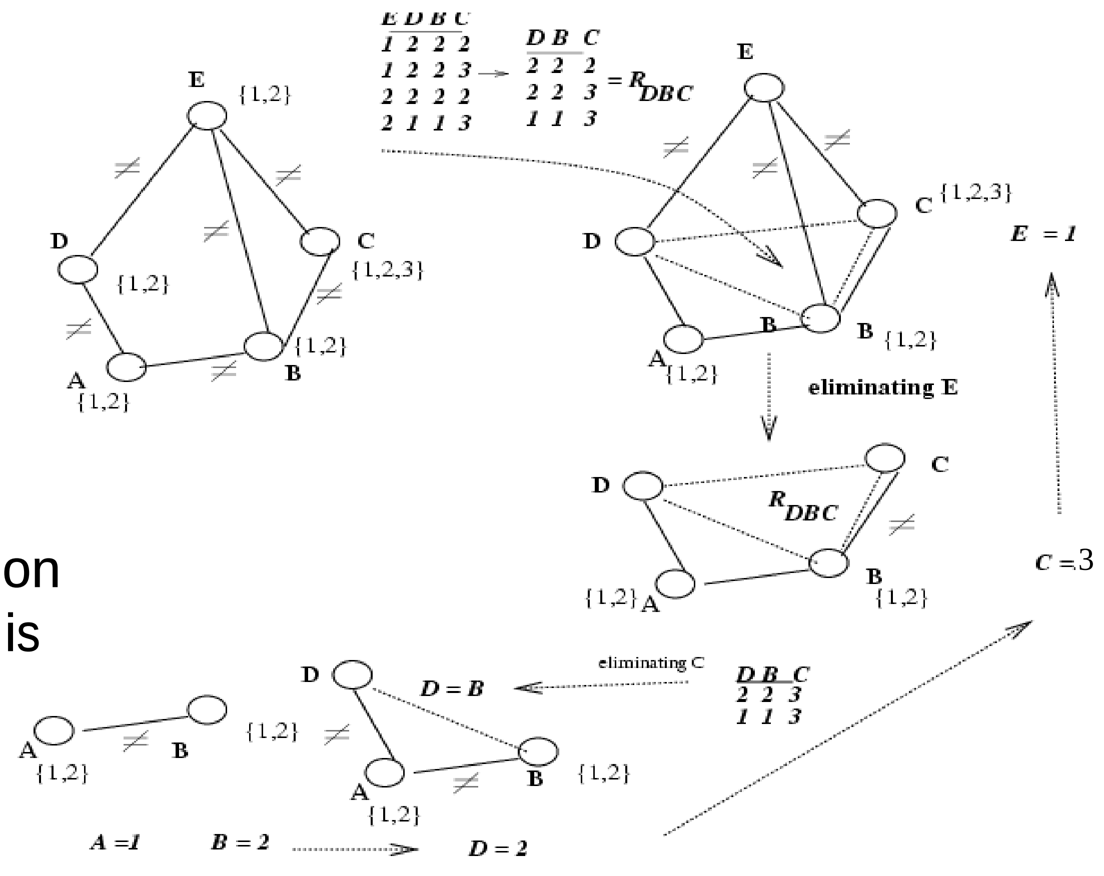


$$R_{DBC} = \prod_{DBC} R_{ED} \bowtie R_{EB} \bowtie R_{EC}$$

Eliminate variable E $\Leftrightarrow$ join and project

Variable Elimination

Eliminate variables one by one: "constraint propagation"



Solution generation after elimination is backtrack-free

Adaptive-consistency, bucket-elimination

ADAPTIVE-CONSISTENCY (AC)

Input: a constraint network $\mathcal{R}$, an ordering $d = (x_1, \ldots, x_n)$
output: A backtrack-free network, denoted $E_d(\mathcal{R})$, along d, if the empty constraint was not generated. Else, the problem is inconsistent

1. Partition constraints into $bucket_1, \ldots, bucket_n$ as follows:
   for $i \leftarrow n$ **downto** 1, put in $bucket_i$ all unplaced constraints mentioning x_i.
2. **for** $p \leftarrow n$ **downto** 1 **do**
3. **for** all the constraints $R_{S_1}, \ldots, R_{S_j}$ in $bucket_p$ **do**
4. $A \leftarrow \bigcup_{i=1}^{j} S_i - \{x_p\}$
5. $R_A \leftarrow \Pi_A(\bowtie_{i=1}^{j} R_{S_i})$
6. **if** R_A is not the empty relation **then** add R_A to the bucket of the latest variable in scope A,
7. **else** exit and return the empty network
8. **return** $E_d(\mathcal{R}) = (X, D, bucket_1 \cup bucket_2 \cup \cdots \cup bucket_n)$

Figure 4.14: Adaptive-Consistency as a bucket-elimination algorithm

Properties of bucket-elimination (adaptive consistency)

- Adaptive consistency generates a constraint network that is **backtrack-free** (can be solved without dead-ends).

- The time and space complexity of adaptive consistency along ordering *d* is $O(n\,(2k)^{w^*+1}), O(n\,(k)^{w^*+1})$ respectively, or $O(r\,k^{w^*+1})$ when r is the number of constraints.

- Therefore, problems having **bounded induced width** are tractable (solved in polynomial time)

- Special cases: *trees* (w*=1), *series-parallel networks* (w*=2), and in general *k-trees* (w*=k).

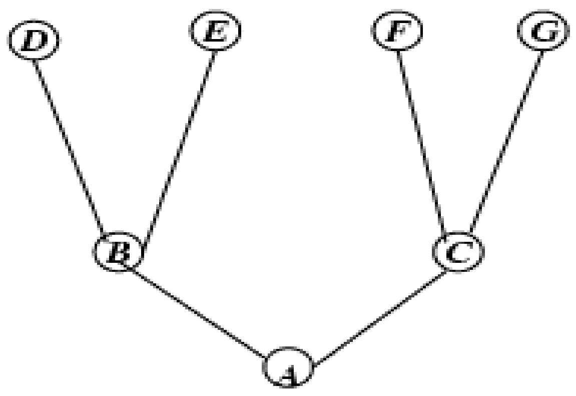
Back to Induced width

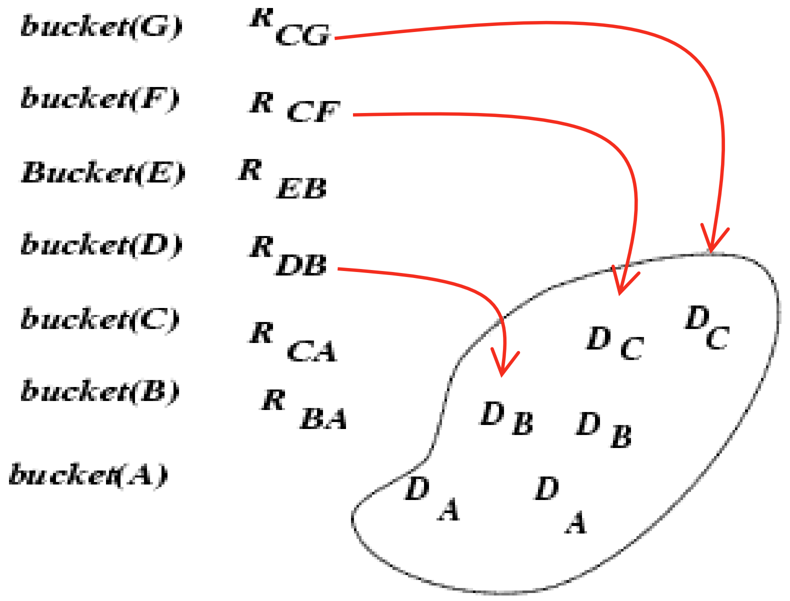
- Finding minimum-w* ordering is NP-complete (Arnborg, 1985)
- Greedy ordering heuristics: *min-width, min-degree, max-cardinality* (Bertele and Briochi, 1972; Freuder 1982), Min-fill.

Solving Trees

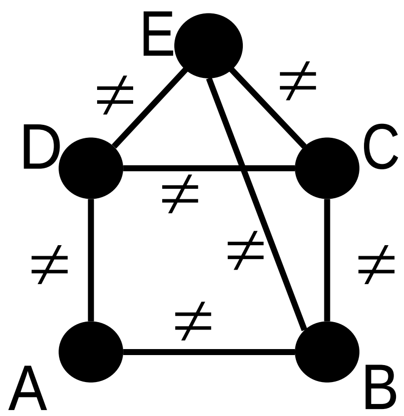
(Mackworth and Freuder, 1985)

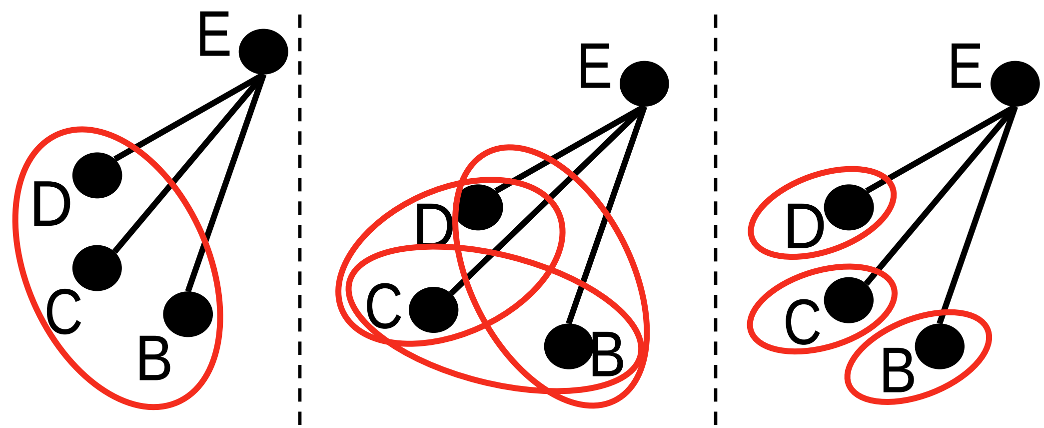
Adaptive consistency is linear for trees and equivalent to enforcing directional arc-consistency (recording only unary constraints)





Summary: directional i-consistency





Adaptive	d-path	d-arc
R_{DCB}	R_{DC}, R_{DB} R_{CB}	R_D R_C R_D

$E: E \neq D, E \neq C, E \neq B$

$D: D \neq C, D \neq A$

$C: C \neq B$

$B: A \neq B$

$A:$

Relational consistency (Chapter 8)

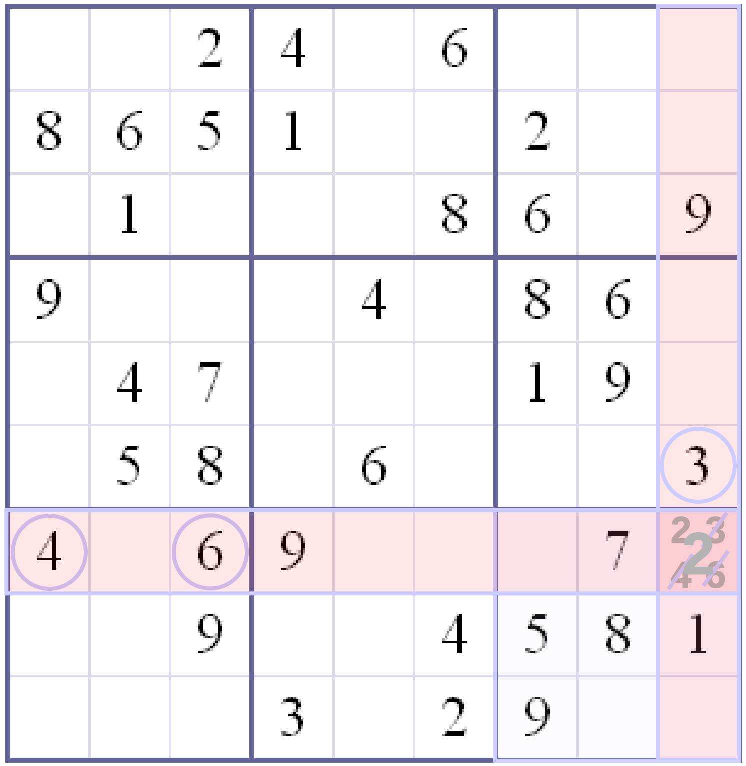
- Relational arc-consistency
- Relational path-consistency
- Relational m-consistency
- Relational consistency for Boolean and linear constraints:
  - Unit-resolution is relational-arc-consistency
  - Pair-wise resolution is relational path-consistency

Sudoku's propagation

- http://www.websudoku.com/
- What kind of propagation we do?

Sudoku

Constraint propagation

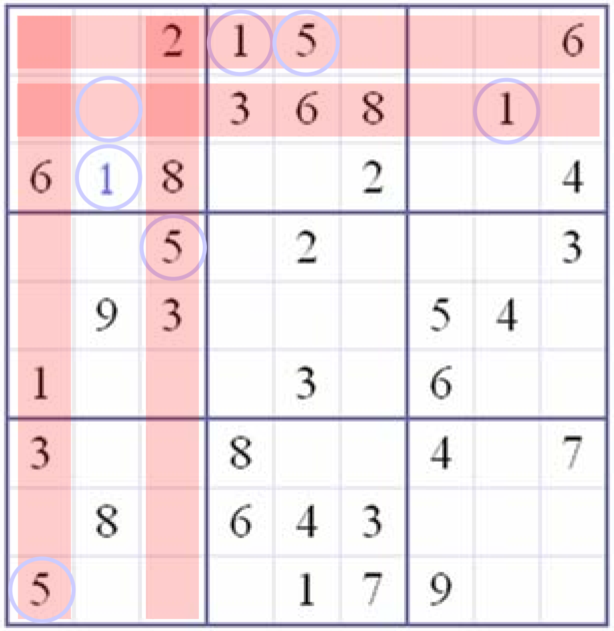


- **Variables: 81 slots**
- **Domains = {1,2,3,4,5,6,7,8,9}**
- **Constraints:**
    - **27 not-equal**

Each row, column and major block must be alldifferent

"Well posed" if it has unique solution: 27 constraints

Sudoku



Each row, column and major block must be alldifferent

“Well posed” if it has unique solution