Relational consistency
(Chapter 8)

- Relational arc-consistency
- Relational path-consistency
- Relational m-consistency

Relational consistency for Boolean and linear constraints:
- Unit-resolution is relational-arc-consistency
- Pair-wise resolution is relational path-consistency
Example

Consider a constraint network over five integer domains, where the constraints take the form of linear equations and the domains are integers bounded by

- $D_x$ in $[-2,3]$
- $D_y$ in $[-5,7]$
- $R_{xyz} : x + y = z$
- $R_{zlt} : z + t = l$
- From $D_x$ and $R_{xyz}$ infer $z-y$ in $[-2,3]$ from this and $D_y$ we can infer $z$ in $[-7,10]$
Relational arc-consistency

Let $R$ be a constraint network, $X = \{x_1, \ldots, x_n\}$, $D_1, \ldots, D_n$, $R_S$ a relation.

$R_S$ in $R$ is relational-arc-consistent relative to $x$ in $S$, iff any consistent instantiation of the variables in $S- \{x\}$ has an extension to a value in $D_x$ that satisfies $R_S$. Namely,

$$\rho(S - x) \subseteq \pi_{S-x} R_S \otimes D_x$$
Enforcing relational arc-consistency

- If arc-consistency is not satisfied add:

\[ R_{S-x} \leftarrow R_{S-x} \cap \pi_{S-x} R_S \otimes D_S \]
Example

- $R_{xyz} = \{(a,a,a),(a,b,c),(b,b,c)\}$.
- This relation is not relational arc-consistent, but if we add the projection:
  
  $R_{xy} = \{(a,a),(a,b),(b,b)\}$, then $R_{xyz}$ will be relational arc-consistent relative to $\{z\}$.
- To make this network relational-arc-consistent, we would have to add all the projections of $R_{xyz}$ with respect to all subsets of its variables.
Relational path-consistency

- Let $R_S$ and $R_T$ be two constraints in a network.
- $R_S$ and $R_T$ are relational-path-consistent relative to a variable $x$ in $S \cup T$ iff any consistent instantiation of variables in $S \cup T - \{x\}$ has an extension to in the domain $D_x$, s.t. $R_S$ and $R_T$ simultaneously;

$$\rho(A) \subseteq \pi_A R_S \otimes R_T \otimes D_x$$

$$A = S \cup T - x$$

- A pair of relations $R_S$ and $R_T$ is relational-path-consistent iff it is relational-path-consistent relative to every variable in $S \cap T$. A network is relational-path-consistent iff every pair of its relations is relational-path-consistent.
We can assign to x, y, l and t values that are consistent relative to the relational-arc-consistent network generated in earlier. For example, the assignment

\((x=2, \ y= -5, \ t=3, \ l=15)\) is consistent, since only domain restrictions are applicable, but no value of z that satisfies \(x+y = z\) and \(z+t = l\).

To make the two constraints relational path-consistent relative to z add: \(x+y+t = l\).
Enforcing relational arc, path and m-consistency

- If arc-consistency is not satisfied add:

\[ R_{S-x} \leftarrow R_{S-x} \cap \pi_{S-x} R_S \otimes D_S \]

\[ \rho(A) \subseteq \pi_A R_S \otimes R_T \otimes D_x \]

\[ A = S \cup T - x \]

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Extended composition

- The extended composition of relation $R_{S_1} \ldots R_{S_m}$ relative to $A$ is defined by
  
  $$EC_A(R_1, \ldots, R_m) = \pi_A(R_1 \otimes R_2 \otimes \ldots \otimes R_m)$$
  
- If the projection operation is restricted to subsets of size $i$, it is called extended $(i,m)$-composition.
- Special cases: domain propagation and relational arc-consistency

$$D_x \leftarrow D_x \cap \pi_x (R_S \otimes D_S)$$

$$R_{S-x} \leftarrow R_{S-x} \cap \pi_{S-x} (R_S \otimes D_S)$$
Example: crossword puzzle, DRC_2

\[ R_{1,2,3,4,5} = \{(H, O, S, E, S), (L, A, S, E, R), (S, H, E, E, T), (S, N, A, I, L), (S, T, E, E, R)\} \]
\[ R_{5,6,9,12} = \{(H, I, K, E), (A, R, O, N), (K, E, E, T), (E, A, R, N), (S, A, M, E)\} \]
\[ R_{5,7,11} = \{(R, U, N), (S, U, N), (L, E, T), (Y, E, S), (E, A, T), (T, E, N)\} \]
\[ R_{8,9,10,11} = R_{3,6,9,12} \]
\[ R_{10,13} = \{(N, O), (B, E), (U, S), (I, T)\} \]
\[ R_{12,13} = R_{10,13} \]
Example: crossword puzzle, Directional-relational-2

\[ R_{1,2,3,4,5} = \{(H, O, S, E, S), (L, A, S, E, R), (S, H, E, E, T), (S, N, A, I, L), (S, T, E, E, R)\} \]
\[ R_{5,6,9,12} = \{(H, I, K, E), (A, R, O, N), (K, E, E, T), (E, A, R, N), (S, A, M, E)\} \]
\[ R_{5,7,11} = \{(R, U, N), (S, U, N), (L, E, T), (Y, E, S), (E, A, T), (T, E, N)\} \]
\[ R_{8,9,10,11} = R_{3,6,9,12} \]
\[ R_{10,13} = \{(N, O), (B, E), (U, S), (I, T)\} \]
\[ R_{12,13} = R_{10,13} \]

\[
\begin{array}{cc}
1 & 2 \\
6 & 7 \\
8 & 9 & 10 & 11 \\
12 & 13 & \\
\end{array}
\]

\[
\begin{array}{c}
\text{bucket}(x_1) \\
\text{bucket}(x_2) \\
\text{bucket}(x_3) \\
\text{bucket}(x_4) \\
\text{bucket}(x_5) \\
\text{bucket}(x_6) \\
\text{bucket}(x_7) \\
\text{bucket}(x_8) \\
\text{bucket}(x_9) \\
\text{bucket}(x_{10}) \\
\text{bucket}(x_{11}) \\
\text{bucket}(x_{12}) \\
\text{bucket}(x_{13}) \\
\end{array}
\]

\[
\begin{array}{c}
R_{1,2,3,4,5} \\
H_{2,3,4,5} \\
R_{3,6,9,12} \\
H_{3,4,5} \\
R_{5,7,11} \\
H_{5,6,9,12} \\
H_{5,6,9,12} \\
R_{8,9,10,11} \\
H_{9,10,11} \\
R_{10,13} \\
H_{10,11,12} \\
\end{array}
\]

Empty relation . . . exit.
Theorem: DRC_2 is exponential in the induced-width.
(because sizes of the recorded relations are exp in w).

Crossword puzzles can be made directional backtrack-free by DRC_2
Domain tightness

- **Theorem**: a strong relational 2-consistent constraint network over bi-valued domains is globally consistent.

- **Theorem**: A strong relational k-consistent constraint network with at most k values is globally consistent.
Inference for Boolean theories

- Resolution is identical to *extended 2 decomposition*
- check: \{ (f \lor x \lor y \lor \neg z), (x \lor y \lor \neg f) \}
- Boolean theories have domain size 2
- Therefore $DRC_2$ makes a cnf globally consistent.
- $DRC_2$ expressed on cnfs is directional resolution
**Directional-resolution**

**Input:** A CNF theory $\varphi$, an ordering $d = Q_1, \ldots, Q_n$ of its variables.

**Output:** A decision of whether $\varphi$ is satisfiable. If it is, a theory $E_d(\varphi)$, equivalent to $\varphi$, else an empty directional extension.

1. **Initialize:** generate an ordered partition of clauses into buckets.
   - $\text{bucket}_1, \ldots, \text{bucket}_n$, where $\text{bucket}_i$ contains all clauses whose highest literal is $Q_i$.
2. **for** $i \leftarrow n \textbf{ downto } 1$ **process** $\text{bucket}_i$:
3. **if** there is a unit clause **then** (the instantiation step)
   - apply unit-resolution in $\text{bucket}_i$ and place the resolvents in their right buckets.
   - **if** the empty clause was generated, theory is not satisfiable.
4. **else** resolve each pair $\{(\alpha \lor Q_i), (\beta \lor \neg Q_i)\} \subseteq \text{bucket}_i$.
   - **if** $\gamma = \alpha \lor \beta$ is empty, return $E_d(\varphi) = \{\}$, theory is not satisfiable
   - **else** determine the index of $\gamma$ and add it to the appropriate bucket.
5. **return** $E_d(\varphi) \leftarrow \bigcup_i \text{bucket}_i$

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Figure 4.20: Directional-resolution
DR resolution = adaptive-consistency=directional relational path-consistency

\[ |bucket_i| = O(\exp(w^*)) \]

DR time and space: \( O(n \exp(w^*)) \)
Directional Resolution $\iff$ Adaptive Consistency

Knowledge compilation

Model generation

bucket A

bucket B

bucket C

bucket D

bucket E

Input

$A \lor B \lor C \lor \neg A \lor B \lor E$

$\neg B \lor C \lor D \lor B \lor C \lor E$

$\neg C \lor C \lor D \lor E$

$D \lor E$

Directional Extension $E_0$

Output

$A = 0$

$B = 1$

$C = 0$

$D = 1$

$E = 0$
Resolution – An Example

\[(x_1 \lor \neg x_2 \lor \neg x_3) \land (\neg x_1 \lor \neg x_2 \lor \neg x_3) \land (x_2 \lor x_3) \land (x_3 \lor x_4) \land (x_3 \lor \neg x_4) \vdash \]

\[(\neg x_2 \lor \neg x_3) \land (x_2 \lor x_3) \land (x_3 \lor x_4) \land (x_3 \lor \neg x_4) \vdash \]
Row convexity

- **Functional constraints**: A binary relation $R_{ij}$ expressed as a $(0,1)$-matrix is functional iff there is at most a single "1" in each row and in each column.

- **Monotone constraints**: Given ordered domain, a binary relation $R_{ij}$ is monotone if $(a,b)$ in $R_{ij}$ and if $c \geq a$, then $(c,b)$ in $R_{ij}$, and if $(a,b)$ in $R_{ij}$ and $c \leq b$, then $(a,c)$ in $R_{ij}$.

- **Row convex constraints**: A binary relation $R_{ij}$ represented as a $(0,1)$-matrix is row convex if in each row (column) all of the ones are consecutive.
Example of row convexity

\[ R_{12} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \]
Theorem:

Let $R$ be a path-consistent binary constraint network. If there is an ordering of the domains $D_1, \ldots, D_n$ of $R$ such that the relations of all constraints are row convex, the network is globally consistent and is therefore minimal.
Linear inequalities

Consider r-ary constraints over a subset of variables $x_1 \ldots x_r$ of the form

$$a_r x_r + \ldots + a_r x_r \leq c,$$

$a_i$ are rational constants. The r-ary inequalities define corresponding r-ary relations that are row convex.

Since r-ary linear inequalities that are closed under relational path-consistency are row-convex, relative to any set of integer domains (using the natural ordering).

**Proposition:** A set of linear inequalities that is closed under $RC_2$ is globally consistent.
Linear inequalities

- Gausian elimination with domain constraint is relational-arc-consistency
- Gaussian elimination of 2 inequalities is relational path-consistency
- **Theorem**: directional relational path-consistency is complete for CNFs and for linear inequalities
**Linear inequalities: Fourier elimination**

**DEFINITION**

**(\varphi, d)**

**Input:** A set of linear inequalities \( \varphi \), an ordering \( d = x_1, \ldots, x_n \).

**Output** A decision of whether \( \varphi \) is satisfiable. If it is, a backtrack-free theory \( E_d(\varphi) \).

1. **Initialize:** Partition inequalities into ordered buckets.
2. **for** \( i \leftarrow n \) **downto** 1 **do**
3. **if** \( x_i \) has one value in its domain **then**
   ```
   substitute the value into each inequality in the bucket and put the resulting inequality in the right bucket.
   ```
4. **else, for each** pair \( \{\alpha, \beta\} \subseteq \text{bucket}_i \), compute \( \gamma = \text{elim}_i(\alpha, \beta) \)
   ```
   if \( \gamma \) has no solutions, return \( E_d(\varphi) = \{\} \), “inconsistency”
   else add \( \gamma \) to the appropriate lower bucket.
5. **return** \( E_d(\varphi) \leftarrow \bigcup_i \text{bucket}_i \)

**Figure 4.22: Fourier Elimination; DLE**
Directional linear elimination, DLE: generates a backtrack-free representation

Theorem 4.8.3 Given a set of linear inequalities \( \varphi \), algorithm DLE (Fourier elimination) decides the consistency of \( \varphi \) over the Rationals and the Reals, and it generates an equivalent backtrack-free representation. \( \square \)
Example

bucket_4 : \[ 5x_4 + 3x_2 - x_1 \leq 5, \ x_4 + x_1 \leq 2, \ -x_4 \leq 0, \]
bucket_3 : \[ x_3 \leq 5, \ x_1 + x_2 - x_3 \leq -10 \]
bucket_2 : \[ x_1 + 2x_2 \leq 0. \]
bucket_1 :

Figure 4.23: initial buckets

bucket_4 : \[ 5x_4 + 3x_2 - x_1 \leq 5, \ x_4 + x_1 \leq 2, \ -x_4 \leq 0, \]
bucket_3 : \[ x_3 \leq 5, \ x_1 + x_2 - x_3 \leq -10 \]
bucket_2 : \[ x_1 + 2x_2 \leq 0 \parallel 3x_2 - x_1 \leq 5, \ x_1 + x_2 \leq -5 \]
bucket_1 : \[ \parallel x_1 \leq 2. \]

Figure 4.24: final buckets