Class Information

• Instructor: Rina Dechter

• Lectures: Monay & Wednesday
• Time: 11:00 - 12:20 pm
• Discussion (optional): Wednesdays 12:30-1:20

• Class page: http://www.ics.uci.edu/~dechter/courses/ics-275a/spring-2014/
Text book (required)

Rina Dechter,

**Constraint Processing**, Morgan Kaufmann
Outline

- Motivation, applications, history
- CSP: Definition, and simple modeling examples
- Mathematical concepts (relations, graphs)
- Representing constraints
- Constraint graphs
- The binary Constraint Networks properties
Outline

- Motivation, applications, history
- CSP: Definition, representation and simple modeling examples
- Mathematical concepts (relations, graphs)
- Representing constraints
- Constraint graphs
- The binary Constraint Networks properties
Graphical Models

Those problems that can be expressed as:

- A set of variables
- Each variable takes its values from a finite set of domain values
- A set of local functions

Main advantage:
- They provide unifying algorithms:
  - Search
  - Complete Inference
  - Incomplete Inference
Many Examples

Combinatorial Problems

MO Optimization

Optimization

Decision

Graphical Models

EOS Scheduling

Bayesian Networks

Graph Coloring

Timetabling

... and many others.
Example: student course selection

- **Context**: You are a senior in college
- **Problem**: You need to register in 4 courses for the Spring semester
- **Possibilities**: Many courses offered in Math, CSE, EE, CBA, etc.
- **Constraints**: restrict the choices you can make
  - Courses have prerequisites you have/don't have
  - Courses/instructors you like/dislike
  - Courses are scheduled at the same time
  - In CE: 4 courses from 5 tracks such as at least 3 tracks are covered

- **You have choices, but are restricted by constraints**
  - Make the right decisions!!
  - [ICS Graduate program](#)
Student course selection (continued)

• **Given**  
  – A set of variables: 4 courses at your college  
  – For each variable, a set of choices (values): the available classes.  
  – A set of constraints that restrict the combinations of values the variables can take at the same time

• **Questions**  
  – Does a solution exist? (classical decision problem)  
  – How many solutions exist? (counting)  
  – How two or more solutions differ?  
  – Which solution is preferable?  
  – etc.
The field of Constraint Programming

• How did it started:
  – Artificial Intelligence (vision)
  – Programming Languages (Logic Programming),
  – Databases (deductive, relational)
  – Logic-based languages (propositional logic)
  – SATisfiability

• Related areas:
  – Hardware and software verification
  – Operation Research (Integer Programming)
  – Answer set programming

• Graphical Models; deterministic
Scene labeling constraint network
Scene labeling constraint network

Fork:

Arrow:

Ell:

Tee:
3-dimensional interpretation of 2-dimensional drawings

Fork:

Arrow:

Ell:

Tee:
The field of Constraint Programming

• **How did it start:**
  – Artificial Intelligence (vision)
  – Programming Languages (Logic Programming),
  – Databases (deductive, relational)
  – Logic-based languages (propositional logic)
  – SATisfiability

• **Related areas:**
  – Hardware and software verification
  – Operation Research (Integer Programming)
  – Answer set programming

• **Graphical Models; deterministic**
Applications

• Radio resource management (RRM)
• Databases (computing joins, view updates)
• Temporal and spatial reasoning
• Planning, scheduling, resource allocation
• Design and configuration
• Graphics, visualization, interfaces
• Hardware verification and software engineering
• HC Interaction and decision support
• Molecular biology
• Robotics, machine vision and computational linguistics
• Transportation
• Qualitative and diagnostic reasoning
Outline

✓ Motivation, applications, history
✓ CSP: Definitions and simple modeling examples
✓ Mathematical concepts (relations, graphs)
✓ Representing constraints
✓ Constraint graphs
✓ The binary Constraint Networks properties
Constraint Networks

Example: map coloring

Variables - countries (A, B, C, etc.)
Values    - colors (red, green, blue)
Constraints: $A \neq B, \ A \neq D, \ D \neq E$, etc.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>red</td>
<td>green</td>
</tr>
<tr>
<td>red</td>
<td>yellow</td>
</tr>
<tr>
<td>green</td>
<td>red</td>
</tr>
<tr>
<td>green</td>
<td>yellow</td>
</tr>
<tr>
<td>yellow</td>
<td>green</td>
</tr>
<tr>
<td>yellow</td>
<td>red</td>
</tr>
</tbody>
</table>
**Example: map coloring**

Variables - countries (A, B, C, etc.)
Values - colors (e.g., red, green, yellow)
Constraints: \(A \neq B, A \neq D, D \neq E, \text{ etc.}\)

|     | A     | B     | C     | D     | E...
|-----|-------|-------|-------|-------|-------
| red | green | red   | green | blue  |
| red | blue  | green | green | blue  |
| ... | ...   | ...   | ...   | green |
| ... | ...   | ...   | ...   | red   |
| red | blue  | red   | green | red   |

**Constraint Satisfaction Tasks**

- Are the constraints consistent?
- Find a solution, find all solutions
- Count all solutions
- Find a good solution
Information as Constraints

• I have to finish my class in 50 minutes
• 180 degrees in a triangle
• Memory in our computer is limited
• The four nucleotides that makes up a DNA only combine in a particular sequence
• Sentences in English must obey the rules of syntax
• Susan cannot be married to both John and Bill
• Alexander the Great died in 333 B.C.
Constraint Network; Definition

• A constraint network is: $R = (X, D, C)$
  - **$X$** variables
    $$X = \{X_1, \ldots, X_n\}$$
  - **$D$** domain
    $$D = \{D_1, \ldots, D_n\}, D_i = \{v_1, \ldots, v_k\}$$
  - **$C$** constraints
    $$C = \{C_1, \ldots, C_t\}, C_i = (S_i, R_i)$$

  - **$R$** expresses allowed tuples over scopes

• **A solution** is an assignment to all variables that satisfies all constraints (join of all relations).

• **Tasks:** consistency?, one or all solutions, counting, optimization
The N-queens problem

The network has four variables, all with domains $D_i = \{1, 2, 3, 4\}$. (a) The labeled chess board. (b) The constraints between variables.

\[
\begin{align*}
R_{12} &= \{(1,3), (1,4), (2,4), (3,1), (4,1), (4,2)\} \\
R_{13} &= \{(1,2), (1,4), (2,1), (2,3), (3,2), (3,4), (4,1), (4,3)\} \\
R_{14} &= \{(1,2), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (3,4), (4,2), (4,3)\} \\
R_{23} &= \{(1,3), (1,4), (2,4), (3,1), (4,1), (4,2)\} \\
R_{24} &= \{(1,2), (1,4), (2,1), (2,3), (3,2), (3,4), (4,1), (4,3)\} \\
R_{34} &= \{(1,3), (1,4), (2,4), (3,1), (4,1), (4,2)\}
\end{align*}
\]
A solution and a partial consistent tuple

Not all consistent instantiations are part of a solution:
(a) A consistent instantiation that is not part of a solution.
(b) The placement of the queens corresponding to the solution (2, 4, 1, 3).
(c) The placement of the queens corresponding to the solution (3, 1, 4, 2).
Example: Crossword puzzle

- Variables: $x_1, \ldots, x_{13}$

- Domains: letters

- Constraints: words from

$$\{\text{HOSES, LASER, SHEET, SNAIL, STEER, ALSO, EARN, HIKE, IRON, SAME, EAT, LET, RUN, SUN, TEN, YES, BE, IT, NO, US}\}$$
Configuration and Design

- Want to build: recreation area, apartment complex, a cluster of 50 single-family houses, cemetery, and a dump

  - Recreation area near lake
  - Steep slopes avoided except for recreation area
  - Poor soil avoided for developments
  - Highway far from apartments, houses and recreation
  - Dump not visible from apartments, houses and lake
  - Lots 3 and 4 have poor soil
  - Lots 3, 4, 7, 8 are on steep slopes
  - Lots 2, 3, 4 are near lake
  - Lots 1, 2 are near highway
Example: Sudoku (constraint propagation)

Each row, column and major block must be alldifferent

“Well posed” if it has unique solution: 27 constraints
Sudoku
(inference)

each row, column and major block must be alldifferent
“Well posed” if it has unique solution
Outline

- Motivation, applications, history
- CSP: Definition, representation and simple modeling examples
- **Mathematical concepts (relations, graphs)**
- Representing constraints
- Constraint graphs
- The binary Constraint Networks properties
Mathematical background

- Sets, domains, tuples
- Relations
- Operations on relations
- Graphs
- Complexity
Two Representations of a relation:
\[ R = \{(\text{black, coffee}), (\text{black, tea}), (\text{green, tea})\}. \]

Variables: Drink, color

\[
\begin{array}{c|c}
  x_1 & x_2 \\
  \hline
  \text{black} & \text{coffee} \\
  \text{black} & \text{tea} \\
  \text{green} & \text{tea} \\
\end{array}
\]

(a) table
Two Representations of a relation: 
\[ R = \{(black, coffee), (black, tea), (green, tea)\}. \]

Variables: Drink, color

(a) table

\[
\begin{array}{c|c}
  x_1 & x_2 \\
  black & coffee \\
  black & tea \\
  green & tea \\
\end{array}
\]

(b) (0,1)-matrix

\[
\begin{array}{c|ccc}
  x_2 & 0 & 1 & 1 \\
apple juice & tea & coffee \\
\end{array}
\]

\[
\begin{array}{c|ccc}
  x_1 & 0 & 1 & 1 \\
black & \text{coffee} & \text{tea} \\
\end{array}
\]

\[
\begin{array}{c|ccc}
  x_1 & 0 & 0 & 1 \\
green & \text{coffee} & \text{tea} \\
\end{array}
\]
### Three Relations

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>s</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>n</td>
<td>n</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>1</td>
</tr>
<tr>
<td>b</td>
<td>c</td>
<td>2</td>
</tr>
<tr>
<td>b</td>
<td>c</td>
<td>3</td>
</tr>
</tbody>
</table>

(a) Relation $R$

(b) Relation $R''$

(c) Relation $R'''$
Operations with relations

• Intersection
• Union
• Difference
• Selection
• Projection
• Join
• Composition
• **Relations are special case of a Local function**

\[ f : \prod_{x_i \in Y} D_i \rightarrow A \]

where

\( \text{var}(f) = Y \subseteq X: \) scope of function \( f \)

\( A: \) is a set of valuations

• **In constraint networks:** functions are boolean

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( f )</th>
<th>relation</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>true</td>
<td></td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>a</td>
<td>b</td>
<td>false</td>
<td></td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>false</td>
<td></td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>true</td>
<td></td>
<td>b</td>
<td>b</td>
</tr>
</tbody>
</table>

Winter 2016
Example of Set Operations: intersection, union, and difference applied to relations.

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>s</td>
</tr>
</tbody>
</table>

(a) Relation $R$

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>e</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>s</td>
</tr>
<tr>
<td>c</td>
<td>n</td>
<td>n</td>
</tr>
</tbody>
</table>

(b) Relation $R''$

<table>
<thead>
<tr>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>1</td>
</tr>
<tr>
<td>b</td>
<td>c</td>
<td>2</td>
</tr>
<tr>
<td>b</td>
<td>c</td>
<td>3</td>
</tr>
</tbody>
</table>

(c) Relation $R'''$

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>s</td>
</tr>
<tr>
<td>c</td>
<td>n</td>
<td>n</td>
</tr>
</tbody>
</table>

(a) $R \cap R''$

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>s</td>
</tr>
<tr>
<td>c</td>
<td>n</td>
<td>n</td>
</tr>
</tbody>
</table>

(b) $R \cup R''$

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>s</td>
</tr>
</tbody>
</table>
## Selection, Projection, and Join

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>s</td>
</tr>
</tbody>
</table>

(a) Relation $R$

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>1</td>
</tr>
<tr>
<td>b</td>
<td>c</td>
<td>2</td>
</tr>
<tr>
<td>b</td>
<td>c</td>
<td>3</td>
</tr>
</tbody>
</table>

(b) Relation $R''$

<table>
<thead>
<tr>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>e</td>
</tr>
<tr>
<td>b</td>
<td>c</td>
<td>e</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>e</td>
</tr>
</tbody>
</table>

(c) $R' \Join R''$

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>e</td>
</tr>
<tr>
<td>b</td>
<td>c</td>
<td>e</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>e</td>
</tr>
</tbody>
</table>

(a) $\sigma_{x_3 = c}(R')$

<table>
<thead>
<tr>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>n</td>
<td>n</td>
</tr>
</tbody>
</table>

(b) $\pi_{\{x_2, x_3\}}(R')$

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>c</td>
<td>3</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>e</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>3</td>
</tr>
</tbody>
</table>
• Join: \( f \otimes g \)

\[
\begin{array}{c|c|c}
  x_1 & x_2 & f \\
  \hline
  a & a & \text{true} \\
  a & b & \text{false} \\
  b & a & \text{false} \\
  b & b & \text{true} \\
\end{array}
\quad \quad
\begin{array}{c|c|c}
  x_2 & x_3 & g \\
  \hline
  a & a & \text{true} \\
  a & b & \text{true} \\
  b & a & \text{true} \\
  b & b & \text{false} \\
\end{array}
\quad \quad
\begin{array}{c|c|c|c|c}
  x_1 & x_2 & x_3 & h \\
  \hline
  a & a & a & \text{true} \\
  a & a & b & \text{true} \\
  a & b & a & \text{false} \\
  a & b & b & \text{false} \\
  b & a & a & \text{false} \\
  b & a & b & \text{false} \\
  b & b & a & \text{true} \\
  b & b & b & \text{false} \\
\end{array}
\]

• Logical AND: \( f \wedge g \)

\[
\begin{array}{c|c|c|c}
  x_1 & x_2 & f & \wedge \\
  \hline
  a & a & \text{true} & \text{true} \\
  a & b & \text{false} & \text{false} \\
  b & a & \text{false} & \text{false} \\
  b & b & \text{true} & \text{false} \\
\end{array}
\]

Winter 2016
Outline

- Motivation, applications, history
- CSP: Definition, representation and simple modeling examples
- Mathematical concepts (relations, graphs)
- Representing constraints/ Languages
- Constraint graphs
- The binary Constraint Networks properties
Modeling; Representing a problem

- If a CSP $M = <X,D,C>$ represents a real problem $P$, then every solution of $M$ corresponds to a solution of $P$ and every solution of $P$ can be derived from at least one solution of $M$.

- The variables and values of $M$ represent entities in $P$.

- The constraints of $M$ ensure the correspondence between solutions.

- The aim is to find a model $M$ that can be solved as quickly as possible.

- **Goal of modeling:** choose a set of variables and values that allows the constraints to be expressed easily and concisely.
Given a proposition theory $\varphi = \{(A \lor B), (C \lor \neg B)\}$ does it have a model?

Can it be encoded as a constraint network?

Variables: $\{A, B, C\}$

Domains: $D_A = D_B = D_C = \{0, 1\}$

Relations:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

If this constraint network has a solution, then the propositional theory has a model.
Constraint’s representations

- Relation: allowed tuples
  
  \[ \begin{array}{ccc} 
  X & Y & Z \\
  1 & 3 & 2 \\
  2 & 1 & 3 \\
  \end{array} \]

- Algebraic expression:
  \[ X + Y^2 \leq 10, X \neq Y \]

- Propositional formula:
  \[ (a \lor b) \rightarrow \neg c \]

- A decision tree, a procedure

- Semantics: by a relation
Outline

☑ Motivation, applications, history
☑ CSP: Definition, representation and simple modeling examples
☑ Mathematical concepts (relations, graphs)
☑ Representing constraints
☑ Constraint graphs
☑ The binary Constraint Networks properties
• A (primal) constraint graph: a node per variable, arcs connect constrained variables.
• A dual constraint graph: a node per constraint’s scope, an arc connect nodes sharing variables = hypergraph
Graph Concepts Reviews:
Hyper Graphs and Dual Graphs

A hypergraph

Primal graphs

Dual graph

Factor graphs
Example: Cryptarithmetic

Variables: \(F, T, U, W, R, O, X_1, X_2, X_3\)

Domains: \(\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}\)

Constraints: \textit{Alldiff} \((F, T, U, W, R, O)\)

\[
\begin{align*}
O + O &= R + 10 \cdot X_1 \\
X_1 + W + W &= U + 10 \cdot X_2 \\
X_2 + T + T &= O + 10 \cdot X_3 \\
X_3 &= F, \ T \neq 0, \ F \neq 0
\end{align*}
\]

What is the primal graph? What is the dual graph?
Propositional Satisfiability

\[ \varphi = \{ (\neg C), (A \lor B \lor C), (\neg A \lor B \lor E), (\neg B \lor C \lor D) \}. \]
Given a telecommunication network (where each communication link has various antennas), assign a frequency to each antenna in such a way that all antennas may operate together without noticeable interference.

**Encoding?**

Variables: one for each antenna

Domains: the set of available frequencies

Constraints: the ones referring to the antennas in the same communication link
Constraint graphs of 3 instances of the Radio frequency assignment problem in CELAR’s benchmark
Examples

Scheduling problem

Five tasks: T1, T2, T3, T4, T5
Each one takes one hour to complete
The tasks may start at 1:00, 2:00 or 3:00
Requirements:
  T1 must start after T3
  T3 must start before T4 and after T5
  T2 cannot execute at the same time as T1 or T4
  T4 cannot start at 2:00

Encoding?

Variables: one for each task
Domains: \( D_{T1} = D_{T2} = D_{T3} = D_{T4} = \{1:00, 2:00, 3:00\} \)
Constraints:

\[
\begin{array}{c}
T4 \\
1:00 \\
3:00
\end{array}
\]
The constraint graph and relations of scheduling problem

Unary constraint
\[ D_{T4} = \{1:00, 3:00\} \]

Binary constraints
\[ R_{\{T1,T2\}}: \{(1:00,2:00), (1:00,3:00), (2:00,1:00), (2:00,3:00), (3:00,1:00), (3:00,2:00)\} \]
\[ R_{\{T1,T3\}}: \{(2:00,1:00), (3:00,1:00), (3:00,2:00)\} \]
\[ R_{\{T2,T4\}}: \{(1:00,2:00), (1:00,3:00), (2:00,1:00), (2:00,3:00), (3:00,1:00), (3:00,2:00)\} \]
\[ R_{\{T3,T4\}}: \{(1:00,2:00), (1:00,3:00), (2:00,3:00)\} \]
\[ R_{\{T3,T5\}}: \{(2:00,1:00), (3:00,1:00), (3:00,2:00)\} \]
A combinatorial circuit: $M$ is a multiplier, $A$ is an adder
Outline

- Motivation, applications, history
- CSP: Definition, representation and simple modeling examples
- Mathematical concepts (relations, graphs)
- Representing constraints
- Constraint graphs
- The binary Constraint Networks properties
Properties of Binary Constraint Networks

A graph $\mathcal{R}$ to be colored by two colors, an equivalent representation $\mathcal{R}'$ having a newly inferred constraint between $x_1$ and $x_3$.

Equivalence and deduction with constraints (composition)
Composition of relations (Montanari'74)

**Input:** two binary relations $R_{ab}$ and $R_{bc}$ with 1 variable in common.

**Output:** a new induced relation $R_{ac}$ (to be combined by intersection to a pre-existing relation between them, if any).

**Bit-matrix operation:** matrix multiplication

$$R_{ac} = R_{ab} \cdot R_{bc}$$

$$R_{ab} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad R_{bc} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R_{ac} = ?$$
Equivalence, Redundancy, Composition

• Equivalence: Two constraint networks are equivalent if they have the same set of solutions.

• Composition in matrix notation

\[ R_{xz} = R_{xy} \cdot R_{yz} \]

• Composition in relational operation

\[ R_{xz} = \pi_{xz}(R_{xy} \boxdot R_{yz}) \]
Relations vs Networks

• Can we represent by binary constraint networks the relations
  
  $R(x_1, x_2, x_3) = \{(0,0,0)(0,1,1)(1,0,1)(1,1,0)\}$
  
  $R(x_1, x_2, x_3, x_4) = \{(1,0,0,0)(0,1,0,0) (0,0,1,0)(0,0,0,1)\}$

• Number of relations $2^{kn}$

• Number of networks: $2^{n^2k}$

• Most relations cannot be represented by binary constraint networks
The N-queens constraint network
Is there a tighter network?

The network has four variables, all with domains $Di = \{1, 2, 3, 4\}$.
(a) The labeled chess board. (b) The constraints between variables.

Solutions are: $(2,4,1,3)$ $(3,1,4,2)$
The 4-queens constraint network:
(a) The constraint graph. (b) The minimal binary constraints. (c) The minimal unary constraints (the domains).

Solutions are: (2,4,1,3) (3,1,4,2)
The projection networks

• The projection network of a relation is obtained by projecting it onto each pair of its variables (yielding a binary network).

• \( Relation = \{(1,1,2)(1,2,2)(1,2,1)\} \)
  – What is the projection network?

• What is the relationship between a relation and its projection network?

• \( \{(1,1,2)(1,2,2)(2,1,3)(2,2,2)\} \) are the solutions of its projection network?
Projection network (continued)

• **Theorem**: *Every relation is included in the set of solutions of its projection network.*

• **Theorem**: *The projection network is the tightest upper bound binary networks representation of the relation.*

Therefore, if a network cannot be represented by its projection network it has no binary network representation.
Partial Order between networks, The Minimal Network

Definition 2.3.10 Given two binary networks, $\mathcal{R}'$ and $\mathcal{R}$, on the same set of variables $x_1, ..., x_n$, $\mathcal{R}'$ is at least as tight as $\mathcal{R}$ iff for every $i$ and $j$, $R'_{i,j} \subseteq R_{i,j}$.

• An intersection of two networks is tighter (as tight) than both
• An intersection of two equivalent networks is equivalent to both

Definition 2.3.14 Let $\{\mathcal{R}_1, ..., \mathcal{R}_l\}$ be the set of all networks equivalent to $\mathcal{R}_0$ and let $\rho = \text{sol}(\mathcal{R}_0)$. Then the minimal network $M$ of $\mathcal{R}_0$ is defined by $M(\mathcal{R}_0) = \cap_{i=1}^l \mathcal{R}_i$.

Theorem 2.3.15 For every binary network $\mathcal{R}$ s.t. $\rho = \text{sol}(\mathcal{R})$, $M(\rho) = P(\rho)$. 
The N-queens constraint network.

The network has four variables, all with domains $D_i = \{1, 2, 3, 4\}$.
(a) The labeled chess board. (b) The constraints between variables.

\[
\begin{align*}
R_{12} &= \{(1,3), (1,4), (2,4), (3,1), (4,1), (4,2)\} \\
R_{13} &= \{(1,2), (1,4), (2,1), (2,3), (3,2), (3,4), (4,1), (4,3)\} \\
R_{14} &= \{(1,2), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (3,4) \\
&\quad (4,2), (4,3)\} \\
R_{23} &= \{(1,3), (1,4), (2,4), (3,1), (4,1), (4,2)\} \\
R_{24} &= \{(1,2), (1,4), (2,1), (2,3), (3,2), (3,4), (4,1), (4,3)\} \\
R_{34} &= \{(1,3), (1,4), (2,4), (3,1), (4,1), (4,2)\}
\end{align*}
\]
The 4-queens constraint network:
(a) The constraint graph. (b) The minimal binary constraints. (c) The minimal unary constraints (the domains).

Solutions are: (2,4,1,3) (3,1,4,2)