A Linear Programming Approach to Max-sum Problem: A Review

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Max-Sum Problem

\[
\max_{x \in X^T} \left[ \sum_{t \in T} g_t(x_t) + \sum_{\{t,t'\} \in E} g_{tt'}(x_t, x_{t'}) \right]
\]

e.g. the MAP problem on MRFs
Formulation of the Problem

\[ G = (T, E) \]

\( T \) is a set of objects, \( x_t \in X \) is a labeling on \( t \)

\[ E \subseteq \binom{T}{2} \]

\[ G' = (T \times X, E_x) \]

\( g_t = (t, x) \) \( g_{tt'} = \{(t, x), (t', x')\} \)
Commutative Semirings

\[(\bigoplus_{x \in X^{\left\{ t \right\}}} g_t(x_t) \otimes g_{tt'}(x_t, x'_t))\]

<table>
<thead>
<tr>
<th>((S, \oplus, \otimes))</th>
<th>task</th>
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</thead>
<tbody>
<tr>
<td>(({0, 1}, \lor, \land))</td>
<td>or-and problem, CSP</td>
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<tr>
<td>([-\infty, \infty), \min, \max)</td>
<td>min-max problem</td>
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<tr>
<td>([-\infty, \infty), \max, +)</td>
<td>max-sum problem</td>
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<tr>
<td>([0, \infty), +, \ast)</td>
<td>sum-product problem</td>
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Rings

\((S, \bullet, \circ)\)

\((Z, +, \ast)\)
Semirings
Semirings

CSP

Denote a problem by \((G,X,\bar{g})\) – Graph, Domain, Constraints
Let \(\bar{g}_t(x),\ \bar{g}_{tt'}(x,x') = \{0,1\}\) say if an assignment is allowed or forbidden

\[
\bar{L}_{G,X}(\bar{g}) = \left\{ x \in X^T \mid \bigwedge_t \bar{g}_t(x_t) \land \bigwedge_{\{t,t'\}} \bar{g}_{tt'}(x_t, x_{t'}) = 1 \right\}
\]
Arc consistency in CSP

\[ \bar{g}_{tt'}(x, x') \in \{0, 1\} \]

\[ \bigvee_{x'} \bar{g}_{tt'}(x, x') = \bar{g}_t(x), \quad \{t, t'\} \in E, \ x \in X \]

The kernel can be obtained by iteratively applying the following relations until no more 0 assignments are made (arc consistency algorithm)

\[ \bar{g}_t(x) := \bar{g}_t(x) \land \bigvee_{x'} \bar{g}_{tt'}(x, x') , \]

\[ \bar{g}_{tt'}(x, x') := \bar{g}_{tt'}(x, x') \land \bar{g}_t(x) \land \bar{g}_{t'}(x') \]
Semirings
Max-sum

Denote a problem by \((G, X, g)\) – Graph, Assignments, Weights

\[
F(x \mid g) = \sum_{t \in T} g_t(x_t) + \sum_{\{t, t'\} \in E} g_{tt'}(x_t, x_{t'})
\]

\[
L_{G,X}(g) = \arg\max_{x \in X^T} F(x \mid g)
\]
Equivalent Transformations

Also known as ERs (Wainwright)

A problem is called equivalent if \((G,X,g)\) and \((G,X,g')\) produce the same problem, denoted as \(g \sim g'\)

The simplest such transformation adds a number \(\phi_{tt'}(x)\) to \(g_t(x)\) while removing from \(g_{tt'}(x,x')\)

This formulation corresponds to potentials or messages from message passing

\[
g_{t}^\phi(x) = g_t(x) + \sum_{t' \in N_t} \phi_{tt'}(x),
\]

\[
g_{tt'}^\phi(x, x') = g_{tt'}(x, x') - \phi_{tt'}(x) - \phi_{t't}(x')
\]
Schlesinger’s Upper Bound

\[ u_t = \max_x g_t(x), \quad u_{tt'} = \max_{x,x'} g_{tt'}(x, x') \]

\[ U(g) = \sum_t u_t + \sum_{\{t,t'\}} u_{tt'} \]

\[ U^*(g) = \min_{\varphi \in \mathbb{R}^p} \left[ \sum_t \max_x g_{t}^\varphi(x) + \sum_{\{t,t'\}} \max_{x,x'} g_{tt'}^\varphi(x, x') \right] \]
Triviality

(t,x) is a maximal node if \( g_t(x) = u_t \)

\{ (t,x), (t',x') \} is a maximal edge if \( g_{tt'}(x,x') = u_{tt'} \)

\( \bar{g}_t(x) = [[g_t(x) = u_t]] \quad \bar{g}_{tt'}(x) = [[g_{tt'}(x,x') = u_{tt'}]] \)

A max-sum problem is **trivial** if a labeling can be formed of a subset of its maximal nodes and edges

**Theorem 4.** Let \( C \) be a class of equivalent max-sum problems. Let \( C \) contain a trivial problem. Then, any problem in \( C \) is trivial if and only if its height is minimal in \( C \).
Triviality

**Theorem 4.** Let $C$ be a class of equivalent max-sum problems. Let $C$ contain a trivial problem. Then, any problem in $C$ is trivial if and only if its height is minimal in $C$.

1. minimize the problem height by equivalent transformations and
2. test the resulting problem for triviality.

Testing for triviality of a max-sum problem is correspondent to solving the CSP generated by its maximal nodes and edges

A CSP is a tight solution to all max-sum problems it can be equivalently transformed into
Equivalent Transformations

- eq. problems with min. height (CSPs)
- eq. problems with equal height
- eq. problems
- all max–sum problems
Linear Programming Relaxation

\[ \alpha_{tt'}(x, x') = \alpha_t(x), \quad \{t, t'\} \in E, \ x \in X, \]

\[ \sum_x \alpha_t(x) = 1, \quad t \in T, \]

\[ \alpha \geq 0, \]

This gives the polytope \( \Lambda_{G,X} \) which has a set of optimal vertices given by

\[ \Lambda_{G,X}(g) = \arg \max_{\alpha \in \Lambda_{G,X}} \langle g, \alpha \rangle \]

\[ \langle g, \alpha \rangle = \sum_t \sum_x \alpha_t(x) g_t(x) + \sum_{\{t, t'\}} \sum_{x, x'} \alpha_{tt'}(x, x') g_{tt'}(x, x') \]
Duality of the Relaxations

\[
\begin{align*}
\langle g, \alpha \rangle & \rightarrow \max_{\alpha} \quad \sum_{t \in T} u_t + \sum_{\{t, t'\} \in E} u_{tt'} & \rightarrow \min_{\varphi, u} \\
\sum_{x' \in X} \alpha_{tt'}(x, x') &= \alpha_t(x) \quad \varphi_{tt'}(x) \in \mathbb{R}, \quad \{t, t'\} \in E, \; x \in X \\
\sum_{x \in X} \alpha_t(x) &= 1 \quad u_t \in \mathbb{R}, \quad t \in T \\
\sum_{x, x' \in X} \alpha_{tt'}(x, x') &= 1 \quad u_{tt'} \in \mathbb{R}, \quad \{t, t'\} \in E \\
\alpha_t(x) &\geq 0 \quad u_t - \sum_{t' \in N_t} \varphi_{tt'}(x) \geq g_t(x), \quad t \in T, \; x \in X \\
\alpha_{tt'}(x, x') &\geq 0 \quad u_{tt'} + \varphi_{tt'}(x) + \varphi_{t't}(x') \geq g_{tt'}(x, x'), \quad \{t, t'\} \in E, \; x, x' \in X
\end{align*}
\]

Theorem 5. The height of \((G, X, g)\) is minimal of all its equivalents if and only if \((G, X, g)\) is relaxed-satisfiable. If it is so, then \(\Lambda_{G,X}(g) = \tilde{\Lambda}_{G,X}(g)\).
More theorems fall out

Theorem 6. Let \((G, X, \bar{g}^*)\) be the kernel of a CSP \((G, X, g)\).
Then, \(\bar{\Lambda}_{G,X}(g) = \bar{\Lambda}_{G,X}(g^*)\).

Theorem 7. A nonempty kernel of \((G, X, g)\) is necessary for its relaxed satisfiability and, hence, for minimal height of \((G, X, g)\).

Finding the kernel does not guarantee finding a solution for the minimal upper bound
Obvious by approach from CSPs

For problems of boolean variables \(|X| = 2\)
finding the kernel is necessary and sufficient for finding the upper bound

\(\bar{g}\) satisfiable \(\Rightarrow\) \(\bar{g}\) relaxed-satisfiable
\(g\) trivial \(\Rightarrow\) height of \(g\) minimal \(\Rightarrow\) kernel of \(\bar{g}\) nonempty
(Super) Submodularity

Known that the (super) submodularity property produces max-sum problems with tractable solutions by conversion to max-flow/min-cut problems.

Has been suggested that supermodularity is the discrete counterpart of convexity. Lots of work shows that the LP relaxation for a supermodular max-sum problem is tight.

Supermodular max-sum problems will always form a **lattice CSP** with a tractable solution.
An application (not just theory!)