Fast Gradient-Descent Methods for Temporal-Difference Learning with Linear Function Approximation

Rich Sutton, University of Alberta
Hamid Maei, University of Alberta
Doina Precup, McGill University
Shalabh Bhatnagar, Indian Institute of Science, Bangalore
David Silver, University of Alberta
Csaba Szepesvari, University of Alberta
Eric Wiewiora, University of Alberta
a breakthrough in RL

• function approximation in TD learning is now straightforward

• as straightforward as it is in supervised learning

• TD learning can now be done as gradient-descent in a novel Bellman error
limitations (for this paper)

• linear function approximation
• one-step TD methods ($\lambda = 0$)
• prediction (policy evaluation), not control
limitations (for this paper)

• linear function approximation
• one-step TD methods ($\lambda = 0$)
• prediction (policy evaluation), not control

all of these are being removed in current work
keys to the breakthrough

• a new Bellman error objective function
• an algorithmic trick—a second set of weights
  • to estimate one of the sub-expectations
• and avoid the need for double sampling
• introduced in prior work (Sutton, Szepesvari & Maei, 2008)
outline

• ways in which TD with FA has not been straightforward

• the new Bellman error objective function

• derivation of new algorithms (the trick)

• results (theory and experiments)
TD+FA was not straightforward

• with linear FA, off-policy methods such as Q-learning diverge on some problems (Baird, 1995)

• with nonlinear FA, even on-policy methods can diverge (Tsitsiklis & Van Roy, 1997)

• convergence guaranteed only for one very important special case—linear FA, learning about the policy being followed

• second-order or importance-sampling methods are complex, slow or messy

• no true gradient-descent methods
Baird’s counterexample

- a simple Markov chain
- linear FA, all rewards zero
- deterministic, expectation-based full backups (as in DP)
- each state updated once per sweep (as in DP)
- weights can diverge to $\pm \infty$

![Graph showing error over sweeps for TD, GTD, GTD2, and TDC methods.](image-url)
outline

• ways in which TD with FA has not been straightforward

• the new Bellman error objective function

• derivation of new algorithms (the trick)

• results (theory and experiments)
e.g. linear value-function approximation in Computer Go

state

$S$

$10^{35}$ states
e.g. linear value-function approximation in Computer Go

$10^{35}$ states $10^5$ binary features
e.g. linear value-function approximation in Computer Go

\[ S \to \Phi_S \times \Theta \]

10^{35} states \quad 10^5 \text{ binary features and parameters}
e.g. linear value-function approximation in Computer Go

\[ \text{state} \]

\[ s \]

\[ \phi_s \]

\[ [0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0] \]

\[ [0.1, -2, 0, 0.5, 0, 0, 0, 5, -0.4] \]

\[ \theta \]

\[ 10^{35} \text{ states} \]

\[ 10^5 \text{ binary features and parameters} \]
e.g. linear value-function approximation in Computer Go

\[ \sum s \phi_s \theta = -2 + 0 + 5 = 3 \]

10³⁵ states 10⁵ binary features and parameters
Notation

• state transitions: \( s \xrightarrow{r} s' \)

• feature vectors: \( \phi, \phi' \in \mathbb{R}^n \quad n \ll \#\text{states} \)

• approximate values: \( V_\theta(s) = \theta^\top \phi \quad \theta \in \mathbb{R}^n \) parameter vector
Notation

- state transitions: \( s \xrightarrow{r} s' \)

- feature vectors: \( \phi, \phi' \in \mathbb{R}^n \quad n \ll \# \text{states} \)

- approximate values: \( V_\theta(s) = \theta^T \phi \quad \theta \in \mathbb{R}^n \text{ parameter vector} \)

- TD error: \( \delta = r + \gamma \theta^T \phi' - \theta^T \phi \quad \gamma \in [0, 1) \)

- TD(0) algorithm: \( \Delta \theta = \alpha \delta \phi \quad \alpha > 0 \)
Notation

- state transitions: $s \xrightarrow{r} s'$
- feature vectors: $\phi \rightarrow \phi' \in \mathbb{R}^n \quad n \ll \#\text{states}$
- approximate values: $V_\theta(s) = \theta^\top \phi \quad \theta \in \mathbb{R}^n$ parameter vector
- TD error: $\delta = r + \gamma \theta^\top \phi' - \theta^\top \phi \quad \gamma \in [0, 1)$
- TD(0) algorithm: $\Delta \theta = \alpha \delta \phi \quad \alpha > 0$
- true values: $V^*(s) = \mathbb{E}[r|s] + \gamma \sum_{s'} P_{ss'} V^*(s')$
- Bellman operator: $TV = R + \gamma PV$ $V^* = TV^*$ over per-state vectors
Value function geometry

The space spanned by the feature vectors, weighted by the state visitation distribution
Value function geometry

$\Phi, D$

The space spanned by the feature vectors, weighted by the state visitation distribution

$T$ takes you outside the space
The space spanned by the feature vectors, weighted by the state visitation distribution.

$T$ takes you outside the space.

$\Pi$ projects you back into it.
Value function geometry

Previous work on gradient methods for TD minimized this objective fn (Baird 1995, 1999)

The space spanned by the feature vectors, weighted by the state visitation distribution

\[ V^* \]

\[ \Pi \]

\[ \Pi TV_\theta \]

\[ TV_\theta \]

\[ \Phi, D \]

\[ \text{RMSBE} \]

\[ \text{RMSPBE} \]

\( T \) takes you outside the space

\( \Pi \) projects you back into it
Value function geometry

The space spanned by the feature vectors, weighted by the state visitation distribution

Previous work on gradient methods for TD minimized this objective fn (Baird 1995, 1999)

Better objective fn?

Mean Square Projected Bellman Error (MSPBE)

$T$ takes you outside the space

$\Pi$ projects you back into it
TD objective functions
(to be minimized)

- Error from the true values
  \[ \| V_\theta - V^* \|_D^2 \]

- Error in the Bellman equation (Bellman residual)
  \[ \| V_\theta - TV_\theta \|_D^2 \]

- Error in the Bellman equation after projection (MSPBE)
  \[ \| V_\theta - \Pi TV_\theta \|_D^2 \]
TD objective functions
(to be minimized)

- Error from the true values \( \| V_\theta - V^* \|_D^2 \) Not TD
- Error in the Bellman equation (Bellman residual) \( \| V_\theta - TV_\theta \|_D^2 \)
- Error in the Bellman equation after projection (MSPBE) \( \| V_\theta - \Pi TV_\theta \|_D^2 \)
TD objective functions (to be minimized)

• Error from the true values \[ \| V_\theta - V^* \|_D^2 \] Not TD

• Error in the Bellman equation (Bellman residual) \[ \| V_\theta - TV_\theta \|_D^2 \] Not right

• Error in the Bellman equation after projection (MSPBE) \[ \| V_\theta - \Pi TV_\theta \|_D^2 \]
TD objective functions
(to be minimized)

• Error from the true values  \[ \| V_{\theta} - V^* \|_D^2 \]  Not TD

• Error in the Bellman equation (Bellman residual)
  \[ \| V_{\theta} - TV_{\theta} \|_D^2 \]  Not right

• Error in the Bellman equation after projection (MSPBE)
  \[ \| V_{\theta} - \Pi TV_{\theta} \|_D^2 \]  Right!
TD objective functions (to be minimized)

- Error from the true values: $\| V_\theta - V^* \|^2_D$ (Not TD)
- Error in the Bellman equation (Bellman residual): $\| V_\theta - TV_\theta \|^2_D$ (Not right)
- Error in the Bellman equation after projection (MSPBE): $\| V_\theta - \Pi TV_\theta \|^2_D$ (Right!)
- Zero expected TD update: $V_\theta = \Pi TV_\theta$
TD objective functions
(to be minimized)

• Error from the true values
  \[ \| V_\theta - V^* \|_D^2 \] 
  Not TD

• Error in the Bellman equation (Bellman residual)
  \[ \| V_\theta - TV_\theta \|_D^2 \] 
  Not right

• Error in the Bellman equation after projection (MSPBE)
  \[ \| V_\theta - \Pi TV_\theta \|_D^2 \] 
  Right!

• Zero expected TD update
  \[ V_\theta = \Pi TV_\theta \] 
  Not an objective
backwards-bootstrapping example

- The two ‘A’ states look the same; they share a single feature and must be given the same approximate value
  \[ V(A1) = V(A2) = \frac{1}{2} \]
- All transitions are deterministic; Bellman error = TD error
- Clearly, the right solution is
  \[ V(B) = 1, \quad V(C) = 0 \]
- But the solution that minimizes the Bellman error is
  \[ V(B) = \frac{3}{4}, \quad V(C) = \frac{1}{4} \]
backwards-bootstrapping example

- The two ‘A’ states look the same; they share a single feature and must be given the same approximate value
  \[ V(A1) = V(A2) = \frac{1}{2} \]

- All transitions are deterministic; Bellman error = TD error

- Clearly, the right solution is
  \[ V(B) = 1, \ V(C) = 0 \]

- But the solution the minimizes the Bellman error is
  \[ V(B) = \frac{3}{4}, \ V(C) = \frac{1}{4} \]
TD objective functions
(to be minimized)

- Error from the true values \( \| V_\theta - V^* \|_D^2 \) Not TD
- Error in the Bellman equation (Bellman residual) \( \| V_\theta - TV_\theta \|_D^2 \) Not right
- Error in the Bellman equation after projection (MSPBE) \( \| V_\theta - \Pi TV_\theta \|_D^2 \) Right!
- Zero expected TD update \( V_\theta = \Pi TV_\theta, \ \mathbb{E}[\Delta \theta_{TD}] = \vec{0} \)
- Norm Expected TD update \( \| \mathbb{E}[\Delta \theta_{TD}] \| \)
- Expected squared TD error \( \mathbb{E}[\delta^2] \) Not an objective
TD objective functions
(to be minimized)

• Error from the true values
  \[ \| V_\theta - V^* \|^2_D \]
  Not TD

• Error in the Bellman equation
  (Bellman residual)
  \[ \| V_\theta - TV_\theta \|^2_D \]
  Not right

• Error in the Bellman equation
  after projection (MSPBE)
  \[ \| V_\theta - \Pi TV_\theta \|^2_D \]
  Right!

• Zero expected TD update
  \[ V_\theta = \Pi TV_\theta, \quad \mathbb{E}[\Delta \theta_{TD}] = \vec{0} \]

• Norm Expected TD update
  \[ \| \mathbb{E}[\Delta \theta_{TD}] \| \]
  previous work

• Expected squared TD error
  \[ \mathbb{E}[\delta^2] \]
TD objective functions  
(to be minimized)

- Error from the true values  
  \[ \| V_\theta - V^* \|^2_D \]

- Error in the Bellman equation  
  (Bellman residual)  
  \[ \| V_\theta - TV_\theta \|^2_D \]

- Error in the Bellman equation  
  after projection  
  (MSPBE)  
  \[ \| V_\theta - \Pi TV_\theta \|^2_D \]

- Zero expected TD update  
  \[ V_\theta = \Pi TV_\theta, \quad \mathbb{E}[\Delta \theta_{TD}] = 0 \]

- Norm Expected TD update  
  \[ \| \mathbb{E}[\Delta \theta_{TD}] \| \]

- Expected squared TD error  
  \[ \mathbb{E}[\delta^2] \]
outline

• ways in which TD with FA has not been straightforward
• the new Bellman error objective function
• derivation of new algorithms (the trick)
• results (theory and experiments)
Gradient-descent learning

1. Pick an objective function $J(\theta)$, a parameterized function to be minimized

2. Use calculus to analytically compute the gradient $\nabla_{\theta} J(\theta)$

3. Find a “sample gradient” that you can sample on every time step and whose expected value equals the gradient

4. Take small steps in $\theta$ proportional to the sample gradient:

$$\Delta \theta = -\alpha \nabla_{\theta} J_t(\theta)$$
Derivation of the TDC algorithm

\[ \Delta \theta = -\frac{1}{2} \alpha \nabla_{\theta} J(\theta) \]
Derivation of the TDC algorithm

\[
\Delta \theta = -\frac{1}{2} \alpha \nabla_\theta J(\theta) = -\frac{1}{2} \alpha \nabla_\theta \| V_\theta - \Pi TV_\theta \|_D^2
\]
Derivation of the TDC algorithm

\[
\Delta \theta = -\frac{1}{2} \alpha \nabla_\theta J(\theta) = -\frac{1}{2} \alpha \nabla_\theta \| V_\theta - \Pi T V_\theta \|_D^2 \\
= -\frac{1}{2} \alpha \nabla_\theta \left( \mathbb{E} [\delta \phi] \mathbb{E} [\phi \phi^\top]^{-1} \mathbb{E} [\delta \phi] \right)
\]
Derivation of the TDC algorithm

\[
\Delta \theta = -\frac{1}{2} \alpha \nabla_\theta J(\theta) = -\frac{1}{2} \alpha \nabla_\theta \| V_\theta - \Pi TV_\theta \|_D^2
\]

\[
= -\frac{1}{2} \alpha \nabla_\theta \left( \mathbb{E} [\delta \phi] \mathbb{E} [\phi \phi^\top]^{-1} \mathbb{E} [\delta \phi] \right)
\]

\[
= -\alpha (\nabla_\theta \mathbb{E} [\delta \phi]) \mathbb{E} [\phi \phi^\top]^{-1} \mathbb{E} [\delta \phi]
\]

This is the main trick!
Derivation of the TDC algorithm

\[
\Delta \theta = -\frac{1}{2} \alpha \nabla_\theta J(\theta) = -\frac{1}{2} \alpha \nabla_\theta \parallel V_\theta - \Pi TV_\theta \parallel^2_D \\
= -\frac{1}{2} \alpha \nabla_\theta \left( \mathbb{E} [\delta \phi] \mathbb{E} [\phi \phi^\top]^{-1} \mathbb{E} [\delta \phi] \right) \\
= -\alpha \left( \nabla_\theta \mathbb{E} [\delta \phi] \right) \mathbb{E} [\phi \phi^\top]^{-1} \mathbb{E} [\delta \phi] \\
= -\alpha \mathbb{E} \left[ \nabla_\theta \left( r + \gamma \theta^\top \phi' - \theta^\top \phi \right) \phi \right] \mathbb{E} [\phi \phi^\top]^{-1} \mathbb{E} [\delta \phi]
\]
Derivation of the TDC algorithm

\[
\Delta \theta = -\frac{1}{2} \alpha \nabla_\theta J(\theta) = -\frac{1}{2} \alpha \nabla_\theta \| V_\theta - \Pi TV_\theta \|_D^2 \\
= -\frac{1}{2} \alpha \nabla_\theta \left( \mathbb{E} [\delta \phi] \mathbb{E} [\phi \phi^\top]^{-1} \mathbb{E} [\delta \phi] \right) \\
= -\alpha \left( \nabla_\theta \mathbb{E} [\delta \phi] \right) \mathbb{E} [\phi \phi^\top]^{-1} \mathbb{E} [\delta \phi] \\
= -\alpha \mathbb{E} \left[ \nabla_\theta (r + \gamma \theta^\top \phi' - \theta^\top \phi) \phi \right] \mathbb{E} [\phi \phi^\top]^{-1} \mathbb{E} [\delta \phi] \\
= -\alpha \mathbb{E} \left[ (\gamma \phi' - \phi) \phi \right] \mathbb{E} [\phi \phi^\top]^{-1} \mathbb{E} [\delta \phi]
\]
Derivation of the TDC algorithm

\[
\Delta \theta = -\frac{1}{2} \alpha \nabla_\theta J(\theta) = -\frac{1}{2} \alpha \nabla_\theta \| V_\theta - \Pi TV_\theta \|_D^2 \\
\quad = -\frac{1}{2} \alpha \nabla_\theta \left( \mathbb{E} [\delta \phi] \mathbb{E} [\phi \phi^T]^{-1} \mathbb{E} [\delta \phi] \right) \\
\quad = -\alpha (\nabla_\theta \mathbb{E} [\delta \phi]) \mathbb{E} [\phi \phi^T]^{-1} \mathbb{E} [\delta \phi] \\
\quad = -\alpha \mathbb{E} [\nabla_\theta (r + \gamma \theta^T \phi - \theta^T \phi) \phi] \mathbb{E} [\phi \phi^T]^{-1} \mathbb{E} [\delta \phi] \\
\quad = -\alpha \mathbb{E} [(\gamma \phi' - \phi) \phi] \mathbb{E} [\phi \phi^T]^{-1} \mathbb{E} [\delta \phi] \\
\quad = \alpha (\mathbb{E} [\phi \phi^T] - \gamma \mathbb{E} [\phi' \phi^T]) \mathbb{E} [\phi \phi^T]^{-1} \mathbb{E} [\delta \phi]
\]

This is the main trick!
Derivation of the TDC algorithm

\[ \Delta \theta = -\frac{1}{2} \alpha \nabla_\theta J(\theta) = -\frac{1}{2} \alpha \nabla_\theta \| V_\theta - \Pi TV_\theta \|_D^2 \]

\[ = -\frac{1}{2} \alpha \nabla_\theta \left( \mathbb{E} [\delta \phi] \mathbb{E} [\phi \phi^\top]^{-1} \mathbb{E} [\delta \phi] \right) \]

\[ = -\alpha \left( \nabla_\theta \mathbb{E} [\delta \phi] \right) \mathbb{E} [\phi \phi^\top]^{-1} \mathbb{E} [\delta \phi] \]

\[ = -\alpha \mathbb{E} \left[ \nabla_\theta \left( r + \gamma \theta^\top \phi' - \theta^\top \phi \right) \phi \right] \mathbb{E} [\phi \phi^\top]^{-1} \mathbb{E} [\delta \phi] \]

\[ = -\alpha \mathbb{E} \left[ (\gamma \phi' - \phi) \phi \right] \mathbb{E} [\phi \phi^\top]^{-1} \mathbb{E} [\delta \phi] \]

\[ = \alpha \left( \mathbb{E} [\phi \phi^\top] - \gamma \mathbb{E} [\phi' \phi^\top] \right) \mathbb{E} [\phi \phi^\top]^{-1} \mathbb{E} [\delta \phi] \]

\[ = \alpha \mathbb{E} [\delta \phi] - \alpha \gamma \mathbb{E} [\phi' \phi^\top] \mathbb{E} [\phi \phi^\top]^{-1} \mathbb{E} [\delta \phi] \]
Derivation of the TDC algorithm

\[ \Delta \theta = -\frac{1}{2} \alpha \nabla_{\theta} J(\theta) = -\frac{1}{2} \alpha \nabla_{\theta} \| V_{\theta} - \Pi TV_{\theta} \|^2_D \]

\[ = -\frac{1}{2} \alpha \nabla_{\theta} \left( \mathbb{E} [\delta \phi] \mathbb{E} [\phi \phi^T]^{-1} \mathbb{E} [\delta \phi] \right) \]

\[ = -\alpha \left( \nabla_{\theta} \mathbb{E} [\delta \phi] \right) \mathbb{E} [\phi \phi^T]^{-1} \mathbb{E} [\delta \phi] \]

\[ = -\alpha \mathbb{E} \left[ \nabla_{\theta} (r + \gamma \theta^T \phi' - \theta^T \phi) \phi \right] \mathbb{E} [\phi \phi^T]^{-1} \mathbb{E} [\delta \phi] \]

\[ = -\alpha \mathbb{E} \left[ (\gamma \phi' - \phi) \phi \right] \mathbb{E} [\phi \phi^T]^{-1} \mathbb{E} [\delta \phi] \]

\[ = \alpha \left( \mathbb{E} [\phi \phi^T] - \gamma \mathbb{E} [\phi' \phi^T] \right) \mathbb{E} [\phi \phi^T]^{-1} \mathbb{E} [\delta \phi] \]

\[ = \alpha \mathbb{E} [\delta \phi] - \alpha \gamma \mathbb{E} [\phi' \phi^T] \mathbb{E} [\phi \phi^T]^{-1} \mathbb{E} [\delta \phi] \]

\[ \approx \alpha \mathbb{E} [\delta \phi] - \alpha \gamma \mathbb{E} [\phi' \phi^T] \approx \alpha \mathbb{E} \left[ \phi \phi^T \right] w \]
Derivation of the TDC algorithm

\[ \Delta \theta = -\frac{1}{2} \alpha \nabla_{\theta} J(\theta) = -\frac{1}{2} \alpha \nabla_{\theta} \| V_{\theta} - \Pi TV_{\theta} \|_{D}^{2} \]

\[ = -\frac{1}{2} \alpha \nabla_{\theta} \left( E[\delta \phi] E[\phi \phi^{T}]^{-1} E[\delta \phi] \right) \]

\[ = -\alpha \left( \nabla_{\theta} E[\delta \phi] \right) E[\phi \phi^{T}]^{-1} E[\delta \phi] \]

\[ = -\alpha E \left[ \nabla_{\theta} \left( r + \gamma \theta^{T} \phi' - \theta^{T} \phi \right) \phi \right] E[\phi \phi^{T}]^{-1} E[\delta \phi] \]

\[ = -\alpha E \left[ (\gamma \phi' - \phi) \phi \right] E[\phi \phi^{T}]^{-1} E[\delta \phi] \]

\[ = \alpha \left( E[\phi \phi^{T}] - \gamma E[\phi' \phi^{T}] \right) E[\phi \phi^{T}]^{-1} E[\delta \phi] \]

\[ = \alpha E[\delta \phi] - \alpha \gamma E[\phi' \phi^{T}] E[\phi \phi^{T}]^{-1} E[\delta \phi] \approx \alpha E[\delta \phi] - \alpha \gamma E[\phi' \phi^{T}] w \]

This is the trick!

\( w \in \mathbb{R}^{n} \) is a second set of weights
Derivation of the TDC algorithm

\[ \Delta \theta = -\frac{1}{2} \alpha \nabla_\theta J(\theta) = -\frac{1}{2} \alpha \nabla_\theta \| V_\theta - \Pi TV_\theta \|^2_D \]

\[ = -\frac{1}{2} \alpha \nabla_\theta \left( \mathbb{E} [\delta \phi] \mathbb{E} [\phi \phi^\top]^{-1} \mathbb{E} [\delta \phi] \right) \]

\[ = -\alpha (\nabla_\theta \mathbb{E} [\delta \phi]) \mathbb{E} [\phi \phi^\top]^{-1} \mathbb{E} [\delta \phi] \]

\[ = -\alpha \mathbb{E} \left[ \nabla_\theta (r + \gamma \theta^\top \phi' - \theta^\top \phi) \phi \right] \mathbb{E} [\phi \phi^\top]^{-1} \mathbb{E} [\delta \phi] \]

\[ = -\alpha \mathbb{E} \left[ (\gamma \phi' - \phi) \phi \right] \mathbb{E} [\phi \phi^\top]^{-1} \mathbb{E} [\delta \phi] \]

\[ = \alpha \left( \mathbb{E} [\phi \phi^\top] - \gamma \mathbb{E} [\phi' \phi^\top] \right) \mathbb{E} [\phi \phi^\top]^{-1} \mathbb{E} [\delta \phi] \]

\[ = \alpha \mathbb{E} [\delta \phi] - \alpha \gamma \mathbb{E} [\phi' \phi^\top] \mathbb{E} [\phi \phi^\top]^{-1} \mathbb{E} [\delta \phi] \]

\[ \approx \alpha \mathbb{E} [\delta \phi] - \alpha \gamma \mathbb{E} [\phi' \phi^\top] w \]  

This is the trick! 

\[ w \in \mathbb{R}^n \] is a second set of weights
The complete TD with gradient correction (TDC) algorithm

- on each transition

\[ s \xrightarrow{r} s' \]

- update two parameters

\[
\theta \leftarrow \theta + \alpha \delta \phi - \alpha \gamma \phi' (\phi^T w)
\]

\[
w \leftarrow w + \beta (\delta - \phi^T w) \phi
\]

- where

\[
\delta = r + \gamma \theta^T \phi' - \theta^T \phi
\]
The complete TD with gradient correction (TDC) algorithm

• on each transition

\[ s \xrightarrow{r} s' \]

\[ \phi \quad \phi' \]

• update two parameters

TD(0)

\[ \theta \leftarrow \theta + \alpha \delta \phi - \alpha \gamma \phi' (\phi^\top w) \]

\[ w \leftarrow w + \beta (\delta - \phi^\top w) \phi \]

• where

\[ \delta = r + \gamma \theta^\top \phi' - \theta^\top \phi \]
The complete \textit{TD with gradient correction (TDC)} algorithm

- on each transition

- update two parameters

\[ \theta \leftarrow \theta + \alpha \delta \phi - \alpha \gamma \phi' (\phi^\top w) \]

\[ w \leftarrow w + \beta (\delta - \phi^\top w) \phi \]

- where

\[ \delta = r + \gamma \theta^\top \phi' - \theta^\top \phi \]
The complete TD with gradient correction (TDC) algorithm

• on each transition

\[ s \xrightarrow{r} s' \]

\[ \phi \xrightarrow{} \phi' \]

• update two parameters

\[ \theta \leftarrow \theta + \alpha \delta \phi - \alpha \gamma \phi' (\phi^\top w) \]

\[ w \leftarrow w + \beta (\delta - \phi^\top w) \phi \]

• where

\[ \delta = r + \gamma \theta^\top \phi' - \theta^\top \phi \]

estimate of the TD error (\( \delta \)) for the current state \( \phi \)
outline

• ways in which TD with FA has not been straightforward

• the new Bellman error objective function

• derivation of new algorithms (the trick)

• results (theory and experiments)
Three new algorithms

• GTD, the original *gradient TD algorithm* (Sutton, Szepesvari & Maei, 2008)
• GTD2, a second-generation GTD
• TDC
Convergence theorems

• For arbitrary on- or off-policy training

• All algorithms converge w.p.1 to the TD fix-point:
  \[ \mathbb{E} [\delta \phi] \to 0 \]

• GTD, GTD2 converge at one time scale
  \[ \alpha = \beta \to 0 \]

• TDC converges in a two-time-scale sense
  \[ \alpha, \beta \to 0 \quad \frac{\alpha}{\beta} \to 0 \]
Summary of empirical results on small problems

Random Walk - Tabular features

Random Walk - Inverted features

Random Walk - Dependent features

Boyan Chain

Usually \( \text{GTD} \ll \text{GTD2} \ll \text{TD, TDC} \)
Computer Go experiment

- Learn a linear value function (probability of winning) for 9x9 Go from self play
- One million features, each corresponding to a template on a part of the Go board
- An established experimental testbed
conclusions

• the new algorithms are roughly the same efficiency as conventional TD on on-policy problems

• but are guaranteed convergent under general off-policy training as well

• their key ideas appear to extend quite broadly, to control, general $\lambda$, non-linear settings, DP, intra-option learning, TD nets...

• $TD$ with $FA$ is now straightforward

• the curse of dimensionality is removed