ICS 6A
Solution to Homework Assignment 7
Winter 2004

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Answer the following questions (explain your answers).


Please see “Solutions to Odd-Numbered Exercises” of Rosen. (Page S-24)


Proof: Let $P(n)$ be $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$,
where $n = 1, 2, 3, \ldots$

- Basis step: for $n = 1$, $1 \cdot 1! = 1 = (1 + 1)! - 1 \Rightarrow P(1)$ is true.
- Inductive step: Assume $P(n)$ is true, i.e. $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$, then:
  $$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! + (n+1) \cdot (n+1)!$$
  $$= (n+1)! - 1 + (n+1) \cdot (n+1)!$$
  $$= (n+1)! - 1$$
  $$= (n+2)! - 1$$

The last equation shows that $P(n+1)$ is true. This completes the inductive step and completes the proof.


Please see “Solutions to Odd-Numbered Exercises” of Rosen. (Page S-24)


Proof: Let $P(n)$ be $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) = n(n+1)(n+2)(n+3)/4$.

- Basis step: $1 \cdot 2 \cdot 3 = 6 = 1 \cdot (1 + 1) \cdot (1 + 2) \cdot (1 + 3)/4 \Rightarrow P(1)$ is true.
- Inductive step: Assume $P(n)$ is true, i.e. $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) = n(n+1)(n+2)(n+3)/4$, then:
  $$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) + (n+1)(n+2)(n+3)$$
  $$= n(n+1)(n+2)(n+3)/4 + (n+1)(n+2)(n+3)$$
  $$= (n+1)(n+2)(n+3)(n+4)/4$$

The last equation shows that $P(n+1)$ is true. This completes the inductive step and completes the proof.


Proof: Let $P(n)$ be $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 2 - \frac{1}{n}$, where $n = 2, 3, \ldots$

- Basis step: for $n = 2$, $1 + \frac{1}{2} = \frac{3}{2} < 2 - \frac{1}{2} = \frac{3}{2} \Rightarrow P(2)$ is true.
- Inductive step: Assume $P(n)$ is true, i.e. $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 2 - \frac{1}{n}$, then:
  $$1 + \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+1} < 2 - \frac{1}{n+1}$$
  $$< 2 - \frac{1}{n+1} + \frac{1}{n+1} \text{ From induction hypothesis}$$
  $$= 2 - \frac{1}{n+1} + \frac{1}{n+1}$$
  $$= 2 - \frac{1}{n+1} + \frac{n(n+1)-(n+1)^2+n}{n(n+1)^2}$$
  $$= 2 - \frac{1}{n+1} + \frac{n^2+n-2n+1+n}{n(n+1)^2}$$
  $$= 2 - \frac{1}{n+1} + \frac{n^2+n-2n+1+n}{n(n+1)^2}$$

1
\[
2 - \frac{1}{n+1} + \frac{-1}{n(n+1)}^2 < 2 - \frac{1}{n+1} \text{ whenever } n > 1, \text{ because } \frac{-1}{n(n+1)}^2 < 0
\]

The last inequality shows that \(P(n+1)\) is true. This completes the inductive step and completes the proof.

   **Proof:** Let \(P(n)\) be “6 divides \(n^3 - n\), where \(n = 0, 1, 2, \cdots\)
   - Basis step: for \(n = 0\), 6 divides \(0^3 - 0 = 0 \Rightarrow P(0)\) is true.
   - Inductive step: Assume \(P(n)\) is true, i.e. \(6\) divides \(n^3 - n\), then:
     \[
     (n + 1)^3 - (n + 1) = n^3 + 3n^2 + 3n + 1 - n - 1 = (n^3 - n) + 3n^2 + 3n
     \]
     \[
     = (n^3 - n) + 3n(n + 1)
     \]
   \((n^3 - n)\) can be divided by 6 from induction hypothesis, either \(n\) or \(n + 1\) are even number, then \(n(n + 1)\) can be divided by 2, so \(3n(n + 1)\) can be divided by 6, and \((n^3 - n) + 3n(n + 1)\) can be divided by 6. This means that \(P(n+1)\) is true.
   
   This completes the inductive step and completes the proof.

   **Please see the “Solutions to Odd-Numbered Exercises” of Rosen. (Page S-26)**

   **Proof:** Let \(P(n)\) be “\(\neg(p_1 \lor p_2 \lor \cdots \lor p_n)\) is equivalent to \(\neg p_1 \land \neg p_2 \land \cdots \land \neg p_n\)”
   - Basis step: \(\neg p_1 \Rightarrow P(2)\) is true.
   - Inductive step: Assume \(P(n)\) is true, i.e. \(\neg(p_1 \lor p_2 \lor \cdots \lor p_n)\) is equivalent to \(\neg p_1 \land \neg p_2 \land \cdots \land \neg p_n\), then:
     \[
     \neg(p_1 \lor p_2 \lor \cdots \lor p_n \lor p_{n+1}) \iff \neg[(p_1 \lor p_2 \lor \cdots \lor p_n) \lor p_{n+1}]
     \]
     is equivalent to \(\neg(p_1 \lor p_2 \lor \cdots \lor p_n) \land \neg p_{n+1}\) by "De Morgan’s laws"  
     is equivalent to \((\neg p_1 \land \neg p_2 \land \cdots \land \neg p_n) \land \neg p_{n+1}\) From induction hypothesis 
     is equivalent to \(\neg p_1 \land \neg p_2 \land \cdots \land \neg p_n \land \neg p_{n+1}\)
   
   The last “equivalent relation” shows that \(P(n+1)\) is true. This completes the inductive step and completes the proof.

   a) \(2^n + 1\): \(a_0 = 2^0 + 1 = 2\)
      \(a_1 = 2^1 + 1 = 3\)
      \(a_2 = 2^2 + 1 = 5\)
      \(a_3 = 2^3 + 1 = 9\)
   
   b) \((n + 1)^{n+1}\): \(a_0 = (0+1)^{0+1} = 1\)
      \(a_1 = (1+1)^{1+1} = 4\)
      \(a_2 = (2+1)^{2+1} = 27\)
      \(a_3 = (3+1)^{3+1} = 256\)
   
   c) \([n/2]\): \(a_0 = [0/2] = 0\)
      \(a_1 = [1/2] = 0\)
      \(a_2 = [2/2] = 1\)
      \(a_3 = [3/2] = 1\)
d) \([n/2] + [n/2]\):
   - \(a_0 = [0/2] + [0/2] = 0 + 0 = 0\)
   - \(a_1 = [1/2] + [1/2] = 0 + 1 = 1\)
   - \(a_2 = [2/2] + [2/2] = 1 + 1 = 2\)
   - \(a_3 = [3/2] + [3/2] = 1 + 2 = 3\)

   **Answer:** The terms could be odd numbers greater than 1;
   the terms could be prime numbers greater than 2;
   the terms could be odd numbers not divisible by 9;
   the terms could be numbers greater than 2 and not divisible by 4 and 6; \(\cdots\)
   There are infinitely many other possibilities.

   a) \(a_n = n^2 + 2 \cdot n + 3\), where \(n = 0, 1, 2, \cdots\)
   b) \(a_n = 4 \cdot n^2 + 7\), where \(n = 0, 1, 2, \cdots\)
   c) \(10^n\) followed by the sum of \(10^n\) and previous terms respectively, where \(n = 0, 1, 2, \cdots\).
   d) The Fibonacci sequence \(f(n + 1)\) listed \(2n - 1\) times.
   e) \(3^n - 1\), where \(n = 0, 1, 2, \cdots\)
   f) \(\frac{(2n+1)!}{2^n n!}\), where \(n = 0, 1, 2, \cdots\)
   g) One 1 followed by two 0s, three 1s, four 0s, and so on.
   h) \(2^n\), where \(n = 0, 1, 2, \cdots\)

   a) \(\log_2 1024 = \log_2 2^{10} = 10\)
   b) \(\log_2 \frac{1}{4} = \log_2 2^{-2} = -2\)
   c) \(\log_2 8 = \frac{\log_{10} 8}{\log_{10} 2} = \frac{3}{2}\) **THEOREM 3** on page A-2 of Rosen.

   **Proof:** Using **THEOREM 2** rule 2: \(\log_b \text{LeftHandSide} = \log_b a^{\log_b c} = \log_b c \cdot \log_b a\)
   \(\log_b \text{RightHandSide} = \log_b c^{\log_b a} = \log_b a \cdot \log_b c\)
   \(\log_b \text{LeftHandSide} = \log_b \text{RightHandSide} \Rightarrow \text{LeftHandSide} = \text{RightHandSide}\)
   So, \(a^{\log_b c} = c^{\log_b a}\)