Suppose that $T_i \sim \text{Exp}(\lambda)$ and $C_i \sim \text{Unif}(a, b)$ where $a$ and $b$ are known constants with $\lambda$ an unknown parameter to estimate. Further assume that the $T_i$’s and $C_i$’s are totally independent.

**Goal:** Find the MLE of $\lambda$ based on $X_i \equiv \min(T_i, C_i)$ and $\delta_i = 1_{T_i = X_i}$ and it’s asymptotic distribution. From this, find the MLE of $S_T(t; \lambda) = \Pr[T > t]$ and it’s asymptotic distribution.

**Solution:** For notational convenience, let $T_i \sim F(\cdot)$ and $C_i \sim G(\cdot)$. Now to write down the likelihood consider the following cases:

$$\delta_i = 1 \implies T_i = X_i, C_i > X_i \implies \text{likelihood contribution is } f_T(X_i)[1 - G_C(X_i)]$$

$$\delta_i = 0 \implies C_i = X_i, T_i > X_i \implies \text{likelihood contribution is } g_C(X_i)[1 - F_T(X_i)]$$

So, we have that

$$L_i(\lambda; X_i, \delta_i) = \delta_i \{f_T(X_i)[1 - G_C(X_i)]\} + (1 - \delta_i) \{g_C(X_i)[1 - F_T(X_i)]\}$$

$$= \{f_T(X_i)[1 - G_C(X_i)]\}^{\delta_i} \{g_C(X_i)[1 - F_T(X_i)]\}^{1 - \delta_i} \propto f_T(X_i)^{\delta_i} S_T(X_i)^{1 - \delta_i}$$

$$= \{\lambda e^{-\lambda X_i}[1 - G_C(X_i)]\}^{\delta_i} \{g_C(X_i)e^{-\lambda X_i}\}^{1 - \delta_i}$$

$$\ell_i(\lambda) = \delta_i \log \lambda - \lambda X_i + k$$

$$U_i(\lambda) = \frac{\delta_i}{\lambda} - X_i$$

$$I_i(\lambda) = -\mathbb{E}\left[\frac{\delta_i}{\lambda^2}\right]$$

From this, the MLE satisfies

$$U(\hat{\lambda}) = \sum_{i=1}^{n} U_i(\hat{\lambda}) = 0$$

$$\Rightarrow \sum_{i=1}^{n} \frac{\delta_i}{\lambda} - \sum_{i=1}^{n} X_i = 0$$

$$\Rightarrow \hat{\lambda} = \frac{\sum_{i=1}^{n} \delta_i}{\sum_{i=1}^{n} X_i} = \frac{\delta}{\bar{X}}$$

Since this is a regular problem, we know from large sample MLE theory that

$$\sqrt{n}(\hat{\lambda} - \lambda) \rightarrow_d N(0, I_1^{-1}(\lambda)),$$

Now, by the invariance property of MLE’s, the MLE of $S(t; \lambda)$ is

$$\hat{S}(t) = S(t; \hat{\lambda}) = e^{-\hat{\lambda} t}, \quad t > 0.$$
So, taking \( g(x) = e^{-xt} \) (i.e. \( g'(x) = -te^{-xt} \)), from the \( \delta \)-method we have

\[
\sqrt{n} \left[ S(t; \lambda) - S(t; \hat{\lambda}) \right] \to_d N \left( 0, t^2 e^{-2\lambda t} I_1^{-1}(\lambda) \right),
\]

or

\[
S(t; \lambda) \sim N \left( S(t; \lambda), t^2 e^{-2\lambda t} n^{-1} I_1^{-1}(\lambda) \right).
\]

**Notes:**

A. Within the asymptotic variance we have

\[
I_1(\lambda) = E \left[ \frac{\delta_i}{\lambda^2} \right] = \frac{\Pr[T_i < C_i]}{\lambda^2},
\]

and

\[
\Pr[T_i < C_i] = 1 - \Pr[T_i > C_i] = 1 - \int_a^b [1 - F_T(c)]g_C(c)dc = 1 - \int_a^b \frac{e^{-\lambda c}}{b-a} dc = 1 + \frac{e^{-\lambda b} - e^{-\lambda a}}{\lambda(b-a)}.
\]

In real applications, we don’t generally know the censoring distribution, so we estimate it’s impact on the information by using the observed information

\[
\hat{I}(\lambda) = \sum_{i=1}^n - \frac{\partial}{\partial \lambda} U_i(\lambda) = \sum_{i=1}^n \frac{\delta_i}{\lambda^2}.
\]

B. In general, under the assumption that censoring is independent of failure times, we specify the \( i \)-th observations contribution to the likelihood function as

\[
L_i(\lambda; X_i, \delta_i) = f_T(X_i)^{\delta_i} S_T(X_i)^{1-\delta_i},
\]