

BRDFs

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1 Background

We briefly review some helpful concepts. Firstly, we need to define solid angle. Solid angle is a generalization of a planar angle to three dimensions. While planar angles are measured in radians, solid angles are measured in steradians. Specifically, a solid angle of 1 steradian is the angle that carves out a unit-area patch on the surface of a unit-radius sphere. Since the total area of a unit-radius sphere is 4π , solid angles take on values from 0 to 4π steradians (much as angles are defined over 0 to 2π radians, where 2π is the circumference of a unit-radius circle). In a spherical coordinate system, an infinitesimal solid angle $\partial\omega$ can be computed as

$$\partial\omega = \sin\theta\partial\theta\partial\phi$$

where θ and ϕ are the standard zenith and azimuth angles (see the wikipedia reference on spherical coordinate systems).

The other concept we need is the notion of the apparent size of a small patch of area ΔA at a distance of R from an observer. We define apparent area as the solid angle subtended by this patch within the viewing hemisphere of the observer. Given the definition of solid angle, this is the area of a patch on a unit-radius sphere that looks to have the same area as ΔA . With a couple of similar-triangle arguments, one can derive apparent surface area as

$$\Omega = \frac{\cos\theta_A\Delta A}{R^2} \tag{1}$$

θ_A is the angle between the patch normal and a vector from the patch to the observer; when $\theta_A = 0$, the patch is facing the observer.

2 Radiance

Radiance is the fundamental unit for measure light. It is defined as

$$L = \frac{\partial^2 P}{\cos\theta\partial A\partial\omega} \tag{2}$$

$$= \frac{P}{\cos\theta\Delta A\Delta\omega} \quad \text{as } \Delta A \rightarrow 0, \Delta\omega \rightarrow 0 \tag{3}$$

P is defined as power, the rate of energy (Q) per unit time $\partial P = \frac{\partial Q}{\partial t}$ ¹. This is measured in watts or joules per second. ∂A refers to the area of a small disk, while $\cos\theta\partial A$ refers to its apparent area when viewed from an angle of θ from its normal. In a spherical coordinate system defined on the disk, θ is the zenith angle. $\partial\omega$ is a small unit of solid angle such that $\partial\omega = \sin\theta\partial\theta\partial\phi$, where ϕ is the corresponding azimuth angle.

Energy can be measured in discrete quanta known as photons. Radiance is then equivalent to the amount of photons per unit time, per unit foreshortened area, per unit solid angle. We can count the photons travelling through a small disk in space along a small set of directions as

$$Q = L \cos\theta \Delta A \Delta\omega \Delta t$$

Oftentimes, we will write radiance as $L(\mathbf{x}, \theta, \phi)$ where x specifies the center of disk as $\Delta A \rightarrow 0$ and (θ, ϕ) specifies the direction of travel as $\Delta\omega \rightarrow 0$. Forsyth advocates the intuition of thinking of radiance as a flashlight, where we must both define the position and direction of the flashlight.

Irradiance is defined as radiant power per unit area, or $L \cos\theta\partial\omega$. This is convenient for computing the amount of radiant power passing through surface ∂A due to radiance arriving from angle (θ, ϕ) . To compute the total power falling on a surface, we have to integrate over all angles in a hemisphere

$$E_0 = \int_{2\pi} L(\mathbf{x}, \cos\theta, \sin\phi) \cos\theta \partial\omega \tag{4}$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} L(\mathbf{x}, \cos\theta, \sin\phi) \sin\theta \partial\theta \partial\phi \tag{5}$$

We use 2π in the integral from (4) because there are 2π steradians in a hemisphere.

Why is radiance the right unit? One may ask why not define radiance differently - perhaps without $\cos\theta$ in the denominator. One answer is that we should measure light using a unit that remains consistent along straight line paths in space. One can derive that radiance is conserved along unoccluded paths between two patches in space. It turns out that $\cos\theta$ is required for the conservation property. This stems from the fact that the solid angle subtended by one patch as seen from the other includes a foreshortening term (1).

3 BRDF

The BRDF is defined as the ratio of outgoing radiance in a particular direction to incoming radiance in a particular direction

$$f(\theta_i, \phi_i, \theta_e, \phi_e) = \frac{L_e(\mathbf{x}, \theta_e, \phi_e)}{L_i(\mathbf{x}, \theta_i, \phi_i) \cos\theta_i \partial\omega_i} \tag{6}$$

¹Many references use the term radiant flux Φ instead of radiant power P as. Both are interchangeable

3.1 Outgoing radiance

We can now define outgoing radiance along a particular direction as

$$L_e(\mathbf{x}, \theta_e, \phi_e) = \int_{2\pi} f(\theta_i, \phi_i, \theta_e, \phi_e) L_i(\mathbf{x}, \theta_i, \phi_i) \cos \theta_i \partial\omega_i \quad (7)$$

A useful measurement is the total radiant power leaving a surface point \mathbf{x} . This is defined as the radiosity

$$B(\mathbf{x}) = \int_{2\pi} L_e(\mathbf{x}, \theta_e, \phi_e) \cos \theta_e \partial\omega_w$$

3.2 Constraints

There are two important physical constraints on the BRDF. One is denoted as the Hemholtz reciprocity condition, which stems from the Second Law of Thermodynamics. Consider two points at a unit distance away from surface point x . We can write their positions as (θ_i, ϕ_i) and (θ_e, ϕ_e) . The radiant power flowing from one point to the other through the BRDF at x must be equal to the radiant power in the other direction. If not, one point would be getting hotter, and radiant power would flow in the other direction until equilibrium.

$$f(\theta_i, \phi_i, \theta_e, \phi_e) = f(\theta_e, \phi_e, \theta_i, \phi_i)$$

The other is that a surface cannot reflect out more light than the incoming light. We can verify this is true by shining a light from a particular direction (θ_i, ϕ_i) and verifying the fraction of radiosity to incoming irradiance is less than 1. This must hold for all incoming directions.

$$\rho_{dh}(\theta_i, \phi_i) = \int_{2\pi} f(\theta_i, \phi_i, \theta_e, \phi_e) \cos \theta_e d\omega_e \quad (8)$$

$$0 \leq \rho_{dh}(\theta_i, \phi_i) \leq 1, \quad \forall \theta_i, \phi_i \quad (9)$$

$\rho_{dh}(\theta_i, \phi_i)$ is called the directional hemispheric reflectance function. It is roughly the measure of the grayscale intensity of the object, independent of the illumination. Ideal absorbers ($\rho_{dh}(\theta_i, \phi_i) = 0$) always look black, while ideal reflectors ($\rho_{dh}(\theta_i, \phi_i) = 1$) tend to look white (though they will look black if no illumination is present!).

Lambertian BRDFs have the property that they are not a function of incoming or outgoing direction, meaning $f(\theta_i, \phi_i, \theta_e, \phi_e)$ is constant. In this case ρ_{dh} is also a constant and is known as the albedo.

4 Specular surfaces

Consider the BRDF for an ideally reflecting mirror is

$$f(\theta_i, \phi_i, \theta_e, \phi_e) = k \delta(\theta_i - \theta_e) \delta(\phi_i + \pi - \phi_e) \quad (10)$$

Since the mirror is ideally reflective,

$$1 = \int_{2\pi} f(\theta_i, \phi_i, \theta_e, \phi_e) \cos \theta_e d\omega_e \quad (11)$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} k \delta(\theta_i - \theta_e) \delta(\phi_i + \pi - \phi_e) \cos \theta_e d \sin \theta_e d\theta_e d\phi_e \quad (12)$$

$$= k \cos \theta_i \sin \theta_i \quad (13)$$

The “constant” k from (10) must be equal to $(\cos \theta_i \sin \theta_i)^{-1}$.