Decentralized Synchronization Protocols with Nearest Neighbor Communication

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ABSTRACT

A class of synchronization protocols for dense, large-scale sensor networks is presented. The protocols build on the recent work of Hong, Cheow, and Scaglione [5, 6] in which the synchronization update rules are modeled by a system of pulse-coupled oscillators. In the present work, we define a class of models that converge to a synchronized state based on the local communication topology of the sensor network only, thereby lifting the all-to-all communication requirement implicit in [5, 6]. Under some rather mild assumptions of the connectivity of the network over time, these protocols still converge to a synchronized state when the communication topology is time varying.

Categories and Subject Descriptors

C.2.2 [**Computer-Communication Networks**]: Network Protocols

General Terms

Algorithms, Theory

Keywords

Sensor networks, Synchronization, Pulse-Coupled Oscillators

1. INTRODUCTION

Wireless ad-hoc sensor networks are typically envisioned to be comprised of a large number of small, low-cost, low-power platforms cooperating in a clever way to infer some desired properties from the environment. Realizing the promise and expectations of this new computing paradigm requires the development of scalable, energy efficient communication and inferencing algorithms that leverage the collective sensing capabilities of the network while operating within the constraints posed by the limited resources of the individual sensors. Thus there has been considerable interest in developing *localized algorithms* with low computational complexity. As a prototypical example motivating the need for local algorithms, consider the well known tradeoff between communication range and power consumption. While an all-to-all communication policy satisfies the need for simplicity, the large number of sensors and their spatial distribution precludes broadcast communication due its heavy draw on power resources. Hence there exists a vast literature on ad-hoc routing and networking.

Synchronization in time is perhaps the most important initial calibration task of the network. There has been considerable interest in developing strategies for time synchronization [2, 3, 9]. This is because several important capabilities of the network depend on the existence of a global clock: MAC layer routing and energy conservation, tracking time varying phenomena, and developing strategies for cooperative *sensor reachback*, to name a few. This latter task is the problem of getting information *out* of the network. Time synchronization allows the network to cooperate as a distributed transmission array capable of broadcasting information to distant users or fusion centers [9]. Thus, in a sense, localized time synchronization protocols provide a workaround to the power/range tradeoff described above.

In this note, we present a localized version of a recent proposal for synchronization in large-scale sensor networks. Inspired by the onset of synchronization in biological systems [15], Scaglione and coworkers have proposed a simple, yet potentially powerful, methodology for distributed time synchronization and change detection [5, 6]. In their proposal, synchronization updates are modeled by the dynamical evolution of a set of pulse-coupled oscillators which have been shown to converge to synchrony in a variety of circumstances. However, a significant drawback of their scheme is the reliance on an all-to-all communication model which can cause instability in the synchronization scheme due to long delays in signal propagation time and increased noise in the communication channel. Moreover, it is unlikely that the biological systems that these equations are intended to model operate in such a manner. In fact, there is empirical evidence supporting the emergence of clusters of synchronized behavior before convergence to synchrony of the entire colony.

Drawing from some recent results in multi-agent control (see for example [11, 18, 19, 20, 16, 17]), we propose a

SenSys'04, November 3–5, 2004, Baltimore, Maryland, USA.

simple method for synchronization without the all-to-all assumption. Specifically, we explore conditions on the update protocol that lead to synchrony with bidirectional nearest neighbor coupling. Noise in the detection model and propagation delay are not explicitly considered here, however one can argue that by extending the results of [5, 6] to nearest neighbor interactions only, these effects may be minimized.

The synchronization scheme is briefly described as follows. Each sensor node emits a pulse signal at a time that is controlled by the evolution of a state variable, x_i, that monotonically increases from 0 to 1. When x_i reaches 1, the sensor emits the signal and resets its state variable to 0. Without coupling to the network, the sensor fires repeatedly with fixed frequency. The collective behavior of a large number of sensors operating in this manner would result in a cacophony of pulse emissions with equal frequency but out of phase. The coupling procedure is designed to drive the phase differences to zero. When a sensor node receives a signal from a neighboring node, it increments its state variable by an amount given by $\epsilon g(x_i)$ where ϵ is a small coupling constant and $g(x_i)$ is the phase response function. In our applications, $g(\cdot)$ is chosen to be positive on the interval $[0, 1]$. Thus the effect of a single coupling decreases the time to fire of the receiving node. Scaglione and coworkers appeal to the following sufficient condition to guarantee convergence (almost always) of the synchronization procedure:

Theorem 1. *[15] If the function governing the evolution of the state variables is smooth, monotonically increasing and concave down, then the set of initial states, $x_i(0)\ \forall i$, that never result in synchrony has measure zero.*

The *leaky integrate-and-fire* model, originally proposed as a model for the cardiac pacemaker, provides such a state function. In this model the state variables evolve according to a set of identical differential equations

$$\dot{x_i}(t) = \alpha - \beta x_i(t),\ 0 < x_i < 1,\ \forall i,\ \alpha > \beta > 0\,. \quad (1)$$

A constant coupling function is chosen in [5, 6] yielding the pulse-coupled equations

$$\dot{x_i}(t) = \alpha - \beta x_i(t) + \epsilon \sum_{j \neq i} \delta(t - t_j^*) \quad (2)$$

where t_j^* is the firing time of the jth oscillator. Note that we have simplified the coupling equations appearing in [5, 6] by not considering the entire history of firing times in the evolution equations. Note also, that the summation in (2) is over all sensors in the network.

An intriguing contribution of [5, 6] is the incorporation of sensor data into the evolution equations. Assume that each node has some ability to classify some phenomenon. Assume that the system is initially synchronized and at a specified sampling time each sensor uniformly perturbs its state dynamics to embed the result of its local classification. For a binary classifier, two out of sync clusters will emerge and thus the coupling procedure will be initiated. By suitable design of the perturbation, the resulting phase shift depends on the proportion of nodes detecting the phenomenon, thereby disseminating the local decisions throughout the network. A similar procedure can be used for sensor reachback. We refer the reader to [5, 6] for a more complete description of this novel idea. We include these remarks here to highlight the versatility of the synchronization protocols and that the methods we propose can accommodate such encoding.

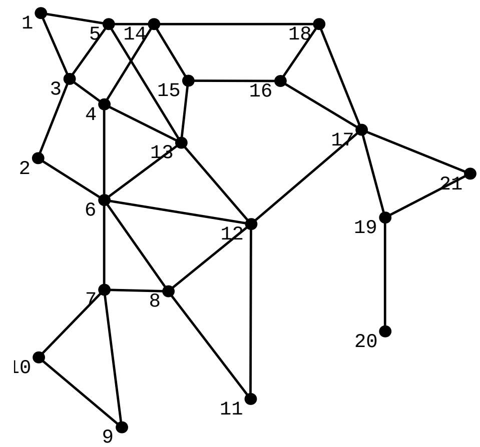

Figure 1: Sensor Net Topology

In this note, we extend this synchronization scheme to where the case where all-to-all communication is not feasible. By casting the pulse-coupled equations as a dynamical system of the phase deviations, a stability result based on nearest neighbor coupling can be obtained.

2. PRELIMINARIES

In this section we provide the introductory background on graph theory [1, 4] and the phase model of pulse-coupled oscillators [7, 10].

2.1 Algebraic Graph Theory

The interconnection topology of the network is specified by a graph with an edge joining neighboring nodes. Formally, let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote a graph with an n-dimensional vertex set, $\mathcal{V}$, and an e-dimensional edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. Vertices are denoted v_i and if $(v_i, v_j) \in \mathcal{E}$, then v_i and v_j are said to be *neighbors*. The set of neighbors of node v_i is denoted by $\mathcal{N}_i = \{j : (v_i, v_j) \in \mathcal{E}\}$. The number of neighbors of a vertex, $|\mathcal{N}_i|$, is the *degree* of the vertex. The $n \times n$ diagonal matrix $\Delta(\mathcal{G})$ whose *i-th* diagonal entry is the degree of node v_i is called the *degree matrix*. A graph $\mathcal{G}$ is given an orientation by choosing a direction for each edge. Define the $n \times e$ *incidence matrix*, $\mathcal{C}(\mathcal{G})$, of an oriented graph with (i, j) entry equal to 1 if edge j terminates in vertex v_i, -1 if edge j originates in vertex v_i, and 0 otherwise. The Laplacian matrix of $\mathcal{G}$ is defined by

$$\mathcal{L}(\mathcal{G}) = \mathcal{C}(\mathcal{G})\mathcal{C}(\mathcal{G})^t = \Delta(\mathcal{G}) - \mathcal{A}(\mathcal{G})$$

where $A(\mathcal{G})$ is the adjacency matrix whose (i, j) entry is equal to 1 if $(v_i, v_j) \in \mathcal{E}$ and 0 otherwise. Finally, a graph $\mathcal{G}$ is said to be *connected* if there exists a path between any two vertices.

The spectrum of the Laplacian captures topological properties of the graph which are crucial in proving the synchronization results cited below. See for example Ref [1]. Most notably, for a connected graph, the Laplacian has a unique zero eigenvalue and the associated eigenvector is given by the vector of ones, $\mathbb{1} \equiv [1, 1, \ldots, 1]^t$.

2.2 Pulse Coupled Oscillators and the Phase Model

Consider a set of n oscillators coupled according to an interconnection graph $\mathcal{G}$. Let $x_i \in [0,1]$ be the state variable of the *i-th* oscillator. When x_i reaches 1, the oscillator emits a pulse and x_i resets to 0. The dynamics of coupled system is given by a system of differential equations of the form

$$\dot{x}_i = f_i(x_i) + \epsilon \sum_{j \in \mathcal{N}_i} g_{ij}(x_i)\delta(t - t_j^*) \tag{3}$$

where $f_i(x_i)$ describes the dynamics of the *i-th* oscillator, $g_{ij}(x_i)$ is the coupling function between oscillators i and j, t_j^* is the firing time of the *j-th* oscillator and ϵ is the (small) coupling constant. Note that the summation in (3) is over the set of neighbors of each oscillator. This embodies the nearest-neighbor coupling among sensors. Due to the δ function in (3), the firing an oscillator increments the state variable of its neighbors by $\epsilon g_{ij}(x_i)$, i.e. if $j \in \mathcal{N}_i$

$$x_j = 1 \Longrightarrow \begin{cases} x_i \to x_i + \epsilon g_{ij}(x_i) & if \quad x_i + \epsilon g_{ij}(x_i) < 1 \\ x_i \to 0 & otherwise\,. \end{cases}$$

Note that once a neighboring oscillator fires, only local information (x_i) is required to implement the update procedure. Our goal is to characterize the synchronization of the firing time of the oscillators without the all-to-all assumption. To do so, it is beneficial to recast (3) in terms of a *phase variable.*

Consider the uncoupled dynamics

$$\dot{\xi}_i = f_i(\xi_i), \qquad \xi_i(0) = 0\,.$$

The solution, $\xi_i(t)$, is periodic with period

$$T_i = \int_0^1 dt = \int_0^1 \frac{dt}{d\xi_i} d\xi_i$$

and frequency

$$\omega_i = \frac{2\pi}{T_i}\,.$$

A phase variable, $\theta_i \in \mathbb{S}^1$, is introduced to simplify the representation of the pulse-coupled oscillators (3). When no oscillator is firing the phase variable, θ, evolves according to

$$\dot{\theta}_i = \omega_i\,,$$

it thus represents the *natural frequency* of the uncoupled oscillator. Coupling among the oscillators can induce a phase shift in the frequency of an oscillator. A phase deviation variable ϕ_i, evolving on $\mathbb{S}^1$, is introduced to capture this phase shift. Now

$$\theta(t) = \omega t + \phi\,.$$

The oscillators (3) evolving in $[0,1]^n$ are transformed to the phase variables θ and then averaged [10, 7, 12] to obtain an evolution equation of the phase deviation variables only. In the following, we show how this representation facilitates the convergence analysis. Note that here we represent $\mathbb{S}^1$ as the interval $[-\pi, \pi]$ by defining $-\pi \equiv \pi$. The direct product of n copies of $\mathbb{S}^1$ is an n-torus and will be denoted $\mathbb{T}^n$. These remarks are summarized by the following lemma, taken directly from Ref. [10], though adapted to the general interconnection case. See also [7].

LEMMA 1. *Let $\mathcal{G}$ be an undirected graph describing the coupling topology of a system of oscillators of the form (3) and let $\mathcal{N}_i$ be the set of neighbors of node i in $\mathcal{G}$. If the oscillators evolve according to identical uncoupled dynamics $f_i(x) = f(x) > 0,\ \forall x \in [0,1]$, then there exists an $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$, there is a continuous change of variables that transforms (3) into the phase model*

$$\dot{\phi}_i = \epsilon \sum_{j \in \mathcal{N}_i} \gamma_{ij}(\phi_j - \phi_i) + \mathcal{O}(\epsilon^2) \tag{4}$$

where

$$\gamma_{ij}(\psi) = \frac{\omega^2}{2\pi} \frac{g_{ij}(\xi(\frac{\psi}{\omega}))}{f(\xi(\frac{\psi}{\omega}))}\,. \tag{5}$$

A similar representation, valid in the limit as $\epsilon \to 0$ and $n \to \infty$, was obtained by Kuramoto [13] with methods from statistical physics and mean field theory.

Synchronization of the oscillators, i.e. all the x_i firing with identical phase and frequency, is equivalent to showing that the coupling procedure drives the phase deviations to a common value, i.e.

$$\phi(t) = [\phi_1(t), \phi_2(t), \ldots, \phi_n(t)]^t \to \phi^*[1, 1, \ldots, 1]^t$$

for some $\phi^* \in \mathbb{S}^1$.

3. SYNCHRONIZATION

Consider a system of n pulse-coupled oscillators of the form

$$\dot{x}_i = f(x_i) + \epsilon \sum_{j \in \mathcal{N}_i} g(x_i)\delta(t - t_j^*), \quad x_i \in [0,1]. \tag{6}$$

Note that each oscillator follows identical uncoupled dynamics and applies an identical coupling function. The phase model for this class of systems is reminiscent of a class of problems studied in the cooperative control literature. Similar results appear in [11, 20, 18, 19, 16, 17]. In particular, consult references [18, 17].

THEOREM 2. *Let $\mathcal{G}^c$ be a connected graph describing the coupling topology of a system of oscillators of the form (6). If $\gamma \propto g/f$ is an uneven function such that $\gamma(0) = 0$ and $\psi \cdot \gamma(\psi) > 0,\ \forall \psi \neq 0$, then in the limit as $\epsilon \to 0$, the system asymptotically synchronizes for any initial condition $x_0 \in \mathcal{D}$, where $\mathcal{D}$ is a compact subset of $[0,1]^n$.*

Proof. In the small ϵ limit, the truncated phase model is an accurate description of the original system (6). The truncated system can be written in vector form as

$$\dot{\phi} = -\epsilon \Omega\, \mathcal{C} \gamma(\mathcal{C}^t \phi), \tag{7}$$

where Ω is a constant depending on the natural frequency ω and $\mathcal{C}$ is the incidence matrix of $\mathcal{G}^c$ with respect to some orientation of $\mathcal{G}^c$. To see this, note that the $|\mathcal{E}|-$dimensional vector $\chi = \mathcal{C}^t \phi$ is the vector of nearest neighbor phase differences and since $\gamma(\cdot)$ is uneven, the expression $\gamma(\mathcal{C}^t \phi)$ does not depend on the (arbitrary) orientation of $\mathcal{G}^c$.

Let $\mathcal{D}_\phi \subset \mathbb{T}^n$ denote the compact image of $\mathcal{D}$ in the new coordinates. Now consider the positive definite function

$$V(\phi) = \frac{1}{2}||\phi||^2. \tag{8}$$

The function $V(\phi)$ is non-increasing along trajectories of (7), since

$$\dot{V}(\phi) = \phi^t \dot{\phi} = -\epsilon\, \Omega \phi^t \mathcal{C} \gamma(\mathcal{C}^t \phi) \quad (9)$$

$$= -\epsilon\, \Omega \chi^t \gamma(\chi) \leq 0\,. \quad (10)$$

Where the last inequality follows from the conditions on γ. Thus $V(\phi)$ is a *Lyapunov function* [12] for the system (7). Lasalle's Invariance Principle [12] asserts that if there exists a Lyapunov function that is negative semidefinite along trajectories of a differential equation, then solutions, originating in $\mathcal{D}_\phi$, converge to the largest invariant set contained in

$$E = \{\phi \,:\, \dot{V}(\phi) = 0\,\}. \quad (11)$$

To show that the system asymptotically synchronizes, we show that the only solution to (11) is given by

$$\phi = \phi^* [1, 1, \ldots, 1]^t\,.$$

To see this note that (9) can be written as

$$\dot{V}(\phi) = -\epsilon\, \Omega \phi^t \mathcal{C} \mathcal{W} \mathcal{C}^t \phi \quad (12)$$

where $\mathcal{C}\mathcal{W}\mathcal{C}^t$ is the *weighted* Laplacian [4] formed by the $|\mathcal{E}| \times |\mathcal{E}|$ positive definite diagonal matrix $\mathcal{W}$ whose *i-th* diagonal entry is given by [1]

$$\mathcal{W}_{ii} = w^i_{uv} = \frac{\gamma(\phi_u - \phi_v)}{\phi_u - \phi_v}$$

where $v \in \mathcal{N}_u$. Now, for a connected graph, by the properties of the graph Laplacian, the unique solution to

$$\mathcal{C}\mathcal{W}\mathcal{C}^t \phi_s = 0$$

is given by the vector of ones. ■

Remark 1. *The Lyapunov function provides a bound on the rate of convergence for the truncated phase model.*

$$\dot{V}(\phi) = -\epsilon\, \Omega \phi^t \mathcal{C}\mathcal{W}\mathcal{C}^t \phi \quad (13)$$

$$= -\epsilon\, \Omega \sum_{(u,v)\in\mathcal{E}} w_{uv} (\phi_u - \phi_v)^2 \quad (14)$$

$$\leq -\epsilon\, \Omega w_{min} \sum_{(u,v)\in\mathcal{E}} (\phi_u - \phi_v)^2 \quad (15)$$

$$= -\epsilon\, \Omega w_{min} \phi^t \mathcal{C}\mathcal{C}^t \phi \quad (16)$$

$$= -\epsilon\, \Omega w_{min} \phi^t \mathcal{L}(\mathcal{G}) \phi\,. \quad (17)$$

Denote the least non-zero eigenvalue of the graph Laplacian by $\lambda_2(\mathcal{G})$, this eigenvalue is known as the algebraic connectivity of the graph. By a corollary to the Rayleigh-Ritz inequality [8], we have

$$\lambda_2(\mathcal{G}) \leq \frac{\phi^t_\perp \mathcal{L}(\mathcal{G}) \phi_\perp}{||\phi_\perp||^2} \quad (18)$$

where $\phi_\perp$ is any vector orthogonal to $\mathbb{1}$. Combining these, we have

$$\dot{V}(\phi) \leq -\epsilon\, \Omega w_{min} \lambda_2(\mathcal{G}) ||\phi_\perp||^2\,. \quad (19)$$

Thus the truncated model converges to the synchronized state exponentially with rate characterized by ϵ, Ω and $\lambda_2(\mathcal{G})$. The algebraic connectivity, $\lambda_2(\mathcal{G})$, is known to be large for dense graphs and relatively small for sparse graphs. In this bound on the convergence rate we see the tradeoff between the connectivity of the graph, the smallness of ϵ (required for our analysis) and the natural frequency of pulse emissions, reflected by the presence of Ω.

[1]Here we define $\gamma(0)/0 \equiv 1$.

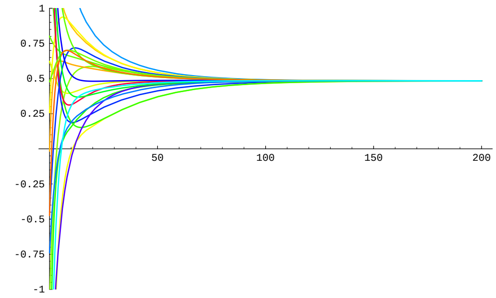

Figure 2: Linear Protocol

Remark 2. *The truncated models can be shown to converge even with a dynamic communication topology. That is, consider the system*

$$\dot{\phi} = -\epsilon\, \Omega \mathcal{C}(\mathcal{G}_{\sigma(t)}) \gamma(\mathcal{C}^t(\mathcal{G}_{\sigma(t)})\phi), \quad (20)$$

where the switching signal, $\sigma(t)$ is a piece-wise constant function taking values in an index set $\mathcal{I}$ and $\{\mathcal{G}_i : i \in \mathcal{I}\}$ is a family of graphs describing the communication topology of the network. If the switching is among connected graphs, then the Lyapunov function of the preceding analysis can be shown to be a so-called common Lyapunov function [14] thereby ensuring convergence of the switched system. In fact a more sophisticated analysis reveals a much weaker condition regarding the connectivity of the family of graphs [17]. Namely, if there exists a time, T, such that the union of all interconnection graphs is connected over any interval of length T, then the system will synchronize. This property is particularly attractive when considering synchronization protocols for sensor networks due to the unavoidable disruptions in service due to noise, node failure, sleep periods, etc.

4. EXAMPLES

4.1 Linear

In this example, the coupled state equations are given by

$$\dot{x}_i = \frac{1}{T} + \epsilon \sum_{j\in\mathcal{N}_i} x_i \delta(t - t^*_j)\,. \quad (21)$$

Thus the firing time updates are given by

$$x_j = 1 \Longrightarrow \begin{cases} x_i \to x_i + \epsilon x_i & if \quad x_i + \epsilon x_i \leq 1 \\ x_i \to 0 & \quad otherwise\,. \end{cases}$$

This corresponds to a nearest neighbor phase model of the form,

$$\dot{\phi}_i = -\frac{\epsilon\omega}{2\pi} \sum_{j\in\mathcal{N}_i} (\phi_i - \phi_j), \quad \phi(0) \in [-\pi, \pi]\,. \quad (22)$$

The system of oscillators can be written in terms of the graph Laplacian, $\mathcal{L}$,

$$\dot{\phi} = -\frac{\epsilon\omega}{2\pi} \mathcal{L}\phi, \quad \phi(0) \in \mathbb{T}^n. \quad (23)$$

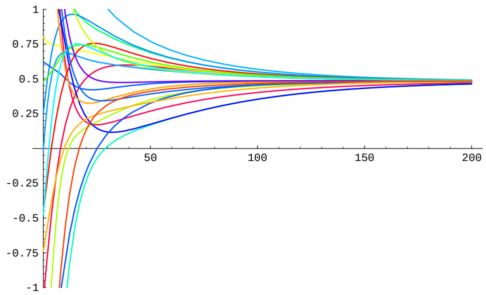

Figure 3: Sine Protocol

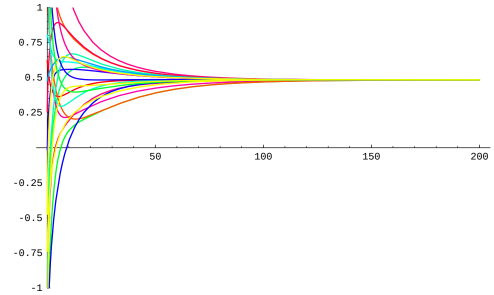

Figure 4: Hyperbolic Sine Protocol

It can be shown [19] that with the linear model, convergence to synchronization is robust with respect to time delay in the communication link, provided the delay is less that

$$\frac{\pi}{2\lambda_{max}(\mathcal{G})}.$$

Figure 2 shows the phase convergence for a network with topology depicted in Figure 1. In this simulation and those that follow, $\epsilon = .1, \Omega = 1$ and

$$\phi^* = \frac{1}{n}\sum_i \phi_i(0) = 0.593974\,.$$

4.2 Sine

In this example, the coupled state equations are given by

$$\dot{x}_i = \frac{1}{T} + \epsilon \sum_{j\in\mathcal{N}_i} \sin(\pi x_i)\delta(t - t_j^*)\,. \qquad (24)$$

Note that an oscillator is insensitive to coupling when $x = 1$ or $x = 0$. This is known as a refractory period for the firing and is implicit in the model proposed in [5].

The firing time updates are given by

$$x_j = 1 \Longrightarrow \begin{cases} x_i \to x_i + \epsilon\sin(\pi x_i) & if \quad x_i + \epsilon\sin(\pi x_i) \le 1 \\ x_i \to 0 & otherwise\,. \end{cases}$$

The phase model in this case is given by

$$\dot{\phi}_i = -\epsilon\omega \sum_{j\in\mathcal{N}_i} \sin\left(\frac{\phi_i - \phi_j}{2}\right), \quad \phi_i(0) \in [-\pi, \pi]\,. \qquad (25)$$

We write the phase model in vector form as

$$\dot{\phi} = -\epsilon\omega\,\mathcal{C}\sin\left(\mathcal{C}^t\frac{\phi}{2}\right), \quad \phi(0) \in \mathbb{T}^n\,. \qquad (26)$$

4.3 Hyperbolic Sine

Here we present a variation on the leaky integrate-and-fire model [15]. The choice of the coupling function was made to ensure an uneven function governing the dynamics in the phase model representation. The coupled oscillators evolve according to,

$$\dot{x}_i = \alpha - \beta x_i + \epsilon \sum_{j\in\mathcal{N}_i} \left(\frac{2\alpha}{\beta}x_i - x_i^2\right)\delta(t - t_j^*) \qquad (27)$$

where $\alpha > \beta > 0$. The solution to the uncoupled dynamics, $\xi_i(t)$, is given by the concave function

$$\xi_i(t) = \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right), \quad \xi_i(0) = 0\,. \qquad (28)$$

In the phase coordinates, the dynamics are governed by

$$\begin{aligned} \gamma(\psi) &= \frac{\omega^2\alpha^2}{2\pi\beta^2}\frac{1 - e^{-2\frac{\beta\psi}{\omega}}}{\alpha e^{-\frac{\beta\psi}{\omega}}} & (29)\\ &= \frac{\omega^2\alpha}{2\pi\beta^2}\left(e^{\frac{\beta\psi}{\omega}} - e^{-\frac{\beta\psi}{\omega}}\right) & (30)\\ &= \frac{\omega^2\alpha}{\pi\beta^2}\sinh\left(\frac{\beta\psi}{\omega}\right). & (31) \end{aligned}$$

Thus the phase model becomes,

$$\dot{\phi} = -\frac{\epsilon\omega^2\alpha}{\pi\beta^2}\,\mathcal{C}\sinh\left(\mathcal{C}^t\frac{\beta\phi}{\omega}\right), \quad \phi(0) \in \mathbb{T}^n\,. \qquad (32)$$

4.4 Simulation with the Linear Model

In this section we present simulation results with the linear model, described by (21). For these simulations, we assume that the pulse propagation time between the neighboring nodes is negligible and hence can be treated as zero. We show convergence results with both static and time-varying topologies. Even though we only present results with a specific initial state, results with other initial conditions have similar characteristics.

Figures 5 and 6 are results for the static topology depicted in Figure 1. Here we use the same parameter as in the phase model simulation given in Figure 2 with $T = \frac{1}{2\pi}$ and $\epsilon = 0.1$. Figure 5 plots the firing time (of any node in the network) versus the time between consecutive firings (or the time elapsed from the previous firing). We can see that the time between firings converges to $T \approx 0.1592$ before 3 time units. This implies that the entire network is synchronized before 3 time units. Figure 6 shows the state trajectories of node 1, 8, and 20. It can be seen that nodes 1 and 8 are synchronized before they are synchronized with node 20, which is at the edge of the network with only one neighbor. Comparing with the simulation of corresponding phase model (Figure 2), we notice that the protocol converges much faster in the time domain. This is due to the property that the state of a node with model (21) would make an instantaneous "jump" with amplitude ϵx when it receives a pulse from a neighboring node. If the state of the node is very close to 1, then the jump would trigger a firing and force the node to synchronize with the neighbor immediately (since we assume the propagation time between neighbors is zero). This discontinuity is not captured in the phase model.

To evaluate the protocol for time-varying topologies, we

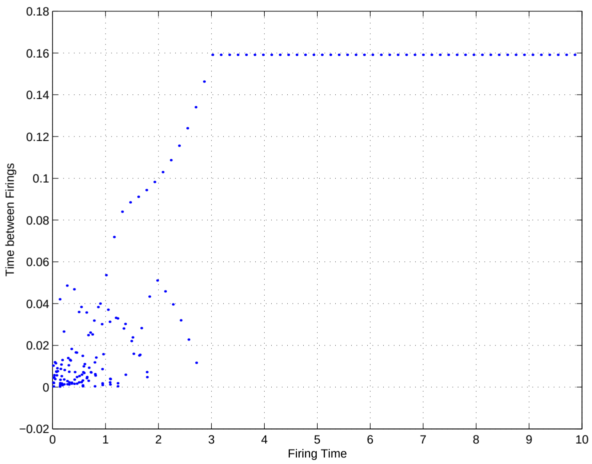

Figure 5: Time between Firings for Static Sensor Net Topology

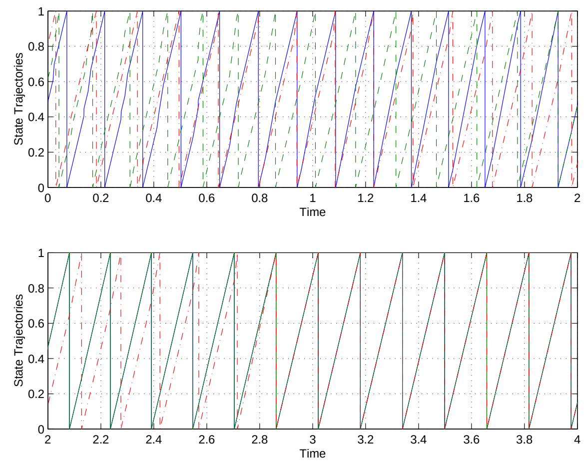

Figure 6: State Trajectory of Node 1, 8, and 20 with Static Topology

consider the scenario where the topology of the sensor network is switching periodically between two topologies every $T_s = 10T$ time units. The two topologies are obtained from the topology in Figure 1 by severing the edges terminating in node 7 and 17. Note that neither of these two topologies is connected. Nevertheless, the union of the sequence of topologies over every $2T_s$ time units is connected. This meets the weaker connectivity condition defined in [16]. The convergence result under such switching topologies is given by Figures 7 and 8.

5. DISCUSSION

By drawing from results in the cooperative control literature, we have extended a recently proposed synchronization scheme to the case where all-to-all communication is not feasible. This method is attractive due to its provable convergence and relative simplicity. Operationally, this localized synchronization scheme is quite simple due to the pulse signal encoding of information. Careful attention must be given to any distributed algorithm mediated by a wireless network, since quantization and encoding can introduce errors in the message passing that may be difficult to characterize and correct. Note that in this approach, there is no signal to transmit except for the pulse. Any quantization errors would occur locally, when the individual processor updates its state equation. We are also confident in this approach with regard to noise. Given a static network, a noisy link would be modeled as a "dropped pulse", which would effectively represent a change in communication topology. However, given that these protocols are robust with respect to changes in communication topology, we are confident that convergence would not be adversely effected. Of course, these simple heuristic remarks must be supplemented with models and rigorous results. We hope to provide these in an extended version of this paper.

We are particularly interested in these techniques for synchronizing information in a sensor network beyond simple time synchronization. Though it should be noted that time synchronization is still an area receiving attention from the research community. The analysis presented here may be

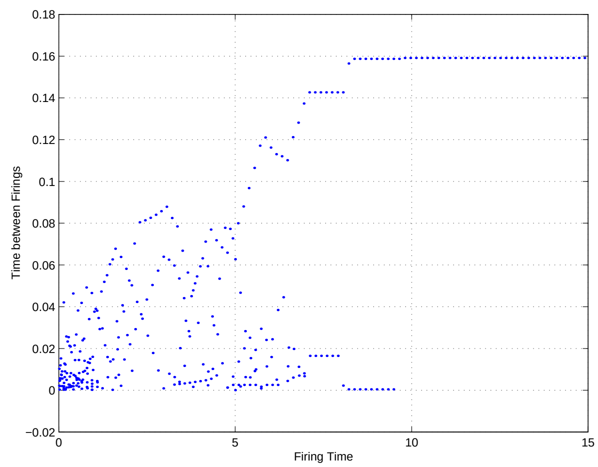

Figure 7: Time between Firings with Switching Topologies

useful in providing a more theoretical understanding of the novel encoding schemes for decentralized binary hypothesis testing and change detection proposed in [5, 6]. It can be shown that these phase models converge to the average of the initial conditions of the phases. A detailed understanding of how this fact manifests in the phase shift experienced at each oscillator may be useful in determining a closed form expression for the proportion of sensors detecting the event that can be evaluated at each sensor. This analysis would require a more rigorous treatment of the $\mathcal{O}(\epsilon^2)$ term in the phase model transformation.

Finally, we note that these ideas could be extended to accommodate directed graphs by following the results of [19, 20, 16]. This, perhaps trivial, extension would provide more confidence in the robustness of this approach by allowing for situations where bidirectional communication cannot be guaranteed. However, these dynamical systems do not solve the average consensus problem.

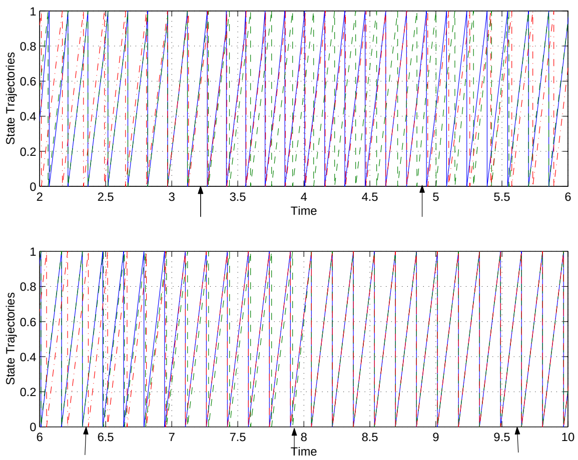

Figure 8: State Trajectory of Node 1, 7, and 20 with Switching Topologies (arrows mark approximated switching times)

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