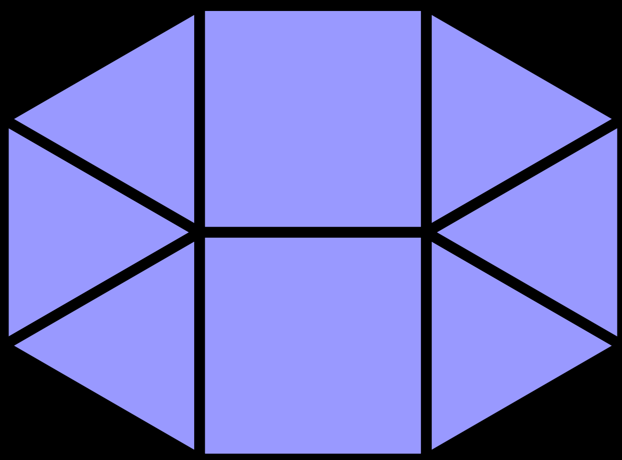
Triangles and Squares

David Eppstein, ICS Theory Group, April 20, 2001

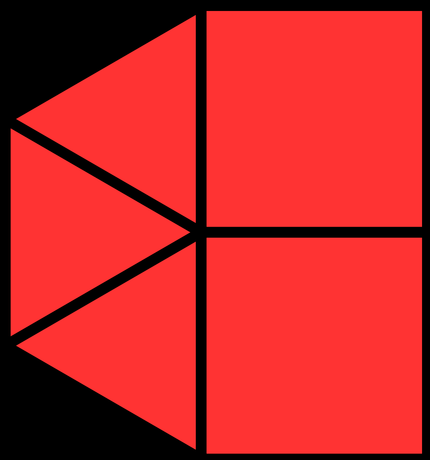
Which unit-side-length convex polygons can be formed by packing together unit squares and unit equilateral triangles? For instance one can pack six triangles around a common vertex to form a regular hexagon. It turns out that there is a pretty set of 11 solutions. We describe connections from this puzzle to the combinatorics of 3- and 4-dimensional polyhedra, using illustrations from the works of M. C. Escher and others.

(Joint work with Günter Ziegler)

Which strictly convex polygons can be made
by gluing together unit squares and equilateral triangles?

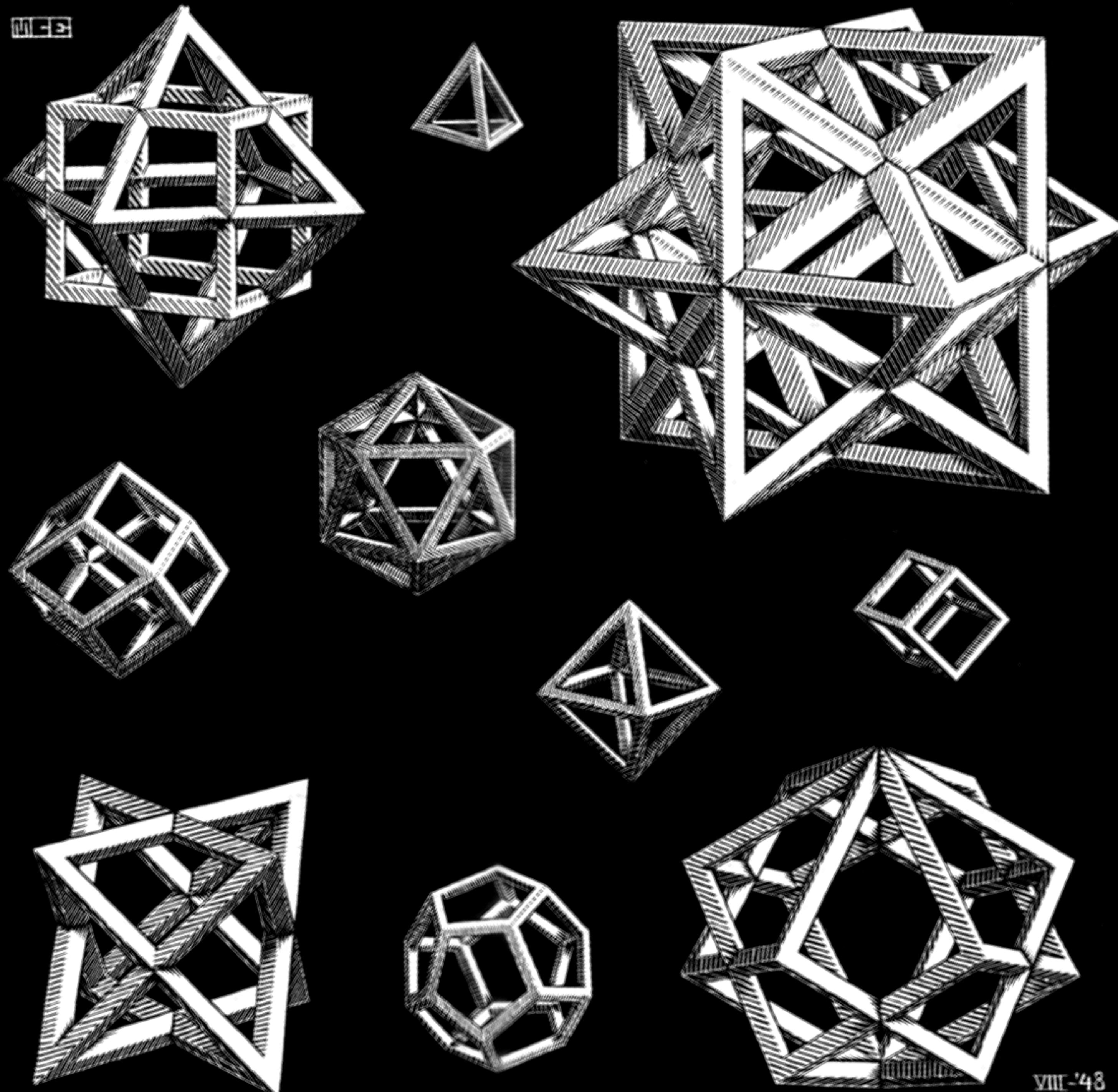


Strictly convex



Not strictly convex

The Five Platonic Solids (and some friends)



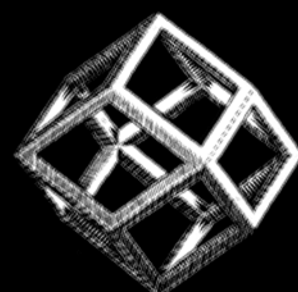
M. C. Escher, Study for Stars, Woodcut, 1948

The Six Regular 4-Polytopes

Simplex, 5 vertices, 5 tetrahedral facets, analog of tetrahedron

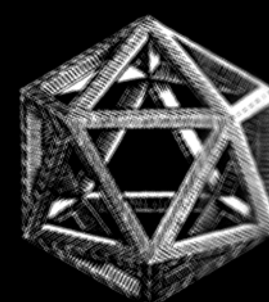
Hypercube, 16 vertices,
8 cubical facets, analog of cube

Cross polytope, 8 vertices,
16 tetrahedral facets, analog of octahedron



24-cell, 24 vertices, 24 octahedral facets, analog of rhombic dodecahedron

120-cell, 600 vertices,
120 dodecahedral facets, analog of dodecahedron



600-cell, 120 vertices,
600 tetrahedral facets, analog of icosahedron

Mysteries of four-dimensional polytopes...

What face counts are possible?

For three dimensions, $f_0 - f_1 + f_2 = 2$, $f_0 \leq f_2 - 4$, $f_2 \leq f_0 - 4$
describe all constraints on numbers of vertices, edges, faces
All counts are within a constant factor of each other

For four dimensions, some similar constraints exist, e.g. $f_0 + f_2 = f_1 + f_3$
but we don't have a complete set of constraints

Is "fatness" $(f_1 + f_2)/(f_0 + f_3)$ bounded?
Known $O((f_0 + f_3)^{1/3})$ [Edelsbrunner & Sharir, 1991]

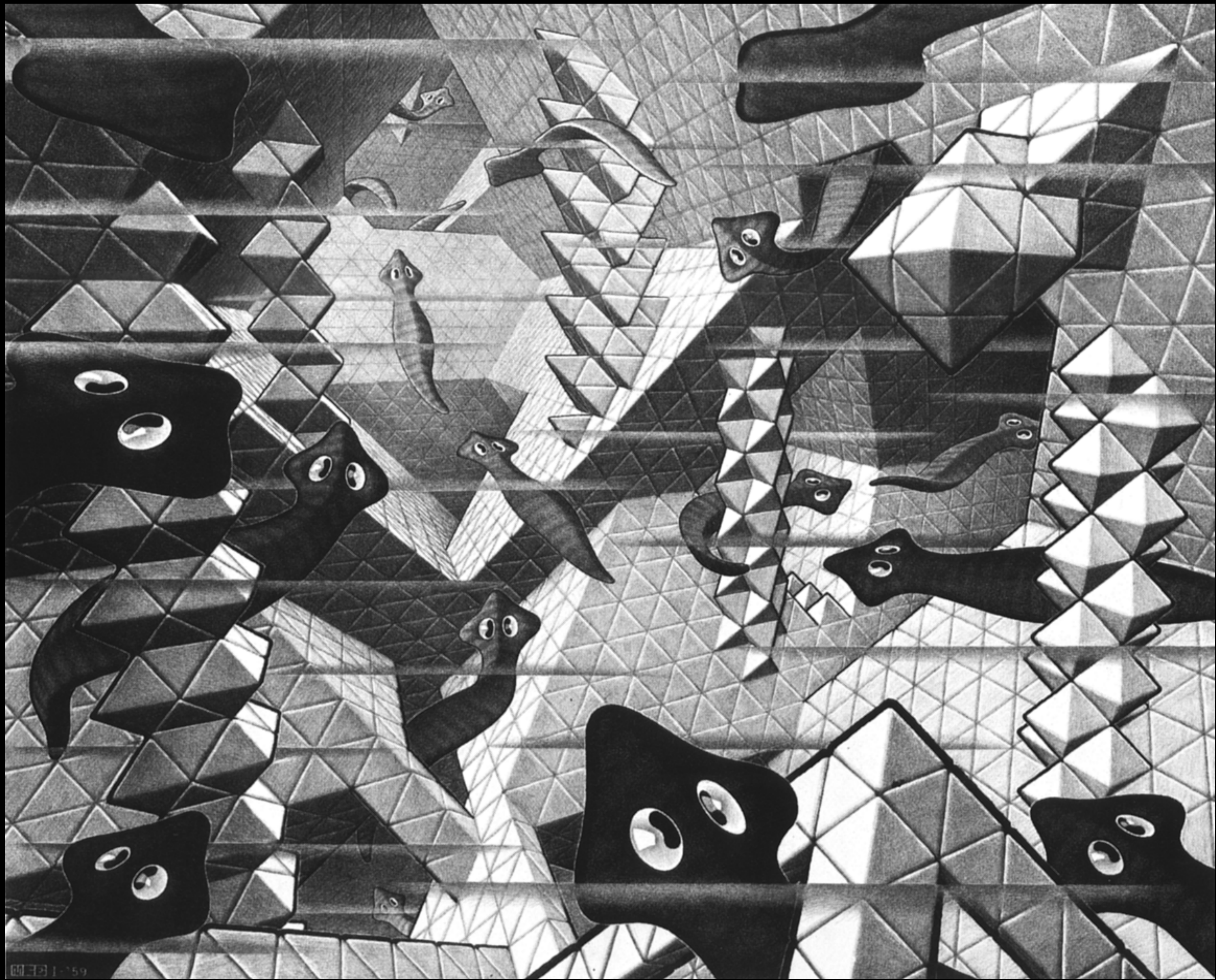
How can we construct more examples like the 24-cell?

All 2-faces are triangles ("2-simplicial")

All edges touch three facets ("2-simple")

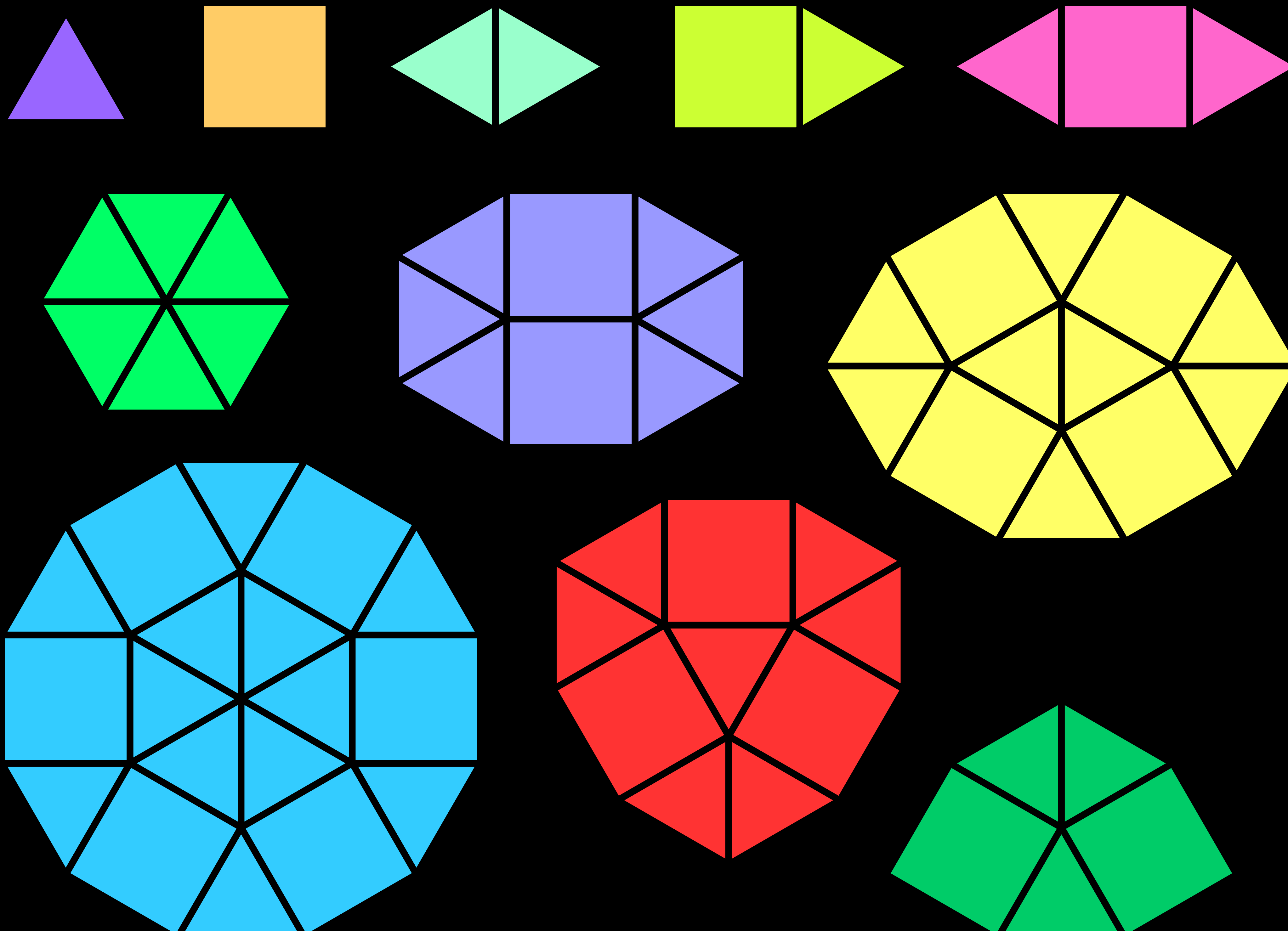
Only few 2-simple 2-simplicial examples were known:
simplex, hypersimplex, 24-cell, Braden polytope

Octahedron and tetrahedron dihedrals add to 180!
So they pack together to fill space



M. C. Escher, Flatworms, lithograph, 1959

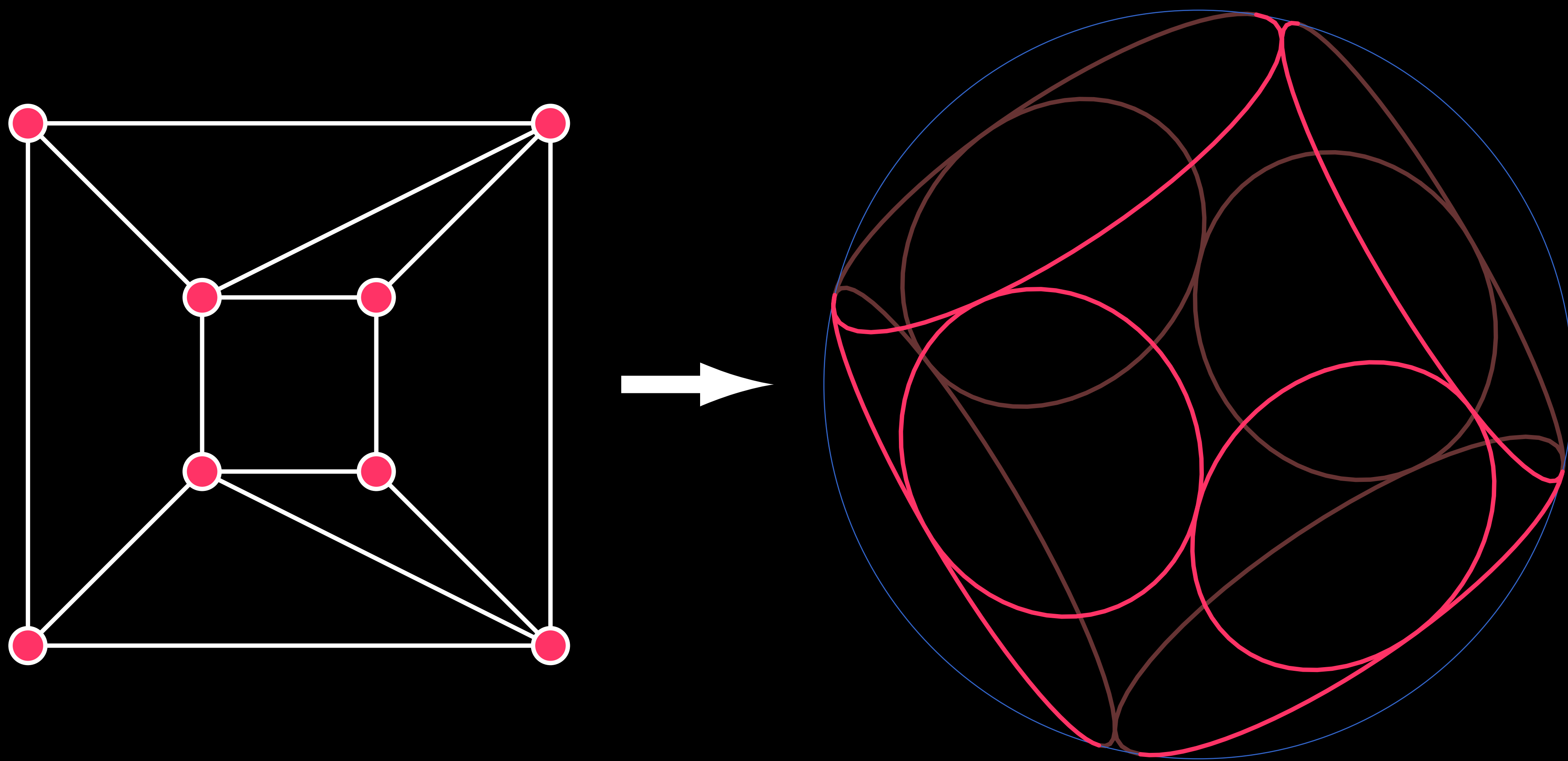
The Eleven Convex Square-Triangle Compounds



Polytopes and spheres

M. C. Escher, Order and Chaos, lithograph, 1950

Theorem [Koebe, 1936]:



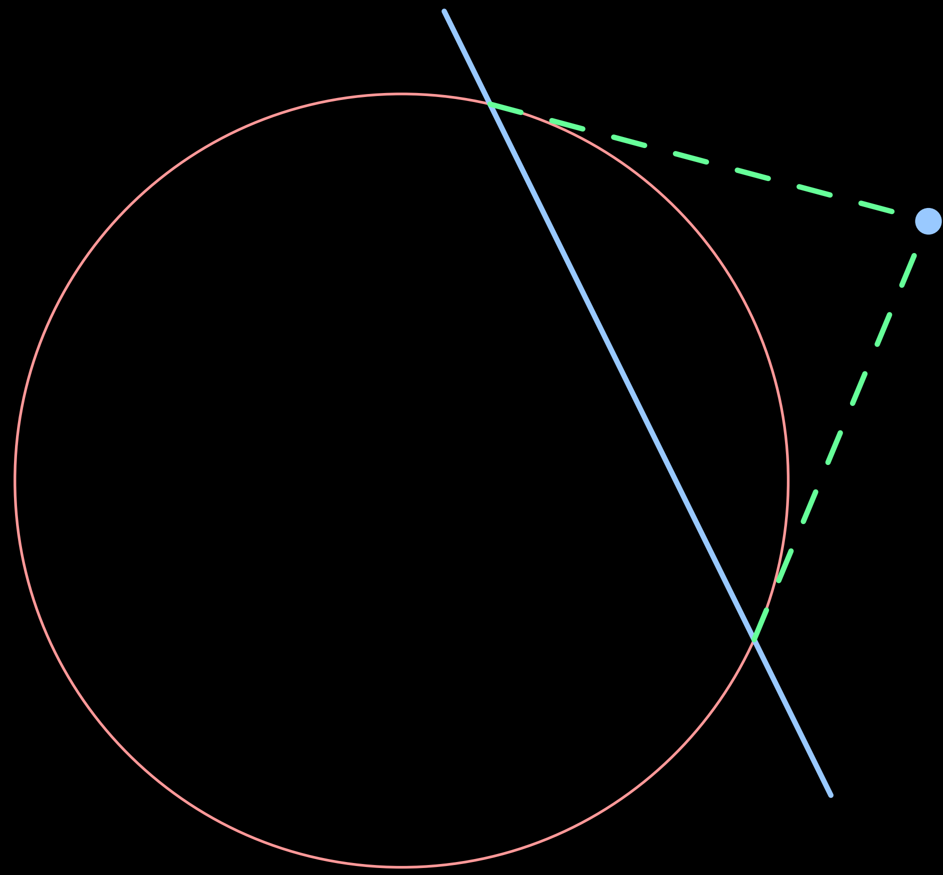
Any planar graph can be represented by circles on a sphere, s.t. two vertices are adjacent iff the corresponding two circles touch

Replacing circles by apexes of tangent cones forms polyhedron with all edges tangent to the sphere

Polarity

Correspond points to lines in same direction from circle center
distance from center to line = 1/(distance to point)

Line-circle crossings equal point-circle horizon
Preserves point-line incidences! (a form of projective duality)



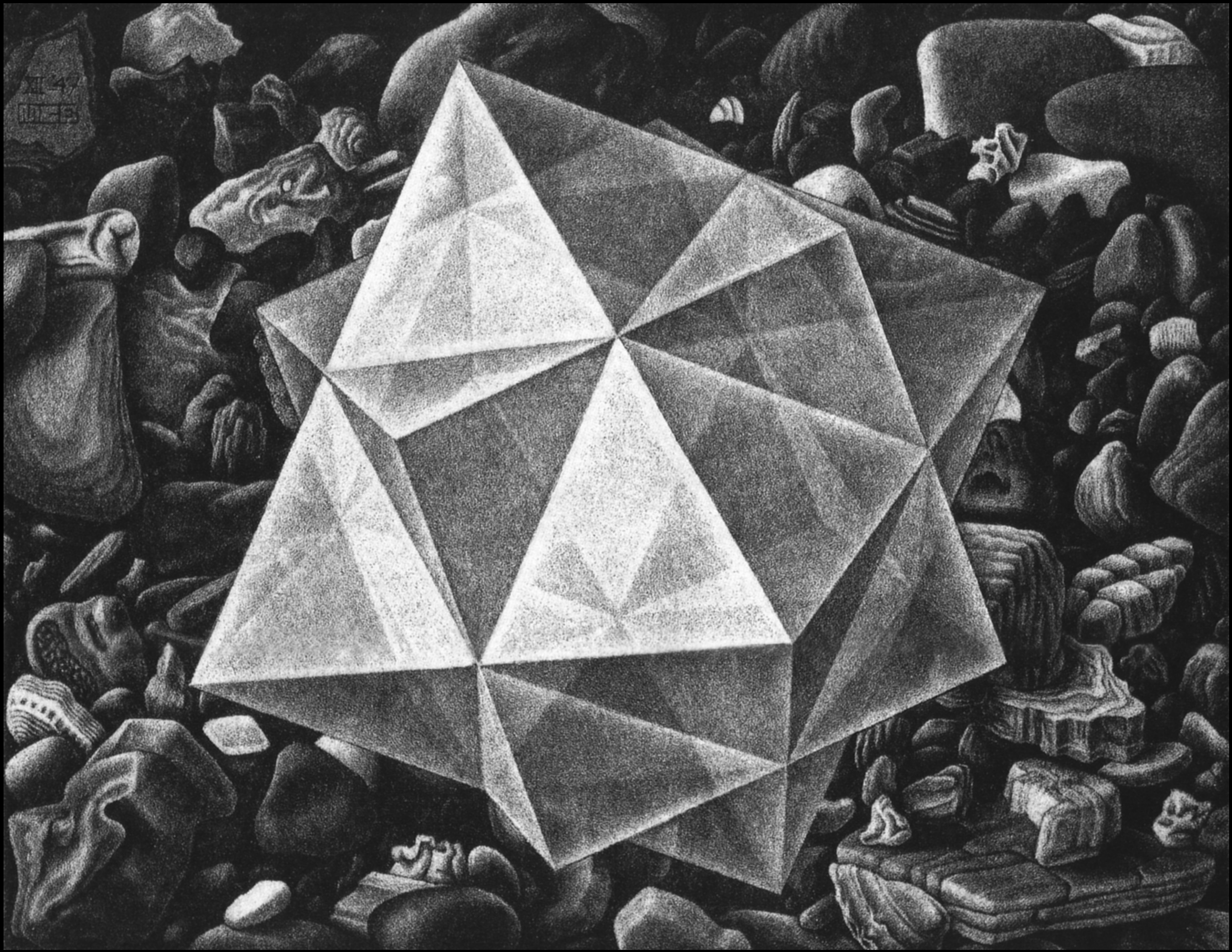
Similar dimension-reversing correspondence in any dimension

Converts polyhedron or polytope (containing center) into its dual

Preserves tangencies with unit sphere

Convex Hull of (P union polar), P edge-tangent

Edges cross at tangencies; hull facets are quadrilaterals



M. C. Escher, Crystal, mezzotint, 1947

Same Construction for Edge-Tangent 4-Polytopes?

Polar has 2-dimensional faces (not edges) tangent to sphere

Facets of hull are dipyramids over those 2-faces

All 2-faces of hull are triangles (2-simple)

Three facets per edge (2-simplicial) if and only if edge-tangent polytope is simplicial

This leads to all known 2-simple 2-simplicial polytopes

Simplex $\Rightarrow$ hypersimplex

Cross polytope $\Rightarrow$ 24-cell

600-cell $\Rightarrow$ new 720-vertex polytope, fatness=5

So are there other simplicial edge-tangent polytopes?

Polars of truncated hypercubes?

Formed by gluing simplexes onto tetrahedral facets of cross polytope

Always simplicial

Many different variations

If we warp the glued simplex to make it edge-tangent, is the result still convex?

Need a space where we can measure convexity independent of warping (projective transformations)

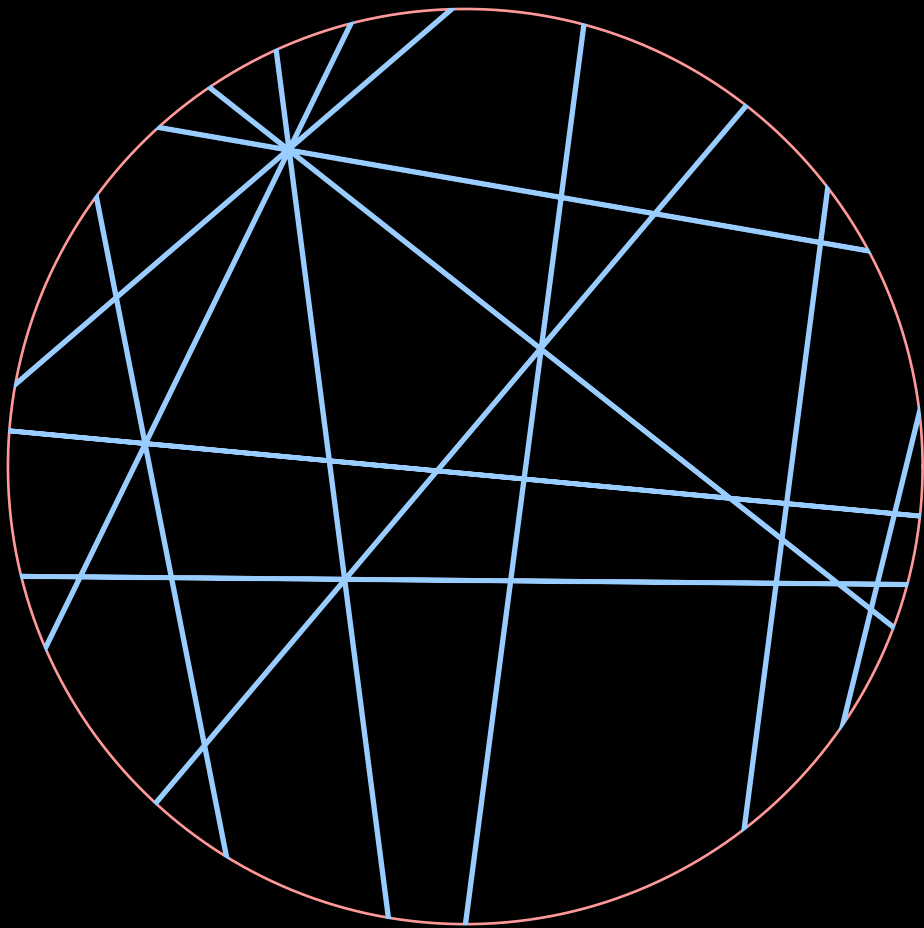
Answer: hyperbolic geometry!

Hyperbolic Space (Poincaré model)

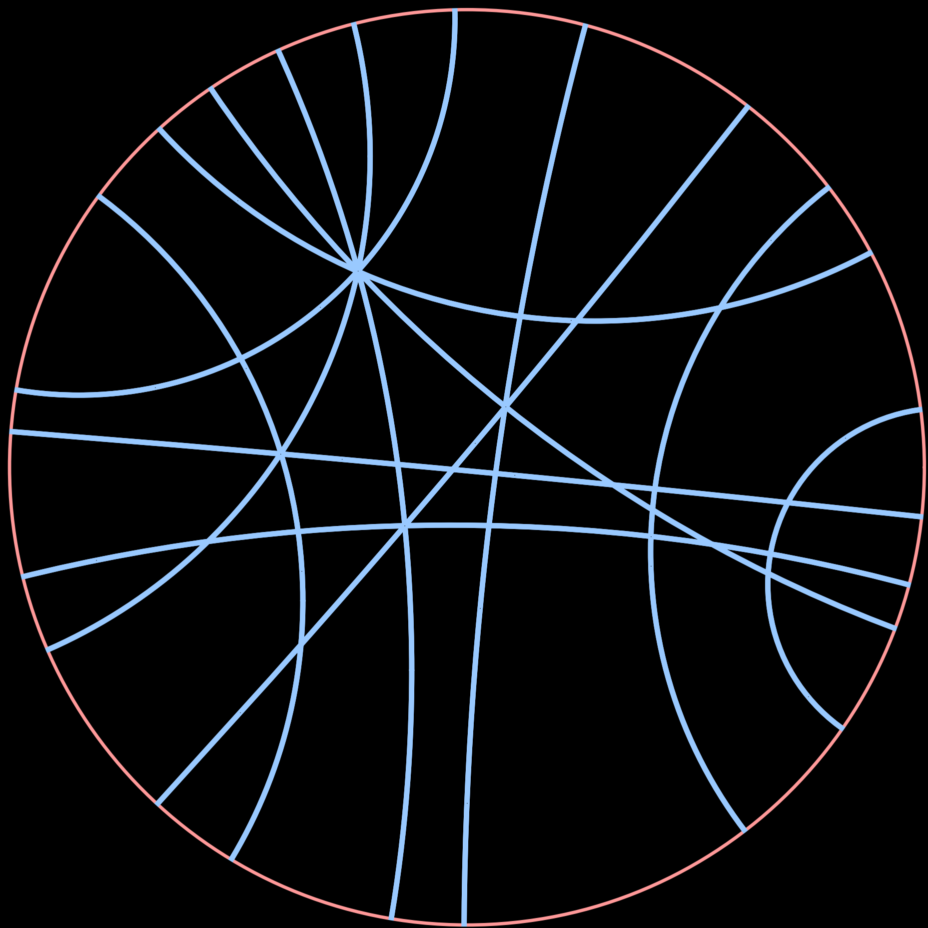
Interior of unit sphere; lines and planes are spherical patches perpendicular to unit sphere

M. C. Escher, Circle Limit II, woodcut, 1959

Two models of Hyperbolic Space



Klein Model
Preserves straightness, convexity
Angles severely distorted



Poincaré Model
Preserves angles
Straightness, convexity distorted

Size versus angle in hyperbolic space

Smaller shapes have larger angles

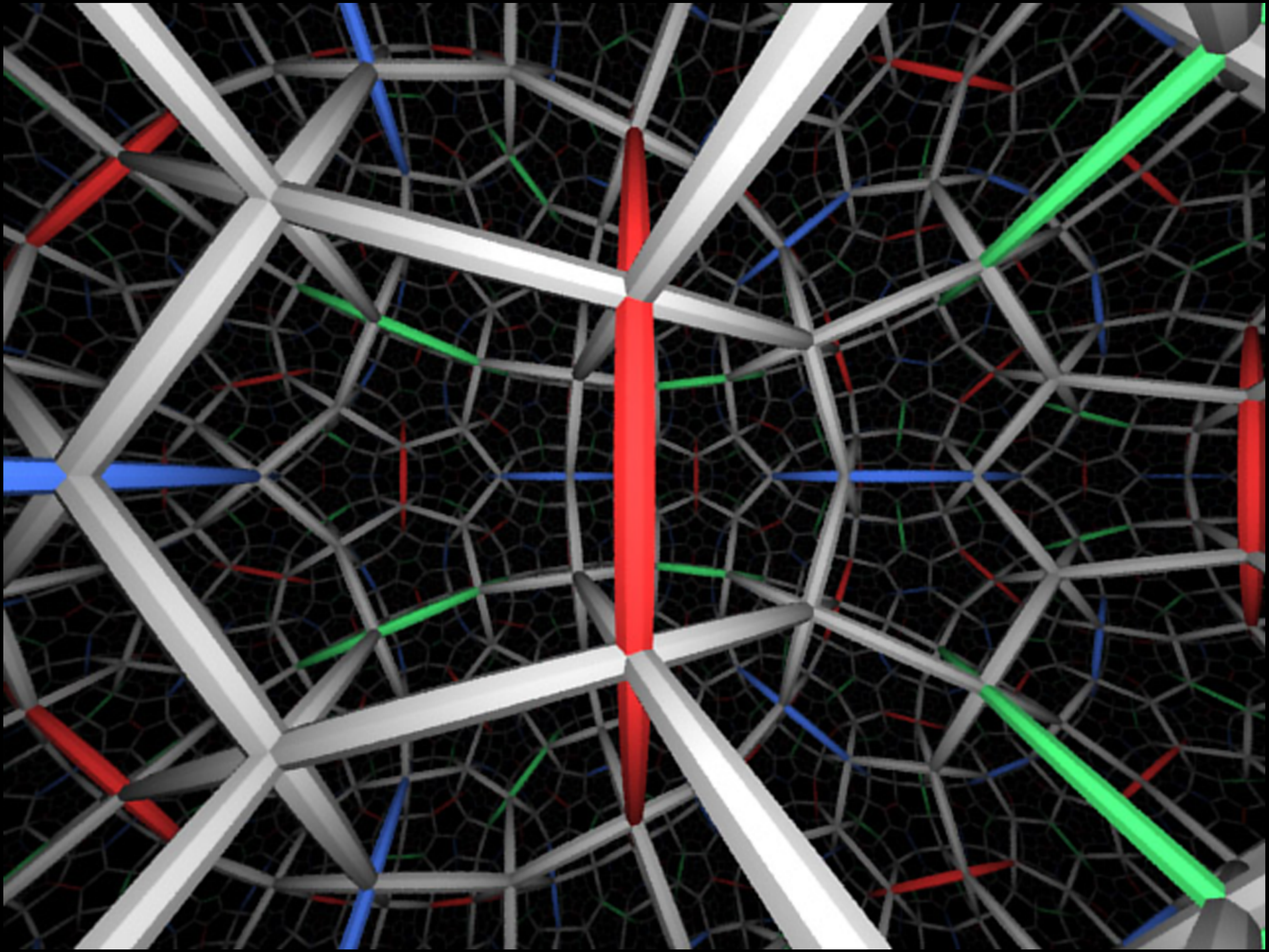
Larger shapes have smaller angles

What are the angles in Escher's triangle-square tiling?

3 triangle + 3 square = 360
2 triangle + 1 square = 180
square < 90, triangle < 60

Another impossible figure!

Right-angled dodecahedra tile hyperbolic space



From Not Knot, Charlie Gunn, The Geometry Center, 1990

Surprise!

Edge-tangent cross polytopes have 90-degree hyperbolic dihedrals

Edge-tangent simplices have 60-degree hyperbolic dihedrals

So truncated cubes work! (new dihedrals are 150 degrees)

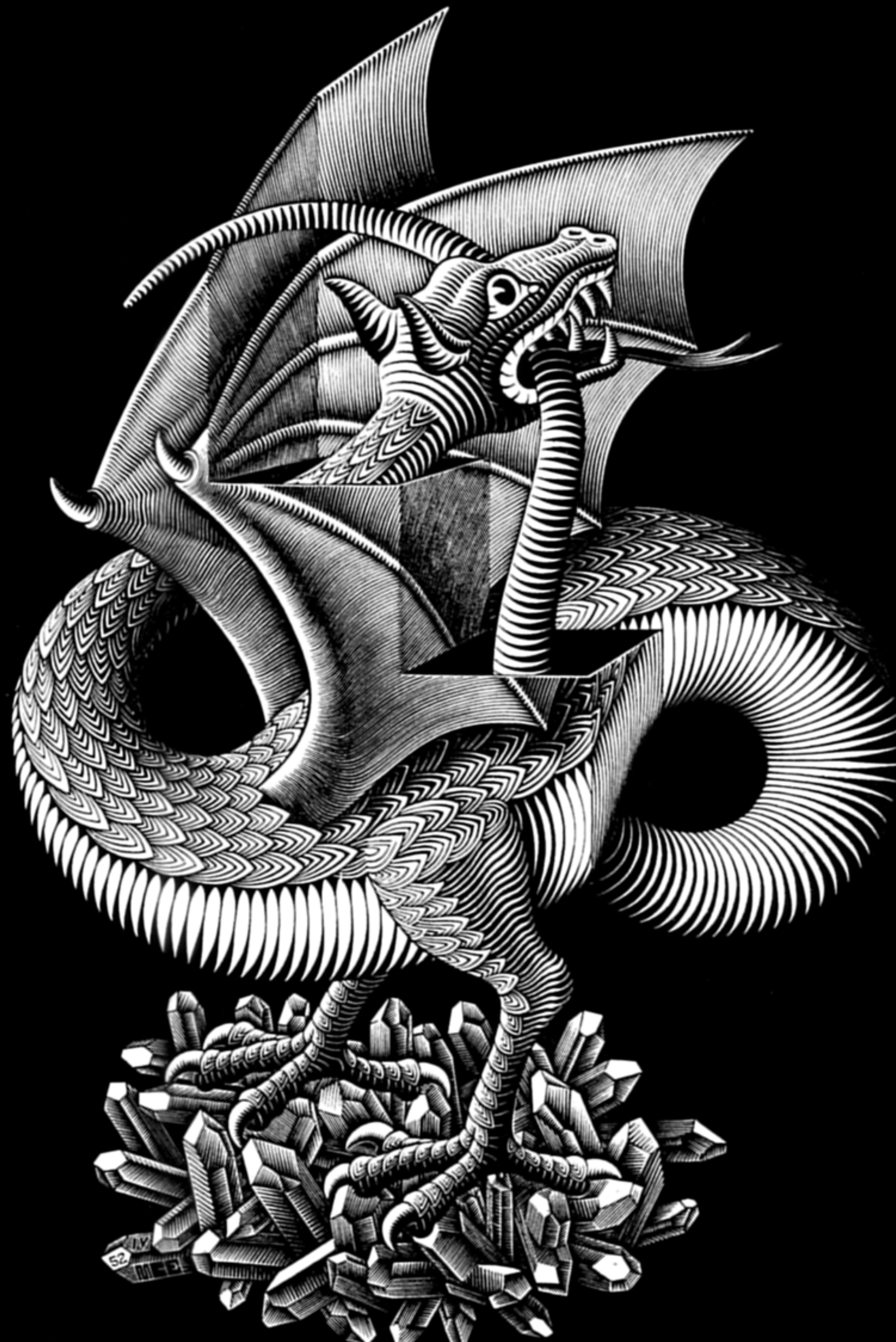
Other examples:

Six simplices around a triangle
(closely related to Soddy's hexlet of nine spheres in 3d)

Glue up to five cross polytopes around a central simplex
then close up nonconvexities by pairs of simplices

Even better, we get infinite families of simplicial edge-tangent polytopes, leading to infinitely many 2-simple 2-simplicial examples!

Glue n cross polytopes end-to-end
forming 4n holes (180-degree dihedrals)
fill with 12n simplices, three per hole



M. C. Escher, Dragon, wood-engraving, 1952