3-Coloring, 3-Edge-Coloring, and Constraint Satisfaction

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These problems are NP-complete
Why do worst-case analysis of exact algorithms?

- With fast computers we can do exponential-time computations of moderate and increasing size

- Algorithmic improvements are even more important than in polynomial-time arena

- Graph coloring is useful e.g. for register allocation and parallel scheduling

- Approximate coloring algorithms have poor approximation ratios

- Interesting gap between theory and practice
Constraint Satisfaction

Given $n$ variables, each with a set of possible values

$m$ constraints forbid certain value combinations

find assignment of values to variables
obeying all constraints

$\text{# values per variable}$

$(a,b)$-CSP

$\text{# variables per constraint}$
Previous CSP Results

Beigel & Eppstein 1995

Messy case analysis
(3, 2)-CSP $\mathcal{O}(1.38028^n)$

Randomized restriction
(k, 2)-CSP $\mathcal{O}\left(\left(\frac{k}{2}\right)^n\right)$

Feder & Motwani 1998

Random permutation of variables
(k, 2)-CSP $\mathcal{O}(k!^{n/k})$

Schöning 1999

Random walk among assignments
(a, b)-CSP $\mathcal{O}\left(\left(\frac{ab - a}{b}\right)^n\right)$
New CSP Results

(3, 2)-CSP $\mathcal{O}(1.36443^n)$
(4, 2)-CSP $\mathcal{O}(1.8072^n)$
(k, 2)-CSP $\mathcal{O}((0.4518k)^n)$

Ideas:

Continued messy case analysis
Stop backtracking when solvable by matching
Define problem size $= n_3 + (2 - \epsilon)n_4$, choose optimal $\epsilon$ for analysis
Combine w/ random restriction for (k, 2)-CSP
3-Vertex-Coloring

Previous results

Lawler 1976
$O(3^{n/3})$

Schiermeyer 1994
$O(1.415^n)$

Beigel & Eppstein 1995
$O(1.3446^n)$

New result

$O(1.3289^n)$
3-Coloring Main Idea

(Same as Beigel & Eppstein 1995)

Find small set $S$ with many neighbors

Choose colors for vertices in $S$

Solve remaining vertices as $(3, 2)$-CSP

Neighbors of colored vertices are restricted to two colors, eliminated from $(3, 2)$-CSP

Time: $O(3^{|S|}1.3645^{|V(G)\setminus(S\cup N(S))|})$
How to Find $S$?

Beigel & Eppstein:

Group all vertices into **height-two trees**

**Local improvement** from **greedy** start

(messy case analysis)

New method:

**Eliminate big clumps** of degree-3 vertices

(else good reduction to smaller coloring instances)

Find **big forest** w/ degree-4 internal nodes

must cover **constant fraction** of graph

Remaining vertices $\Rightarrow$ **height-two trees**

few grandchildren per tree

start with **fractional assignment** nodes-trees

then use **integer flow** to make 0-1 assignment

$S = $ **big forest internal nodes**

$+ \text{ height-two tree roots}$
3-Edge-Coloring

Previous result

Beigel & Eppstein 1995
$\mathcal{O}(1.5039^n)$

(minor mods to vertex coloring alg)

New result

$\mathcal{O}(2^{n/2})$
3-Edge-Coloring Main Idea

Generalize problem:

Add *constraints* forcing pairs of edges to have different colors

Eliminate edges with four neighbors

Reduce to *two subproblems* with *two fewer* vertices (but one more constraint)

Find many independent reductions by matching

Transform remaining problem to vertex coloring

Edge intersection graph + edge per *constraint*
3-Edge-Coloring Analysis

Let \( m_i = \# \) edges with \( i \) neighbors

Can find \( \frac{m_4}{3} \) indep. reductions (else not 3-colorable)

\( 2^{\frac{m_4}{3}} \) subproblems after reduction
\( m_3 \) edges per subproblem

Time = \( \mathcal{O}(1.3289^{m_3} 2^{\frac{m_4}{3}}) \)

Maximized when \( m_3 = 0, m_4 = 3n/2 \):

\( \mathcal{O}(2^{n/2}) \)
Conclusions

More efficient algorithms for several important NP-complete problems

Many other problems for further work

Recent progress:
  General graph chromatic number
  $O(2.4422^n)$ [Lawler 1976] $\Rightarrow O(2.4150^n)$