On the Planar Split Thickness of Graphs

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Definition by example, I

Draw a graph (here, $K_{7,8}$) with:

- Each vertex $\Rightarrow O(1)$ points (here, 2 points/vertex)
- Each edge $\Rightarrow$ curve between representatives of its endpoints
- No crossings
Split thickness: max points/vertex (here, 2)

$G$ is $k$-splittable: it has a drawing with split thickness $\leq k$

E.g. this drawing shows that $K_{6,10}$ is 2-splittable
Motivation: Maps of clustered social networks

Network itself drawn conventionally (no split vertices)
Clusters drawn as regions with $\leq k$ connected components
To construct drawing, need to show cluster graph is $k$-splittable
Related research

Rephrased into our terminology:

Heawood 1890:
$K_{12}$ is 2-splittable

Ringel and Jackson 1984:
Optimal $k$-splittability for $K_n$ ($n > 6$) is $k = \lceil n/6 \rceil$

Hartsfield et al 1985 and later researchers:
Split to planar minimizing total $\#$ splits rather than splits/vertex

Knauer and Ueckerdt 2012:
Split vertices to transform graph into several types of trees
Complete bipartite graphs

Theorem: $K_{a,b}$ is 2-splittable if and only if $ab \leq 4(a + b) - 4$

Proof:

$ab \leq 4(a + b) - 4 \implies G \subset K_{4,b}, K_{5,16}$ (above), $K_{6,10}$, or $K_{7,8}$

$ab > 4(a + b) - 4 \implies$ too many edges for bipartite planar drawing
Let max degree $= \Delta(G)$

Then every graph $G$ is $\lceil \Delta(G)/2 \rceil$-splittable

Regular graphs with odd $\Delta$, high girth are not $\lfloor \Delta/2 \rfloor$-splittable:

- high-girth planar graphs have edges/vertices $\leq 1 + o(1)$
- but any $\lfloor \Delta/2 \rfloor$-split would have edges/vertices $= 1 + \frac{1}{\Delta-1}$. 
Splittability by genus

Theorem: Toroidal and projective-planar graphs are 2-splittable
Theorem: 2-splittability is NP-complete

Reduction from planar 3-SAT with a cycle through clause vertices
(shown NPC by Kratochvíl, Lubiw, & Nešetřil 1991)
Approximation

Part of a family of graph parameters (arboricity, thickness, degeneracy, etc) all within constant factors of each other

Arboricity $a(G)$: minimum number of trees whose union is the given graph

Every graph is $a(G)$-splittable: draw the trees disjointly

Every $n$-vertex $k$-splittable graph has $\leq (3k + 1)(n - 1)$ edges $\Rightarrow$ (Nash-Williams 1964) $a(G) \leq 3k + 1$

So arboricity is a $(3 + \frac{1}{k})$-approximation to splittability (can improve to 3-approximation using pseudoarboricity)
Fixed-parameter tractability

Theorem: can test $k$-splittability of graphs of treewidth $\leq w$ in time $O(f(k, w) \cdot n)$

Main ideas:

► Use monadic second-order logic (MSO) to represent graph properties as quantified formulae over vertex and edge sets

$$\forall S \subseteq E(G) : \exists T \subseteq G(V) : \ldots$$

► A standard DFS-tree trick distinguishes endpoints of each edge
► Use edge-set variables to partition the edges according to the vertex-copies that each endpoint connects to
► Simulate any MSO formula on the split graph by a more complex formula on the original graph
► Planarity $=$ absence of $K_5$ and $K_{3,3}$ minors

► Use Courcelle’s theorem to construct an automaton that tests whether tree-decompositions obey the formula
Conclusions

Defined a new concept of $k$-splittability, used it to draw nonplanar graphs in a planar way

Tight bounds for complete graphs, complete bipartite graphs, and graphs of bounded maximum degree

NP-complete but $O(1)$-approximable, FPT for bounded treewidth

Future work: splitting vertices to produce near-planar graphs (e.g. low genus or bounded local crossing number)