Area-Universal Rectangular Layouts

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Rectangular layout

partition of a rectangle into finitely many interior-disjoint rectangles, such that no four rectangles meet in one point.
Applications: floor planning
Applications: rectangular cartograms

(Rectangular Cartograms [Raisz 1934]
visualize statistical data about sets of regions;
regions are rectangles;
area proportional to some geographic variable)
Rectangular cartograms

Given a plane triangulated graph $G = (V,E)$ and a positive weight for each vertex.

Construct a partition of a rectangle into rectangular regions

- $G$ is the dual graph of the partition (that is, the partition is a rectangular dual of $G$)
- The area of each region = the weight of the corresponding vertex
Constructing a cartogram

1. Find a rectangular dual $L$ for $G$
2. Give rectangles correct areas
Rectangular dual

[Kozminski & Kinnen ’85]

A planar graph $G$ has a **rectangular dual** $\iff$ we can complete with four outer vertices to obtain a graph $E(G)$ s.t.

1. every interior face of $E(G)$ is a triangle
2. the exterior face of $E(G)$ is a quadrangle
3. $E(G)$ has no separating triangles
Constructing a cartogram

1. Find a rectangular dual \( L \) for \( G \)

2. Give rectangles correct areas = turn it into an equivalent layout whose regions have given areas
Equivalent layouts

E(G)

L

equivalent to L

NOT equivalent to L

Equivalent layout
a rectangular dual of L such that the adjacencies of the regions have the same orientations
Constructing a cartogram

Solution does not always exist
When it does it is unique [Wimer, Koren, and Cederbaum ‘88]
Finding a suitable layout

- There are potentially exponentially many rectangular duals for a given graph
- There are layouts that “work” for any set of weights:

Area-universal layout L
for every choice of weights for the regions of L there is a layout L’ equivalent to L such that the areas of rectangles in L’ are equal to the given weights.
Finding a suitable layout

Theorem

A layout is area-universal, if an only if it is one-sided.

Area-universal layout $L$

for every choice of weights for the regions of $L$ there is a layout $L'$ equivalent to $L$ such that the areas of rectangles in $L'$ are equal to the given weights.
One-sided layouts

One-sided layout $L$: every maximal line segment of $L$ must be the side of at least one rectangle.
Finding one-sided layouts
One-sided duals

[Rinsma ‘87]

There exists an outer-planar triangulated graph that does have rectangular duals, but no one-sided dual.
Regular edge labelings

- **Horizontal adjacency**
- **Vertical adjacency**

**Regular edge labeling [Kant and He’97]**

- **Inner vertex**
- **Outer vertices**
  - **Top vertex**
  - **Left vertex**
  - **Right vertex**
  - **Bottom vertex**
Regular edge labelings

Theorem [Kant and He’97]

Every rectangular dual for $E(G)$ corresponds to a regular edge labeling of $E(G)$ and vice versa.
Non-one-sided layouts

Look for RELs without the patterns above
Distributive lattice of RELs

[Fusy’05]
Distributive lattice of RELs

[Fusy’05]
Distributive lattice of RELs

*For graphs with trivial separating 4-cycles:
Distributive lattice of RELs

Exponential size
Birkhoff’s representation theorem
Birkhoff’s representation theorem
Birkhoff’s representation theorem

O(n^2) size can be constructed in polynomial time
Finding area-universal layouts

- Fixed parameter tractable algorithm that runs in $O(2^{O(K^2)} \cdot n^{O(1)})$ time

$K =$ number of degree-four vertices in the graph $E(G)$
Summary

Results

- We can find an area-universal layout in $O(2^{O(K^2)} n^{O(1)})$ time
- Perimeter cartograms
- Area-universal layouts for dual spanning trees in $O(n)$ time

Open problems

- Is there a polynomial algorithm for area-universal layouts?
- Can we efficiently find a layout that realizes a given area assignment in case when a graph has no area-universal layout?