

CS 177, Solutions to Sample Midterm Questions

Applications of Probability in Computer Science: Winter 2007

Problem 1

Answer each of the following questions, where X and Y are two discrete-valued random variables, with X taking m values x_1, \dots, x_m , and Y taking n values y_1, \dots, y_n . You can use lower case x_i and y_j to denote generic values of variables X and Y respectively.

1. Define precisely $E[X]$, the expected value of X .

Solution: $E[X] = \sum_{i=1}^m x_i P(x_i)$.

2. Use Bayes rule to write down an expression for $P(y_j|x_i)$.

Solution:

$$P(y_j|x_i) = \frac{P(x_i|y_j)P(y_j)}{P(x_i)}$$

Note that the numerator could also be written as $P(y_j, x_i)$ and the denominator could be written as $\sum_{j=1}^n P(x_i, y_j)$.

3. Let Z be a 3rd variable taking values z_1, \dots, z_k . Show (using 1 or more equations) how to compute $P(z_s|x_i, y_j)$, assuming that you are given the joint distribution $P(x_i, y_j, z_s)$ for all values x_i, y_j, z_s .

Solution: by Bayes rule we can write

$$P(z_s|x_i, y_j) = \frac{P(x_i, y_j|z_s)P(z_s)}{P(x_i, y_j)}$$

The numerator can be written as $P(x_i, y_j, z_s)$ which we can just “read off” from our joint distribution table. The denominator can be written (by the law of total probability) as

$$P(x_i, y_j) = \sum_{s=1}^k P(x_i, y_j, z_s)$$

4. State precisely a set of necessary conditions for X and Y to be independent.

Solution: a set of necessary conditions is that $P(x_i, y_j) = P(x_i)P(y_j)$ for all values of i and j .

5. If we assume X and Y are independent (just for this part of the problem), precisely how many separate probability numbers are required to specify the joint probability mass function $P(X, Y)$. If we don't assume independence, how many probability numbers are needed?

Solution: without independence we need a table $m \times n$ numbers for the joint distribution—actually since these numbers must sum to 1, we really only need $mn - 1$ since once we specify these probabilities the remaining probability is already determined by the sum-to-1 constraint. So $O(mn)$ in total.

If we assume independence, then we don't need a full joint distribution, we just need the individual probability distributions for X and Y , which require (respectively) m and n probability values (again technically its only $m - 1$ and $n - 1$ given that we know these numbers must sum to 1. So $O(m + n)$ in total.

Problem 2

You are a computer consultant who runs a very successful series of 1-day workshops on JAVA programming techniques. For each individual that registers there is a 3% chance that they will not show up (probability 0.03). Assume decisions are made independently by each registrant. The conference room where you hold the course can seat 100 people. If you decide to accept reservations for 104 people what is the probability that there will be a seat available for all of the registered people who show up?

Solution: Let X denote the number of registered people that show up at the workshop. Thus, the probability that there will be a seat available for all of them can be expressed as:

$$\begin{aligned} P(X \leq 100) &= 1 - \left[P(X = 104) + P(X = 103) + P(X = 102) + P(X = 101) \right] \\ &= 1 - \left[(0.97)^{104} + \sum_{k=1}^3 \binom{104}{104-k} (0.97)^{104-k} (0.03)^k \right] \end{aligned}$$

The first term, $(0.97)^{104}$ is about 0.042, so we know the probability is upper bounded by $1 - 0.042 = 0.958$. The rest of the terms can be evaluated using the code you wrote for Homework 1.

Problem 3

A professor randomly selects 1 question to grade out of 8 questions on a homework

1. a student has answered 6 questions. What is the probability that none of the students' questions are graded?
2. a student answered k questions, where k varies from 1 to 8. Derive a general expression (i.e., a mathematical expression involving k) for the probability that none of the students' questions are graded.
3. A different professor grades 2 questions out of 8. What is the probability that a student who answered 6 questions out of the 8 will have none of those questions graded? what is the probability that a student will have only 1 question graded? what is the probability they will have 2 questions graded?
4. Solve part 3 for the general case where the student answered k questions, where $k = 1, 2, \dots, 8$. Again your answers should be mathematical expressions expressed as a function of k .

Solution:

1. The probability that the selected question was not answered by the student is $\frac{2}{8} = \frac{1}{4}$
2. If the student answered k questions, the probability that none of the answered questions are graded is $\frac{8-k}{8}$
3. The student answered 6 questions out of 8. Consider these 6 to be marked a certain color, e.g., red, and the non-answered questions marked blue. So we can rephrase the question as that of drawing 2 colored balls from a hat (which corresponds to the professor randomly selecting 2 questions to grade) and then asking what the probability is of getting no reds, 1 red, and 2 reds. The probability of a randomly-selected ball being red (or that a randomly-selected question was answered by the student) is $6/8$ and being blue is $2/8$.

(a) The probability of both balls selected being blue (neither of the selected questions were answered by the student) can be written as

$$P(\text{ball1=blue, ball2=blue}) =$$

$$P(\text{ball2=blue}|\text{ball1=blue})P(\text{ball1=blue}) =$$

$$1/7 \quad \text{times} \quad 2/8 \quad = \quad 2/56.$$

(b) The probability of getting a red ball and a blue ball can happen in 2 different ways: red first and blue second or blue first and red second. The probability of getting red and blue is the sum over these 2 possible events (since they are mutually exclusive), i.e.,

$$\begin{aligned} &P(\text{ball1}=\text{red}, \text{ball2}=\text{blue}) + P(\text{ball1}=\text{blue}, \text{ball2}=\text{red}) = \\ &P(\text{ball2}=\text{blue}|\text{ball1}=\text{red})P(\text{ball1}=\text{red}) + \\ &P(\text{ball2}=\text{red}|\text{ball1}=\text{blue})P(\text{ball1}=\text{blue}) = \\ &2/7 \quad \text{times} \quad 6/8 \quad + \quad 6/7 \quad \text{times} \quad 2/8 \quad = \quad 24/56 \end{aligned}$$

(c) Similarly, the probability of getting 2 reds is $6/8$ times $5/7 = 30/56$.

As a check, the 3 probabilities should sum to 1 (do they?) since the outcomes 0,1, and 2 are mutually exclusive and exhaustive.

4. We can generalize the above equations by making the probabilities in the various expressions above depend on k . For example, the probability of obtaining two red balls when k questions have been answered can be written as $(k/8)(k-1/7)$, $k = 1, \dots, 8$.

You might imagine that we could model this as using a binomial random variable with $n = 2$ and $p = k/8$. Why would a binomial model **not** be correct here?

Problem 4

Consider the following very simple Markov model for an e-commerce Web-site with 4 states: S for the start state, B for a browsing state, D for a check-out state, and E for an end-state.

- From the start state S the user always goes to state B .
- Given that a user is in state B , the user next goes to state D with probability 0.1, to state E with probability 0.2, and has a 0.7 probability of staying in state B .
- Given that a user is in state D , the user next goes to state B with probability 0.2, to state E with probability 0.7, and has a 0.1 probability of staying in state D .
- The end-state is an “absorbing” state—once the user reaches this state they stay there forever.

1. Define the transition matrix P for the corresponding Markov chain (where row 1 = S , row 2 = B , row 3 = D , row 4 = E).
2. If a user is in state S at time t , what is the probability of being in State E at time $t + 3$
3. What is the probability that a user makes a purchase (i.e., ever visits state D)?

Solution:

1. Matrix P is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.7 & 0.1 & 0.2 \\ 0 & 0.2 & 0.1 & 0.7 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2. Paths: $SBEE + SBBE + SBDE = P(X_{t+3} = E | X_t = S) = P_{SE}^3 = 0.41$

$$P^3 = \begin{pmatrix} 0 & 0.510 & 0.080 & 0.410 \\ 0 & 0.373 & 0.059 & 0.568 \\ 0 & 0.118 & 0.019 & 0.863 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3. At least one purchase path: $1 - \left[S\{B\}^+E \right]$. That is, 1 - No purchase.

$$\begin{aligned} 1 - P(\text{No purchase}) &= 1 - \sum_{i=0}^{\infty} P_{SE}P_{BB}^iP_{BE} = 1 - P_{SE}P_{BE} \sum_{i=0}^{\infty} P_{BB}^i \\ &= 1 - (1)(0.2) \sum_{i=0}^{\infty} (0.7)^i = 1 - (0.2) \left(\frac{10}{3} \right) = \frac{1}{3} \end{aligned}$$