Guard Placement For Wireless Localization

David Eppstein   Michael T. Goodrich   Nodari Sitchinava

Department of Computer Science
University of California, Irvine
{eppstein, goodrich, nodari}(at)ics.uci.edu

Abstract

Motivated by secure wireless networking, we consider the problem of placing fixed localizers that enable mobile communication devices to prove they belong to a secure region that is defined by the interior of a polygon. Each localizer views an infinite wedge of the plane, and a device can prove membership in the secure region if it is inside the wedges for a set of localizers whose common intersection contains no points outside the polygon. This model leads to a broad class of new art gallery type problems, for which we provide upper and lower bounds.

1 Introduction

Localization is becoming an important topic in wireless mobile computing (e.g., see [6]), where we wish to determine with certainty the position of a wireless device in a geometric environment. Such localization problems are typically facilitated by *locators*, which are wireless base stations placed at fixed locations that aid the wireless devices to determine their positions. In this paper, we are interested in such a localization problem, where we are asked to deploy a collection of locators in what can be viewed as a two-dimensional space so that a wireless device can prove that it belongs to a given polygonal environment. In this case, the locators are simple, fixed base stations that can broadcast information in certain directions, such that devices outside of the broadcast angle for a station cannot receive the transmissions from that station. Viewed geometrically, such a locator is a point-based *guard* with a fixed angle of view oriented in a fixed range of directions; hence, we typically refer to such locators in this paper as "angle guards" or simply as "guards" if the context is clear.

From the standpoint of a mobile device, which we model as a point p in the plane, an angle guard g can be viewed as a Boolean predicate, $B_g(p)$, which is true if p is inside the angle associated with g and is false otherwise. Moreover, we assume that if $B_g(p)$ is true, then the mobile device associated with p can produce a certificate for this fact. For example, the angle guard g could periodically broadcast a secret key K in its transmission angle, so that only a wireless device in this angle would have knowledge of this key (in which case a zero-knowledge non-interactive proof-of-knowledge of K would suffice as a certificate that $B_g(p)$ is true).

Put strictly as a geometric problem, then, we are given a polygon P and are asked to place angle guards in, around, and outside P in such a way that we can define a monotone[1] Boolean formula, $F(p)$, built from the angle-guard predicates, $B_g(p)$, so that $F(p)$ is true if and only if p is inside P. Moreover, we desire that the number of angle guards needed to define such a formula be small, since there may be a non-trivial expense in deploying such a collection of guards. Thus, this problem can be viewed as a kind of art gallery problem [19, 20, 23], where it is not sufficient that the guards merely see all of the art gallery, but instead they must collectively define the geometry of the art gallery. More specifically, this problem can be viewed

---

[1]A Boolean formula is monotone if it contains only AND (·) and OR (+) operators; hence, has no NOT operations.

as a "sculpture garden" problem, where the guards and the formula F distinguish the space of the sculpture garden from the surrounding land (without the use of walls or fences). (See Figure 1.)
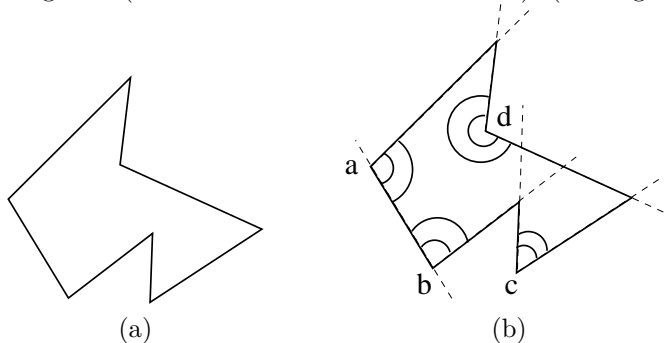

(a)   (b)

Figure 1: Illustrating the sculpture garden problem. (a) an example 7-vertex polygon P; (b) a set of 4 angle guards that solve the sculpture garden problem for P. The Boolean formula in this case is $F = d(ab + c)$.

Ideally, we would like the formula F to be *concise*, meaning that $O(1)$ certificates always suffice to prove that $F(p)$ is true for any point p inside P. The motivation for this desire is that we wish to prove that a point p is inside P using only angle-guard predicates, and we would like that proof to be as short as possible. For example, if F were in disjunctive normal form (DNF), that is, F was a disjunction of conjunctive clauses, and each clause in F contained a constant number of angle-guard predicates, then F would be concise.

1.1   Related Prior Work

As mentioned above, localization is becoming an important topic in wireless mobile computing, where a number of research teams are interested in solutions that avoid the use of GPS, which has a number of practical drawbacks. For example, Bulusu *et al.* [3] study how RF strength and angle can be used for sensor localization, and Savvides *et al.* [21] show how to improve the consistency of such an approach by iterative algorithms. Alternatively, Howard *et al.* [15] use a potential-field based approach and Chakrabarty *et al.* [4] use a grid-based technique for deploying locators. On the other hand, He *et al.* [14] use a random deployment and use point-in-triangle tests to determine location based on audible signals.

Of considerable relevance, of course, is prior work on using directional antennas in wireless communication networks. For example, Ko and Vaidya [16] discusses how to use base stations with directional antennas (as in our angle guards) to improve network protocols, but they assume that the mobile agents already know their locations. Bao and Garci-Luna-Aceves [1], on the other hand, use directional antennas for adaptively discovering connection directions in an ad hoc network. We are not familiar with any existing prior work, however, that uses directional antennas for localization itself. Nevertheless, using the results of our paper as a combinatorial justification, a companion paper [6] addresses the implementation issues of using locators with directional antennas for mobile device localization.

Art gallery problems are a classic topic in Computational Geometry and much has been written about them (e.g., see [19, 20, 23]). The starting point for this related research is a result of Chvátal [7] that $\lfloor n/3 \rfloor$ point guards are sufficient and sometimes necessary to be able to fully see a simple polygon having n vertices. More related to the topic of this paper, "prison yard" problems [10, 19, 20, 23] seek a set of guards that can simultaneously see both the interior and exterior of a simple polygon, in which case $\lceil n/2 \rceil$ guards are sufficient and sometimes necessary [10]. Relating to angle guards, Estivill-Castro *et al.* [9] show that vertex angle guards (which they call "floodlights") with angles of $180°$ are sufficient to see any simple polygon and there are polygons such that any fixed angle less than this will not. Likewise, Steiger and Streinu [22] and Bose *et al.* [2] study the complexity of illuminating wedges with angle-restricted floodlights placed at a fixed set of points.

Unfortunately, solutions to art gallery or prison yard problems do not translate into solutions to sculpture garden problems like the ones we study in this paper, since we are interested in more than simply seeing the inside and outside of a polygon—we wish to prove when a point is inside a polygon using only the guards as witnesses.

Even more related to the topic of this paper is prior work on finding a constructive solid geometry (CSG) representation of a simple polygon, since CSG representations can be used to prove polygon containment. Dobkin *et al.* [8] describe a method for constructing a formula F that defines a simple polygon using primitives that are halfplanes defined by lines through polygonal edges, so that each halfplane is used exactly once. Using our terminology, this is equivalent to a formula F for a set of n angle guards, with each guard placed on an edge of the polygon with a 180° degree angle defined by the edge. Such a formula would not, in general, be concise, however. More recently, Walker and Snoeyink [24] study the problem of using polygonal CSG representations, *a la* Dobkin *et al.* [8], for performing point-in-polygon tests. They experimentally consider several interesting heuristics for improving the efficiency of such tests, by "flattening" the CSG tree defined by the formula, but they are not able to produce proofs that are guaranteed to be concise in the sense of this paper. Likewise, Goodrich [12] shows how any CSG formula tree can be transformed into an equivalent DAG of depth $O(\log n)$, but this again is not sufficient to guarantee conciseness in the sense of this paper (in that we desire constant-depth formulas).

Of course, one can always triangulate [5] any polygon, P, and use two angle guards to define each of the resulting $n + 2(h - 1)$ triangles, where h is the number of holes in P. This would give rise to a concise formula F for defining P, but it uses at least $2n + 4(h - 1)$ angle guards, which is much higher than we would like. Thus, the challenge is to find ways of producing polygon-defining formulas that use fewer than n angle guards and are hopefully also concise.

1.2   Our Results

In this paper, we present a number of results concerning the kinds and number of angle guards needed to define various polygons (we use n throughout to refer to the number of vertices of a given polygon). Specifically, we show the following:

1. Define a *natural* angle-guard vertex placement to be one where we place each angle guard at a vertex of the polygon with the angle of that vertex as the angle of the guard (as in Figure 1). We show there is a polygon P such that a natural angle-guard vertex placement cannot fully distinguish between points on the inside and outside of P (even if we place a guard at every vertex of P). This negative result implies that there are cases when we must use Steiner points or Steiner angles for sculpture garden problems.

2. We show that, for any polygon P, there is a set of $n + 2(h - 1)$ angle guards and an associated concise formula F for solving the sculpture garden problem for P, where h is the number of holes in P (so a simple polygon can be defined with $n - 2$ guards).

3. We give a class of simple polygons that we conjecture require $n - 2$ angle guards for any solution to the sculpture garden problem.

4. We observe that, for any convex polygon P, there is a natural angle-guard vertex placement such that $\lceil n/2 \rceil$ guards are sufficient to solve the sculpture garden problem for P, and we show this bound is optimal for any general-position polygon (for which no two edges belong to the same line).

5. We show that $\lceil n/2 \rceil + O(1)$ angle guards are sufficient to solve the sculpture garden problem for pseudo-triangles.

6. We show how any solution to the sculpture garden problem can be made concise with a small blow-up in the number of guards.

7. We give an example of a class of simple (non-general-position) polygons that have sculpture garden solutions using $O(\sqrt{n})$ guards, and we show this bound is optimal to within a constant factor.

8. We show how to find a guard placement whose size is within a factor of 2 of the optimal number for any particular polygon.

9. We show that, for any orthogonal polygon P (which is probably the most likely real-world application), there is a set of $\lceil 3(n - 2)/4 \rceil$ angle guards and an associated concise formula F for solving the sculpture garden problem for P.

Thus, we feel this paper begins an interesting new branch of work on polygon guarding problems.

2 Natural Angle-Guard Placements

As defined above, a *natural* angle-guard vertex placement is one where we place each angle guard at a vertex of the polygon with the angle of that vertex as the angle of the guard. (See Figure 2a.)
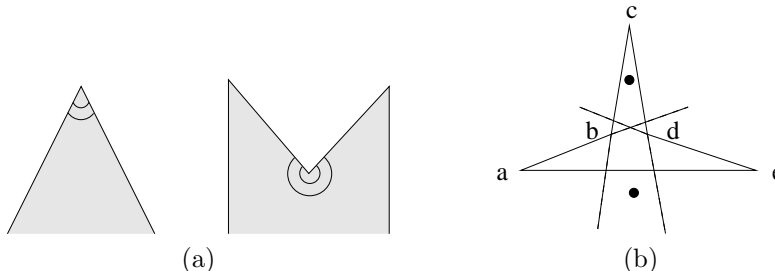

(a)　　　　　　　　　　　　　　(b)

Figure 2: Natural angle-guard placements. (a) examples of natural angle-guard vertex placements for convex and reflex angles of a polygon; (b) an example polygon that cannot be defined using a natural angle-guard placement, for the point, p, inside the polygon cannot be distinguished from the point, q, outside the polygon.

A natural angle-guard placement has an obvious aesthetic appeal. Unfortunately, the sculpture garden problem cannot be solved using natural guards for some polygons.

Theorem 1. *There is a pentagon P such that it is impossible to solve the sculpture garden for P using a natural angle-guard vertex placement.*

Proof. Let P be the pentagon illustrated in Figure 2b, and let p be the highlighted point inside of P and let q be the highlighted point outside of P. Then the natural guards cannot distinguish between the two points, p and q. For natural guards a and e, both points are outside the angles they cover, while, for guards b, c and d, both points are inside the angles which they cover. That is, $B_x(p) = B_x(q)$, for $x = a, b, c, d, e$. Therefore, any formula built using predicates B_x, for $x = a, b, c, d, e$, will have identical values on p and q. Since p and q are on opposite sides of the boundary of P, this implies that it is impossible to solve the sculpture garden problem for P using a natural angle-guard vertex placement. □

This theorem implies that some sculpture garden solutions must use Steiner points or Steiner angles. Nevertheless, for orthogonal polygons, natural guard placements suffice.

Theorem 2. *Natural guards (one on each vertex) suffice to guard any orthogonal polygon.*

Proof. Let p be any point inside the polygon P. We wish to show that the intersection of the guarded regions for the natural guards containing p, so for any q outside P we will find a guard separating p from q. To do so, let R be the rectangle having p and q as opposite corners. The boundary ∂P may cross R many times, but there is at least one component B of $\partial P \cap R$ that crosses R and separates p from q in R, with p on the interior side of B; for instance, we may choose B as the component of $\partial P \cap R$ that is farthest from p among the components reachable from p via paths in $P \cap R$. Let e be an edge of B that crosses the boundary of R, and let v be the endpoint of e that is outside R; then the guard at v separates p from q. □

3 An Upper Bound For Arbitrary Polygons

In this section we show that the sculpture garden problem can be solved for any h-hole polygon with at most $n + 2(h - 1)$ guards and a concise formula. To prove this bound we need to establish some preliminary results presented in the following lemmas.

Lemma 1. *The sculpture garden problem can be solved with two guards for any tetragon (quadrilateral).*

Proof. If the tetragon is convex, place the two natural angle-guards in any two opposite corners. If the tetragon has a reflex vertex, place one natural angle-guard in the reflex vertex and the other in the opposite vertex (see Figure 3). The conjunction of the two angle guards defines the tetragons in each case. ◻

Figure 3: Solutions for the sculpture garden problem for tetragons.

Lemma 2. *The sculpture garden problem can be solved with three guards for any pentagon P.*

Proof. Consider a tetragon T which fully contains the pentagon P and shares at least 3 consecutive edges of P. (We show later how to find T.) By Lemma 1 we can solve the sculpture garden problem for the tetragon T using exactly 2 guards.

Now, since T shares 3 consecutive edges of P, it means that at least 4 vertices of P lie on T or, equivalently, there is at most 1 vertex $v \in P$ that does not lie on T. That means that there are at most 2 edges of P which lie inside T and which might not have been covered by guards. To complete the solution to the sculpture garden problem, place a natural angle guard at vertex v (If there is no such vertex, i.e. only 1 edge is not covered by the guards, it means that the pentagon P is convex and we can place a natural angle guard on either of the vertices incident on such an edge).

The final solution to the sculpture garden problem on the pentagon will be the conjunction of all the guards placed for a total of 3 guards.

To complete the proof we now describe how to find the tetragon T which fully contains the pentagon P and shares at least 3 edges with it. Consider the convex hull H of P.

- If H consists of 5 vertices (i.e. P is convex), pick any 4 edges of H. T is the tetragon which is constructed by the intersection points of the lines on which those 4 edges lie (see Figure 4(a)). Note that 3 vertices of the tetragon will be shared with the original pentagon.

- If H consists of 4 vertices, then T is equal to H.

- If H consists of 3 vertices, then there are two cases to consider:

  1. *Two edges of the pentagon P are also edges of H.* Note that the two edges have to be adjacent since P is a pentagon. Let $ABCDE$ be the pentagon P with vertices A, B, and E comprising the vertices of the convex hull H (see Figure 4(c)). Consider the edge $BE \in H$ which is not part of the pentagon P. Of the two pentagon vertices $C, D \notin H$ at least one of them can be connected to both B and E without intersecting P. Without loss of generality let D be such a vertex. Since each one of C and D are adjacent to either vertex B or E, one of the segments DB or DE is also an edge of the pentagon P (in our example $DE \in P$). Then the desired tetragon T consists of the pentagon edges AB, AE and DE as well as the segment DB. As desired, T fully contains the pentagon P (no edge of T intersects P) and T shares 3 consecutive edges of P (AB, AE and DE in our example).

  2. *Only one edge of the pentagon P is also an edge of H.* Let $ABCDE$ be the pentagon P with ACE being the convex hull H (see Figure 4(d)). Pick one of the two vertices $B, D \notin H$. In our example we pick vertex B. The desired tetragon T consists of the pentagon edges AE, AB and BC and the edge CE of the convex hull H. Note that the two vertices which are not on the convex hull (B and D in our example) will never be adjacent if the convex hull shares only 1 edge with the pentagon. Thus, both neighbors of each of those vertices are the vertices of the convex hull. Therefore, the two rays originating from those vertices and shooting along the edges of the pentagon (BA and BC in our example, since we picked B) don't intersect the pentagon P. Thus, the tetragon $ABCE$ fully contains the pentagon P and shares 3 consecutive edges (AE, AB and BC) as desired.
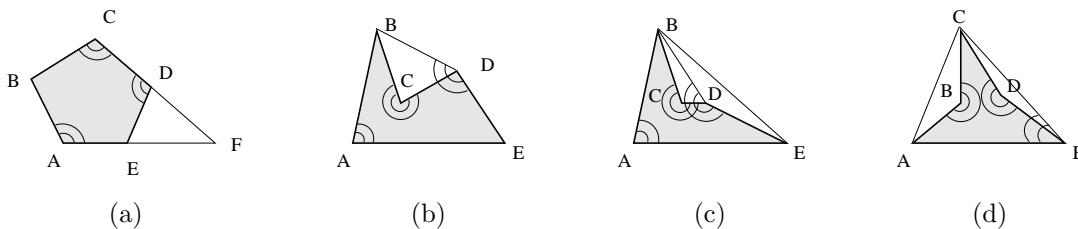
Figure 4: Various pentagons $P = ABCDE$, as well as the corresponding convex hulls H and the containing tetragons T: (a) $H = ABCDE$, $T = ABCF$; (b) $T = H = ABDE$; (c) $H = ABE$, $T = ABDE$; (d) $H = ACE$, $T = ABCE$. The corresponding solutions for the sculpture garden problem is the conjunction of all the guards placed.

Lemma 3. *The sculpture garden problem can be solved with at most 4 guards for any hexagon.*

Proof. Any hexagon whose dual graph of the triangulation is not a star graph or whose triangulation can be modified to have a non-star dual graph, can be split into two tetragons each of which (by Lemma 1) can be solved with two angle guards for a total of four. Thus, the only interesting case is when a hexagon has a single triangulation and its dual graph is a star graph.

Let H be such a hexagon and consider its triangulation. Since this is the only triangulation, combining any pair of triangles produces non-convex tetragons. (If that wasn't the case, we could combine two triangles into a convex tetragon and switch the diagonal to obtain a different triangulation, which would violate the assumption of the uniqueness of the triangulation.) Consider triangle BDF which corresponds to the center vertex of the dual star graph. The lines on the boundary of the triangle BDF partition the plane into 6 regions. For all pairs of adjacent triangles to construct a non-convex tetragon it must be true that the vertices A, C and E lie in one of the three shaded regions. Since at most 2 of these vertices can lie in the same shaded region, there are two cases to consider:

1. Each vertex A, C and E lie in its own region (Figure 5(a)). The vertices B, D and F are all reflex vertices and the rays originating at these vertices and shooting along the edges of the polygon intersect each other only at the polygon vertices A, C and E. Thus, the conjunction of natural angle guards placed at the reflex vertices of the polygon (a total of 3) will define the polygon (See Figure 5(b)).
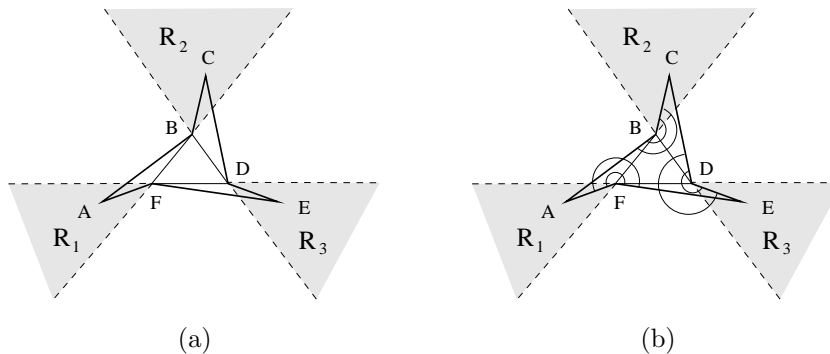

Figure 5: An example of a hexagon with each of the vertices A, C and E in their own region (a) and the corresponding solution for the sculpture garden problem (b).

2. Two of the three vertices A, C, E lie in the same region. Without loss of generality let A and C lie in the same region R_1 and vertex E lie in region R_3 (Figure 6(a)). Rays originating at vertices A and C and shooting along the edges of the polygon all intersect the polygon edge DF. Consequently, they

will all intersect edge EF and will never intersect edge DE except at vertex D. Thus, the polygon defined by the conjunction of two natural angle guards at vertices A and C and an edge guard[2] on the edge EF is fully contained inside the hexagon H. Moreover, the only part of the hexagon that is not covered by the above 3 guards is a part of triangle DEF near the vertex E. Since we already have an edge guard at the edge EF we can cover the whole triangle
DEF by placing one additional angle guard at vertex D whose wedge is defined by the rays DF and DE (See Figure 6(b)). Thus, a total of 4 guards is required to guard this hexagon.

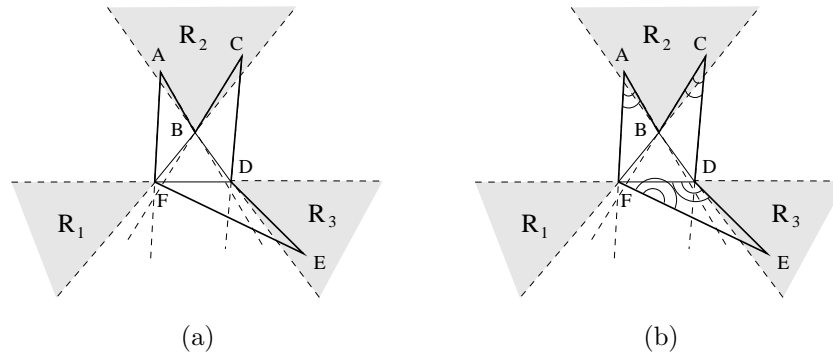(a)   (b)

Figure 6: An example of a hexagon with two vertices A and C in the same region (a) and the corresponding solution for the sculpture garden problem (b).

Thus, we conclude that at most 4 guards are required to guard any hexagon. □

Lemma 4. *Any polygon P with more than three vertices can be partitioned into a collection of tetragons, pentagons and at most one hexagon whose dual triangulation tree is star-shaped.*

Proof. Consider a dual spanning tree of a triangulation of the polygon P, which is necessarily a degree-three tree. If the tree is a two-, three- or star-shaped four-node tree, we are done because the corresponding polygon is a tetrahedron, a pentagon or a hexagon.

If there are more than four nodes in the tree or the four-node tree is not star-shaped, recursively trim the tree in the following way. Pick a leaf v such that v's neighbor u has one of the following properties:

1. u has degree 2 and u's neighbor $w \neq v$ is not a leaf.

2. u has degree 3 and *exactly one* of u's other neighbors $w, z \neq v$ is also a leaf. Without loss of generality, let w be an internal node, i.e. not a leaf.

(Note, unless the tree is one of the base cases, a leaf v with one of the two properties always exists because the tree is a binary one.)

If u has property 1, then remove v and u from the tree and add the tetragon, associated with the removed two nodes of the tree into the collection. If u has property 2, then remove u, v, and z from the tree and add the pentagon associated with the removed three nodes of the tree into the collection.

Continue the trimming until the tree is a two-, three- or star-shaped four-node tree. At each step we removed a tetragon or a pentagon from the polygon P. Since we were removing only leaves with their (common) neighbors at each step, the tree stays connected throughout the trimming process. Therefore, the star-shaped four-node tree could have emerged only at the end of the trimming process, i.e. there will be only one hexagon.

There cannot be a single triangle left after the partitioning for the following reason. A single triangle corresponds to a single node in the dual tree. If there is any single node left after the trimming process it would be w. However, in both properties 1 and 2 node w is not a leaf and, therefore, cannot be the only node left after the trimming.

---

[2]An *edge guard* is an angle guard with a 180° angle defined by the edge on which it is placed.

Therefore, we can always partition the polygon into a collection of tetragons, pentagons and at most one hexagon with a star-shaped dual triangulation tree. □

Theorem 3. $n + 2(h - 1)$ *guards are sufficient to solve the sculpture garden problem with a concise formula with the length of the proof certificate at most three for any polygon with h holes.*

Proof. Consider a triangulation of the polygon. Partition the polygon into the collection of tetragons, pentagons and at most one hexagon as in Lemma 4. Each tetragon will consist of two triangles and by Lemma 1 can be covered by two guards. Each pentagon will consist of three triangles and by Lemma 2 can be covered by three guards. The hexagon (if there is one) will consist of four triangles and by Lemma 3 can be covered by three guards. Thus, the number of required guards will be no larger than the number of triangles in the triangulation, which is $n + 2(h - 1)$. The formula for the whole polygon will be the disjunction of the formulas for each of the smaller polygons, which (by Lemmas 1, 2, and 3) are conjunctions of length at most three. Thus, each proof certificate will be at most of length three. □

4 Lower Bounds

In this section we discuss some lower bounds for sculpture garden problems. We begin, however, with a conjecture.

Conjecture 1. *There is a polygon that requires $n - 2$ angle guards to solve the sculpture garden problem.*

The following theorem establishes a lower bound on the number of guards for arbitrary polygons.

Theorem 4. *At least $\lceil \frac{n}{2} \rceil$ guards are required to solve the sculpture garden problem for any polygon with no two edges lying on the same line.*

Proof. Assume less than $\lceil \frac{n}{2} \rceil$ guards can guard a particular polygon. Then there exists an edge e which is not collinear with any of the guards' boundary lines of the angle which they guard. This implies that there exists a non-empty region R which is fully located on one side (inside or outside) of each guards' guarded region and such that edge e splits R into two subregions R_1 and R_2. Without loss of generality assume R_1 is inside the polygon and R_2 is outside the polygon. Then no guard can distinguish whether a point is in R_1 and R_2, i.e., no guard can distinguish between points inside and outside the polygon. Thus, less than $\lceil \frac{n}{2} \rceil$ guards cannot guard a polygon. □

Theorem 4 provides a general lower bound on the number of guards for an arbitrary general-position polygon, which is off by a factor of 2 from the upper bound established above. For non-general-position polygons the following lower bound applies.

Theorem 5. *Any n-sided polygon requires $\Omega(\sqrt{n})$ guards.*

Proof. If a polygon P is defined by g angle guards, then P can have at most $g(2g - 1)$ polygon vertices, as each vertex occurs at the intersection of two of the $2g$ rays bounding guard regions. □

5 Polygon Classes that Require Fewer than $n - 2$ Guards

In this section we consider classes of polygons for which the general upper bound of $n - 2$ guards for arbitrary polygons can be considerably improved.

5.1 Convex Polygons

We begin with an observation that, for convex polygons, only $\lceil \frac{n}{2} \rceil$ guards are required to solve the sculpture garden problem.

Theorem 6. $\lceil \frac{n}{2} \rceil$ *guards are always sufficient to solve the sculpture garden problem for any convex polygon by placing the natural angle-guards in every other vertex of the polygon.*

Proof. Each natural angle-guard guards a region which fully contains the polygon. The intersection of these regions is the convex hull of the polygon, which is the polygon itself, since it is convex. Thus, the conjunction of the guards placed in every other corner of the convex polygon will define the polygon itself. $\square$

Together with the general lower bound on the number of guards, the above theorem shows that $\lceil \frac{n}{2} \rceil$ is a tight bound on the number of guards required to solve the sculpture garden problem for convex polygons. The formula is not concise, of course, but we show in Section 6 how to make it concise with a small blow-up in the number of guards.

5.2 Pseudo-triangles

A *pseudo-triangle* is a polygon with only 3 convex vertices and the rest of the vertices being reflex.

Theorem 7. *Only $\lceil \frac{n}{2} \rceil + O(1)$ guards are required to solve the sculpture garden problem for pseudo-triangles.*

Proof. Insert a Steiner vertex v anywhere in the kernel of the pseudo-triangle[3] and connect v to each of the convex vertices of the polygon. This partitions the polygon into three fans.

Guard each fan separately by placing the natural angle-guards on every other reflex vertex of the fan, as well as at the newly created Steiner vertex v. The formula for each fan will be the conjunction of the guard at v and the disjunction of all the guards on the reflex vertices.

The final formula will be the disjunction of the formulae of each separate fan. See Figure 7 for an example.
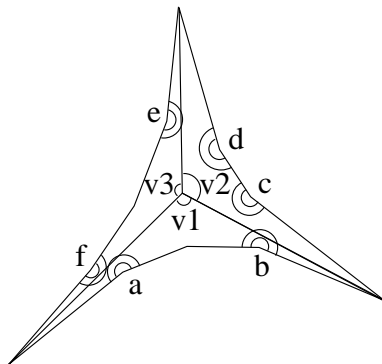

Figure 7: An example of a pseudo-triangle and its guard coverage. The corresponding formula is $F = v_1(a + b) + v_2(c + d) + v_3(e + f)$

The number of total guards to cover a pseudo-triangle is the number of guards on the reflex vertices, which is $\lceil \frac{n}{2} \rceil + O(1)$, plus three more guards on the Steiner vertex v. Thus the total number of guards is $\lceil \frac{n}{2} \rceil + O(1)$. $\square$

5.3 Polygons with a Sublinear Number of Guards

We now present a class of polygons for which a square-root number of guards is sufficient to solve the sculpture garden problem, providing an upper bound within a constant factor of the lower bound of Theorem 5.

Theorem 8. *There exist n-sided simple polygons that can be guarded concisely by $O(\sqrt{n})$ guards in a natural vertex placement.*

Proof. Form a line arrangement in the form of a grid with $4k$ horizontal lines and $4k$ vertical lines, and let P be a polygon with boundaries that zigzag between pairs of vertical lines in the grid, as shown in Figure 8. With such a construction we can form a vertex of P at every arrangement vertex except for some of the

---

[3]A kernel is a region of a star polygon (which pseudo-triangle is) from which one can draw a line to any vertex without crossing the boundary of the polygon.

vertices on the top and bottom horizontal lines of the arrangement, so P has $\Omega(k^2)$ vertices; by finding the next larger polygon of this form and then simplifying it we can find for any n a polygon with n vertices, the edges of which belong to a grid with $k = O(\sqrt{n})$. We place $16k$ guards, one on each side of each line of the arrangement (using natural angle guards placed at vertices). Using these guards, we can separately guard each rectangle of the arrangement, and hence P, with four guards per point. $\square$
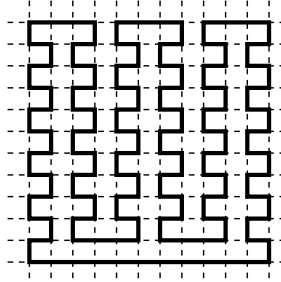

Figure 8: An example polygon that can be defined with $O(\sqrt{n})$ angle guards.

A natural question raised by this example is whether it is always possible to find an angle-guard placement that minimizes the number of guards for a particular polygon. Although this problem may be NP-hard, we show in the next theorem that we can always achieve a 2-approximation for this problem.

Theorem 9. *For any polygon P, we can find in linear time a collection of guards for P, using a number of guards that is within a factor of two of optimal.*

Proof. For each halfplane for which a portion of the boundary of the halfplane is used as one of the boundary edges of P, place an edge guard on the line bounding the halfplane, and construct the Peterson CSG formula [8] for P. In any collection of guards for P, each such halfplane must be guarded by one of the two rays from one of the guards, so the optimal number of guards is at least half the number of guards used. $\square$

It's tempting to look for an exact algorithm for guarding, using graph matching to find halfplanes that can be paired up and covered by a single guard, but it may not be apparent which halfplanes can be paired or whether such pairings are independent from each other. In addition, it is unclear whether the optimal set of guards always has guard rays that lie along polygon edges, as would be produced by such a matching algorithm.

5.4   Orthogonal Polygons

We now consider the case when the input is a polygon with axis-parallel sides (i.e., an *orthogonal polygon*). In this case we can considerably improve our $n - 2$ bound on the number of guards needed for general polygons. Our construction may place guards interior to the polygon, as well as on the boundary, and is based on the following result known for its application to art gallery theory:

Lemma 5 (O'Rourke et al. [13, 17, 18, 19]). *Any simple orthogonal polygon with n sides may be partitioned into $\lceil(n - 2)/4\rceil$ orthogonal polygons, each having at most six sides. Such a partition may be found in time bounded by that for finding a horizontal visibility diagram of the polygon ($O(n)$ time, by Chazelle's algorithm [5]).*

We require a slight refinement of this lemma. Note that, for an n-sided orthogonal polygon, n must always be even (there are exactly as many horizontal edges as vertical) so we need only distinguish between the case when $(n - 2)/2$ is even and when it is odd.

Lemma 6. *Any simple orthogonal polygon with n sides may be partitioned into $\lceil(n - 2)/4\rceil$ orthogonal polygons, each having at most six sides. If $(n - 2)/2$ is odd, at least one of the polygons in the partition has only four sides. Such a partition may be found in time $O(n)$.*

10

Proof. If $(n-2)/2$ is even, we are done. Otherwise, form the vertical visibility decomposition of the polygon by extending a vertical line segment across the polygon from each concave vertex to the nearest opposite boundary. This is a partition of the polygon into rectangles; if we form a graph with one vertex per rectangle and one edge between any two rectangles sharing a visibility edge, this graph is a tree. Let R be a rectangle forming a leaf in this tree, use R as one node in the partition, and apply Lemma 5 to the remaining $(n-2)$-vertex polygon. $\qquad\square$
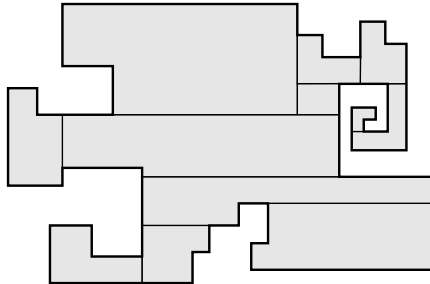

Figure 9: Partition of an orthogonal polygon into rectangular and hexagonal pieces.

Figure 9 depicts the partition of an orthogonal polygon into rectangular and hexagonal pieces according to these lemmas. Note that some of the vertices of the pieces may lie interior to the polygon.

Theorem 10. *In any simple orthogonal polygon with n sides, we may place at most $\lceil 3(n-2)/4 \rceil$ right-angle guards, in such a way that any point within the polygon has a concise proof of membership in the polygon, involving only two guards. This placement may be accomplished in time $O(n)$.*

Proof. We perform the partition of Lemma 6, and guard each piece of the partition separately. Each six-vertex polygon must be an L-shaped union of two rectangles sharing a common corner; it can be guarded by one guard at the common corner and one guard at each of the two opposite rectangle corners, for a total of three guards. Each four-vertex polygon is a rectangle and may be guarded with only two guards. If $(n-2)/2$ is even, we obtain at most $3(n-2)/4 = \lceil 3(n-2)/4 \rceil$ guards. If $(n-2)/2$ is odd, we obtain at most $2 + 3(n-4)/4 = \lceil 3(n-2)/4 \rceil$ guards. $\qquad\square$

6   Conciseness Trade-offs

The formula we provided for convex polygons in the proof of Theorem 6 is optimal as far as the number of required guards goes. However, it is not concise; in fact, the proof certificate is as long as the formula itself, i.e. $\lceil \frac{n}{2} \rceil$. This is far from the desired $O(1)$ bound for conciseness provided with other polygons in this paper. The following theorem provides a trade-off between the number of required guards and the conciseness of the formula.

Theorem 11. *Let P be a polygon taken from a class of polygons that is closed under partitioning via diagonals and such that n-vertex polygons of this class can be defined with $f(n)$ angle guards. Then there is a concise solution to the sculpture garden problem for P that uses $O(nf(c)/c)$ guards, where c is the maximum desired size of a proof a point is inside P.*

Proof. Triangulate P. If P is not simple, then add diagonals so that the dual to the triangulation is a tree T. Perform a recursive centroid decomposition [11] of T, stopping as soon as a subtree has size at most c. Each cut of T corresponds to our adding diagonals to P and this entire process introduces $O(n/c)$ subpolygons (of the same class as P), each of size at most c. Thus, each subpolygon can be defined with $f(c)$ angle guards, and we can define a concise formula for P that is the disjunction of the formulas for the subpolygons. $\qquad\square$

For example, we can produce a concise guarding of a convex polygon P using $\lceil n/2 \rceil (1+\epsilon)$ guards so that any point can prove it is inside P using $O(1/\epsilon)$ guards, for any constant $\epsilon > 0$.

11

7   Conclusion and Open Problems

In this paper, we introduced the sculpture garden problem for placing angle guards in such a way as to define a polygon P and prove when points are inside P. We presented a number of results concerning the kinds and number of guards needed to define various polygons. We provided the $n-2$ upper and $\frac{n}{2}$ lower bounds for general polygons, as well as conjectured the existence of some polygons which require as many as $n-2$ angle guards. We also provided several classes of polygons which require substantially fewer guards than the general upper bound. We feel this paper begins an interesting new branch of work on polygon guarding problems and hope that it will inspire future work in this direction. In particular, we leave the following open problems:

1. Is there a simple polygon that requires $n-2$ angle guards to define it (our conjecture is "yes")?

2. Our results also apply to the inverse sculpture garden problem, so that mobile devices outside a polygon can prove they are outside. What are the best upper and lower bounds for a generalization of the sculpture garden problem so that devices inside or outside the polygon can prove their respective locations?

3. Establish tight upper and lower bounds for solving the sculpture garden problem for orthogonal polygons.

4. Is the problem of finding the minimum number of angle guards for a particular polygon NP-hard?

Acknowledgment

We would like to thank Matthew Dickerson for helpful comments regarding several of the proofs contained in this paper.

References

[1] L. Bao and J. Garcia-Luna-Aceves. Transmission scheduling in ad hoc networks with directional antennas, 2002.

[2] P. Bose, L. J. Guibas, A. Lubiw, M. H. Overmars, D. L. Souvaine, and J. Urrutia. The floodlight problem. *International Journal of Computational Geometry and Applications*, 7(1/2):153–163, 1997.

[3] N. Bulusu, J. Heidemann, and D. Estrin. Gps-less low cost outdoor localization for very small devices, 2000.

[4] K. Chakrabarty, S. Iyengar, H. Qi, and E. Cho. Grid coverage of surveillance and target location in distributed sensor networks, 2002.

[5] B. Chazelle. Triangulating a simple polygon in linear time. *Discrete Comput. Geom.*, 6:485–524, 1991.

[6] Y. Cho, L. Bao, and M. Goodrich. Protocols for Region Locality Among Mobile Devices. *Technical Report*, 2005.

[7] V. Chvátal. A combinatorial theorem in plane geometry. *J. Combin. Theory Ser. B*, 18:39–41, 1975.

[8] D. P. Dobkin, L. Guibas, J. Hershberger, and J. Snoeyink. An efficient algorithm for finding the CSG representation of a simple polygon. *Algorithmica*, 10:1–23, 1993.

[9] V. Estivill-Castro, J. O'Rourke, J. Urrutia, and D. Xu. Illumination of polygons with vertex lights. *Information Processing Letters*, 56:9–13, 1995.

[10] Z. Füredi and D. Kleitman. The prison yard problem. *Combinatorica*, 14:287–300, 1994.

[11] M. T. Goodrich. Planar separators and parallel polygon triangulation. *J. Comput. Syst. Sci.*, 51(3):374–389, 1995.

[12] M. T. Goodrich. An improved ray shooting method for constructive solid geometry models via tree contraction. *Internat. J. Comput. Geom. Appl.*, 8(1):1–24, 1998.

[13] E. Györi. A short proof of the rectilinear art gallery theorem. *SIAM J. Alg. Disc. Meth.*, 7:452–454, 1986.

[14] T. He, C. Huang, B. Blum, J. Stankovic, and T. Abdelzaher. Range-free localization schemes in large scale sensor networks, 2003.

[15] A. Howard, M. Mataric, and G. Sukhatme. Mobile sensor network deployment using potential fields: A distributed, 2002.

[16] Y.-B. Ko, V. Shankarkumar, and N. H. Vaidya. Medium access control protocols using directional antennas in ad hoc networks. In *INFOCOM (1)*, pages 13–21, 2000.

[17] H. Mannila and D. Wood. A simple proof of the rectilinear art gallery theorem. Technical Report C-1984-16, Univ. of Helsinki, 1984.

[18] J. O'Rourke. An alternate proof of the rectilinear art gallery theorem. *J. Geometry*, 21:118–130, 1983.

[19] J. O'Rourke. *Art Gallery Theorems and Algorithms*. Number 3 in Int. Ser. Monographs on Computer Science. Oxford Univ. Press, 1987.

[20] J. O'Rourke. Visibility. In J. E. Goodman and J. O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, chapter 25, pages 467–480. CRC Press LLC, Boca Raton, FL, 1997.

[21] A. Savvides, C.-C. Han, and M. B. Strivastava. Dynamic fine-grained localization in ad-hoc networks of sensors. In *Mobile Computing and Networking*, pages 166–179, 2001.

[22] W. Steiger and I. Streinu. Illumination by floodlights. *Computational Geometry. Theory and Applications*, 10(1):57–70, 1998.

[23] J. Urrutia. Art gallery and illumination problems. In J.-R. Sack and J. Urrutia, editors, *Handbook of Computational Geometry*, pages 973–1027. North-Holland, 2000.

[24] R. J. Walker and J. Snoeyink. Practical point-in-polygon tests using CSG representations of polygons. In M. T. Goodrich and C. C. McGeoch, editors, *Algorithm Engineering and Experimentation (Proc. ALENEX '99)*, volume 1619 of *Lecture Notes Comput. Sci.*, pages 114–123. Springer-Verlag, 1999.