

# Geometric Fingerprint Recognition via Oriented Point-Set Pattern Matching

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## Abstract

Motivated by the problem of fingerprint matching, we present geometric approximation algorithms for matching a *pattern* point set against a *background* point set, where the points have angular orientations in addition to their positions.

## 1 Introduction

Fingerprint recognition typically involves a three-step process: (1) digitizing fingerprint images, (2) identifying *minutiae*, which are points where ridges begin, end, split, or join, and (3) matching corresponding minutiae points between the two images. An important consideration is that the minutiae are not pure geometric points: besides having geometric positions, defined by  $(x, y)$  coordinates in the respective images, each minutiae point also has an *orientation* (the direction of the associated ridges), and these orientations should be taken into consideration in the comparison, e.g., see [13, 9, 16, 19, 10, 11, 17, 15, 12] and Figure 1.

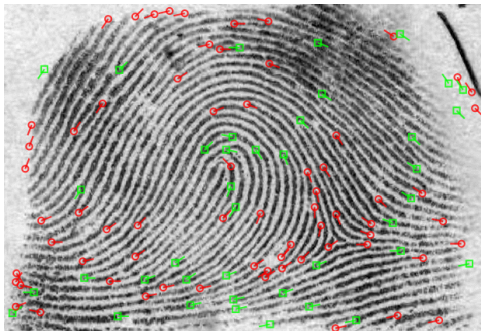


Figure 1: Screenshot of the display of fingerprint minutiae in NIST’s Fingerprint Minutiae Viewer (FpMV).

In this paper, we consider computational geometry problems inspired by this fingerprint matching problem. The problems we consider are all instances of point-set pattern matching problems, where we are given a “pattern” set,  $P$ , of  $m$  points in  $\mathbb{R}^2$  and a “background” set,  $B$ , of  $n$  points in  $\mathbb{R}^2$ , and we are asked to find a transformation of  $P$  that best aligns the points of  $P$  with a subset of the points in  $B$ , e.g., see [3, 4, 5, 6, 7].

A natural choice of a distance measure to use in this case, between a transformed copy,  $P'$ , of the pattern,  $P$ , against the background,  $B$ , is the *directed Hausdorff distance*, defined as  $h(P', B) = \max_{p \in P'} \min_{q \in B} \rho(p, q)$ , where  $\rho$  is an underlying distance metric for points, such as the Euclidean metric. In other words, the problem is to find a transformation of  $P$  that minimizes the farthest any point in  $P$  is from its nearest neighbor in  $B$ . Rather than only considering the positions of the points in  $P$  and  $B$ , however, in this paper we consider instances in which each point in  $P$  and  $B$  also has an associated *orientation* defined by an angle, as in the fingerprint matching application.

It is important in such *oriented point-set pattern matching* problems to use an underlying distance that combines information about both the locations and the orientations of the points, and to use this distance in finding a good transformation. Our goal is to design efficient algorithms that can find a transformation that is a good match between  $P$  and  $B$  taking both positions and orientations into consideration.

**Previous Work.** In the domain of fingerprint matching, past work tends to focus on matching fingerprints heuristically or as pixelated images, taking into consideration both the positions and orientation of the minutiae or other features, e.g., see [13, 9, 16, 19, 10, 11, 17, 15, 12]. We are not aware of past work on studying fingerprint matching as a computational geometry problem, however.

Geometric pattern matching for point sets without orientations, on the other hand, has been well studied from a computational geometry viewpoint, e.g., see [1, 4, 6, 18]. For such unoriented point sets, existing algorithms can find an optimal solution minimizing Hausdorff distance, but they generally have high polynomial running times. Several existing algorithms give approximate solutions to geometric pattern matching problems [3, 5, 7, 8], but we are not aware of previous approximation algorithms for oriented point-set pattern matching. Goodrich *et al.* [7] present approximation algorithms for geometric pattern matching in multiple spaces under different types of motion, achieving approximation ratios ranging from 2 to  $8 + \epsilon$ , for constant  $\epsilon > 0$ . Cho and Mount [5] show how to achieve improved approximation ratios for such matching problems, at the expense of making the analysis more complicated.

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Other algorithms give approximation ratios of  $1 + \epsilon$ , allowing the user to define the degree of certainty they want. Indyk *et al.* [8] give a  $(1 + \epsilon)$ -approximation algorithm whose running time is defined in terms of both the number of points in the set as well as  $\Delta$ , which is defined as the distance between the farthest and the closest pair of points. Cardoze and Schulman [3] offer a randomized  $(1 + \epsilon)$ -approximation algorithm for  $\mathbb{R}^d$  whose running time is also defined in terms of  $\Delta$ . These algorithms are fast when  $\Delta$  is relatively small, which is true on average for many application areas, but these algorithms are much less efficient in domains where  $\Delta$  is likely to be arbitrarily large.

**Our Results.** In this paper, we present a family of simple algorithms for approximate oriented point-set pattern matching problems, that is, computational geometry problems motivated by fingerprint matching. Each of our algorithms uses as a subroutine a *base algorithm* that selects certain points of the pattern,  $P$ , and “pins” them into certain positions with respect to the background,  $B$ . This choice determines a transformed copy  $P'$  of the whole point set  $P$ . We then compute the directed Hausdorff distance for this transform by querying the nearest neighbor in  $B$  for each point of  $P'$ . To find nearest neighbors for a suitably-defined metric on oriented points that combines straight-line distance with rotation amounts, we adapt balanced box decomposition (BBD) trees [2] to oriented point sets, which may be of independent interest. The general idea of this adaptation is to insert two copies of each point such that, for any query point, if we find its nearest neighbor using the  $L_1/L_2$ -norm, we will either find the nearest neighbor based on  $\mu_1/\mu_2$  or we will find one of its copies. The full details of this approach are described in Appendix B. The output of the base algorithm is the transformed copy  $P'$  that minimizes this distance. We refer to our base algorithms as *pin-and-query* methods.

These base algorithms are all simple and effective, but their approximation factors are larger than 2, whereas we seek  $(1 + \epsilon)$ -approximation schemes for any constant  $\epsilon > 0$ . To achieve such results, our approximation schemes call the base algorithm twice. The first call determines an approximate scale of the solution. Next, our schemes apply a *grid-refinement* strategy that expands the set of background points by convolving it with a fine grid at that scale, in order to provide more candidate motions. Finally, they call the base algorithm a second time on the expanded input. This allows us to leverage the speed and simplicity of the base algorithms, gaining greater accuracy while losing only a constant factor in our running times.

The resulting approximation algorithms run in the same asymptotic time bound as the base algorithm (with some dependence on  $\epsilon$  in the constants) and

achieve approximations that are a  $(1 + \epsilon)$  factor close to optimal, for any constant  $\epsilon > 0$ . For instance, one of our approximation schemes, designed in this way, guarantees a worst case running time of  $O(n^2 m \log n)$  for rigid motions defined by translations and rotations. Thus, our approach results in polynomial-time approximation schemes (PTASs), where their running times depend only on combinatorial parameters. Specifically, we give the runtimes and approximation ratios for our algorithms in Table 1.

Algorithm	Running Time	Approx. Ratio
T	$O(nm \log n)$	$1 + \epsilon$
TR	$O(n^2 m \log n)$	$1 + \epsilon$
TRS	$O(n^2 m \log n)$	$1 + \epsilon$

Table 1: Running times and approximation ratios for our approximation algorithms.

The primary challenge in the design of our algorithms is to come up with methods that achieve an approximation factor of  $1 + \epsilon$ , for any small constant  $\epsilon > 0$ , without resulting in a running time that is dependent on a geometric parameter like  $\Delta$ . The main idea that we use to overcome this challenge is for our base algorithms in some cases to use two different pinning schemes, one for large diameters and one for small diameters. We show that one of these pinning schemes always finds a good match, so choosing the best transformation found by either of them allows us to avoid a dependence on geometric parameters in our running times. As mentioned above, all of our base algorithms are simple, as are our  $(1 + \epsilon)$ -approximation algorithms. Moreover, proving each of our algorithms achieves a good approximation ratio is also simple, involving no more than “high school” geometry. Still, for the sake of our presentation, we postpone some proofs and simple cases to appendices.

## 2 Formal Problem Definition

Let us formally define the *oriented point-set pattern matching* problem. We define an *oriented point set* in  $\mathbb{R}^2$  to be a finite subset of the set  $O$  of all oriented points, defined as

$$O = \{(x, y, a) \mid x, y, a \in \mathbb{R}, a \in [0, 2\pi)\}.$$

We consider three sets of transformations on oriented point sets, corresponding to the usual translations, rotations, and scalings on  $\mathbb{R}^2$ . In particular, we define the set of *translations*,  $\mathcal{T}$ , as the set of functions  $T_v : O \rightarrow O$  of the form

$$T_v(x, y, a) = (x + v_x, y + v_y, a),$$

where  $v = (v_x, v_y) \in \mathbb{R}^2$  is referred to as the *translation vector*.

Let  $R_{p,\theta}$  be a rotation in  $\mathbb{R}^2$  where  $p$  and  $\theta$  are the center and angle of rotation, respectively. We extend the action of  $R_{p,\theta}$  from unoriented points to oriented points by defining

$$R_{p,\theta}(x, y, a) = (R_{p,\theta}(x, y), (a + \theta) \bmod 2\pi),$$

and we let  $\mathcal{R}$  denote the set of rotation transformations from  $O$  to  $O$  defined in this way.

Finally, we define the set of *scaling* operations on an oriented point set. Each such operation  $S_{p,s}$  is determined by a point  $p = (x_p, y_p, a_p)$  at the center of the scaling and by a scale factor,  $s$ . If a point  $q$  is Euclidean distance  $d$  away from  $p$  before scaling, the distance between  $q$  and  $p$  should become  $sd$  after scaling. In particular, this determines  $S_{p,s} : O \rightarrow O$  to be the function

$$S_{p,s}(x, y, a) = (x_p + s(x - x_p), y_p + s(y - y_p), a).$$

We let  $\mathcal{S}$  denote the set of scaling functions defined in this way.

As in the unoriented point-set pattern matching problems, we use a directed Hausdorff distance to measure how well a transformed patten set of points,  $P$ , matches a background set of points,  $B$ . That is, we use

$$h(P, B) = \max_{p \in P} \min_{q \in B} \mu(p, q),$$

where  $\mu(p, q)$  is a distance metric for oriented points in  $\mathbb{R}^2$ . Our approach works for various types of metrics,  $\mu$ , for pairs of points, but, for the sake of concreteness, we focus on two specific distance measures for elements of  $O$ , which are based on the  $L_1$ -norm and  $L_2$ -norm, respectively. In particular, for  $(x_1, y_1, a_1), (x_2, y_2, a_2) \in O$ , let

$$\begin{aligned} \mu_1((x_1, y_1, a_1), (x_2, y_2, a_2)) = \\ |x_1 - x_2| + |y_1 - y_2| + \min(|a_1 - a_2|, 2\pi - |a_1 - a_2|), \end{aligned}$$

and let

$$\begin{aligned} \mu_2((x_1, y_1, a_1), (x_2, y_2, a_2)) = \\ \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + \min(|a_1 - a_2|, 2\pi - |a_1 - a_2|)^2}. \end{aligned}$$

Intuitively, one can interpret these distance metrics to be analogous to the  $L_1$ -norm and  $L_2$ -norm in a cylindrical 3-dimensional space where the third dimension wraps back around to 0 at  $2\pi$ . Thus, for  $i \in \{1, 2\}$ , and  $B, P \subseteq O$ , we use the following directed Hausdorff distance:

$$h_i(P, B) = \max_{p \in P} \min_{b \in B} \mu_i(p, b).$$

Therefore, for some subset  $\mathcal{E}$  of  $\mathcal{T} \cup \mathcal{R} \cup \mathcal{S}$ , the *oriented point-set pattern matching* problem is to find a composition  $E$  of one or more functions in  $\mathcal{E}$  that minimizes  $h_i(E(P), B)$ .

### 3 Translations and Rotations

In this section, we present our base algorithm and approximation algorithm for approximately solving the oriented point-set pattern matching problem where we allow translations and rotations. Given two subsets of  $O$ ,  $P$  and  $B$ , with  $|P| = m$  and  $|B| = n$ , our goal here is to minimize  $h_i(E(P), B)$  where  $E$  is a composition of functions in  $\mathcal{T} \cup \mathcal{R}$ . In the case of translations and rotations, we actually give two sets of algorithms—one set that works for point sets with large diameter and one that works for point sets with small diameter. Deciding which of these to use is based on a simple calculation (which we postpone to the analysis below), which amounts to a normalization decision to determine how much influence orientations have on matches versus coordinates.

#### Base Algorithm Under Translation and Rotation with Large Diameter.

In this subsection, we present an algorithm for solving the approximate oriented point-set pattern matching problem where we allow translations and rotations. This algorithm provides a good approximation ratio when the diameter of our pattern set is large. Given two subsets  $P$  and  $B$  of  $O$ , with  $|P| = m$  and  $|B| = n$ , we wish to minimize  $h_i(E(P), B)$  over all compositions  $E$  of one or more functions in  $\mathcal{T} \cup \mathcal{R}$ . Our algorithm is as follows (see Figure 2).

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#### Algorithm BaseTranslateRotateLarge( $P, B$ ):

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Find  $p$  and  $q$  in  $P$  having the maximum value of
 $\|(x_p, y_p) - (x_q, y_q)\|_2$ .
for every pair of points  $b, b' \in B$  do
  Pin step: Apply the translation,  $T_v \in \mathcal{T}$ , that takes
   $p$  to  $b$ , and apply the rotation,  $R_{p,\theta}$ , that makes  $p$ ,
   $b'$ , and  $q$  collinear.
  Let  $P'$  denote the transformed pattern set,  $P$ .
  for every  $q \in P'$  do
    Query step: Find a nearest-neighbor of  $q$  in  $B$ 
    using the  $\mu_i$  metric, and update a candidate
    Hausdorff distance accordingly.
  end for
return the smallest candidate Hausdorff distance
found as the smallest distance,  $h_i(R_{p,\theta}(T_v(P)), B)$ .
end for

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The points  $p$  and  $q$  can be found in  $O(m \log m)$  time [14]. The pin step iterates over  $O(n^2)$  translations and rotations, respectively, and, for each one of these transformations, we perform  $m$  BBD queries, each of which takes  $O(\log n)$  time. Therefore, our total running time is  $O(n^2 m \log n)$ . Our analysis for this algorithm's approximation factor uses the following simple lemma.

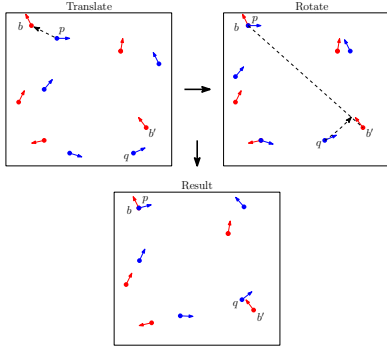


Figure 2: Illustration of the translation and rotation steps of the base approximation algorithm for translation and rotation in  $O$  when diameter is large.

**Lemma 1** Let  $P$  be a finite subset of  $O$ . Consider the rotation  $R_{c,\theta}$  in  $\mathcal{R}$ . Let  $q = (x_q, y_q, a_q)$  be the element in  $P$  such that  $\|(x_q, y_q) - (x_c, y_c)\|_2 = D$  is maximized. For any  $p = (x_p, y_p, a_p) \in P$ , denote  $R_{c,\theta}(x_p, y_p, a_p)$  as  $p' = (x_{p'}, y_{p'}, a_{p'})$ . Let  $i \in \{1, 2\}$ . Then for all  $p \in P$ ,  $\mu_i(p, p') \leq \|(x_q, y_q) - (x_{q'}, y_{q'})\|_i + \pi \|(x_q, y_q) - (x_{q'}, y_{q'})\|_2 / (2D)$ .

**Theorem 2** Let  $h_{opt}$  be  $h_i(E(P), B)$  where  $E$  is the composition of functions in  $\mathcal{T} \cup \mathcal{R}$  that attains the minimum of  $h_i$ , for  $i \in \{1, 2\}$ . The algorithm above runs in time  $O(n^2 m \log n)$  and produces an approximation to  $h_{opt}$  that is at most  $(A_1 + \epsilon)h_{opt}$  for  $h_1$  and at most  $(A_2 + \epsilon)h_{opt}$  for  $h_2$ , where  $\epsilon > 0$  is a fixed constant,  $A_1 = 6 + \sqrt{2}\pi/D$ , and  $A_2 = 2 + \sqrt{2}(2 + \pi/D)$ .

**Grid Refinement.** In this subsection, we describe our grid refinement process, which allows us to use a base algorithm to obtain an approximation ratio of  $1 + \epsilon$ . To achieve this result, we take advantage of an important property of the fact that we are approximating a Hausdorff distance by a pin-and-query algorithm. Our base algorithm approximates  $h_{opt}$  by pinning a reference pattern point,  $p$ , to a background point,  $b$ . Reasoning backwards, if we have a pattern in an optimal position, where every pattern point,  $p$ , is at distance  $d \leq h_{opt}$  from its associated nearest neighbor in the background, then one of the transformations tested by the base pin-and-query algorithm moves each pattern point by a distance of at most  $(A_i - 1)d$  away from this optimal location when it performs its pinning operation.

Suppose we could define a constant-sized “cloud” of points with respect to each background point, such that one of these points is guaranteed to be very close to the optimal pinning location, much closer than the distance  $d$  from the above argument. Then, if we use these cloud points to define the transformations checked by the base algorithm, one of these transformations will move each point from its optimal position by a much smaller distance.

To aid us in defining such a cloud of points, consider the set of points  $G(p, l, k) \subset \mathbb{R}^2$  (where  $p = (x_p, y_p)$  is some point in  $\mathbb{R}^2$ ,  $l$  is some positive real value, and  $k$  is some positive integer) defined by the following formula:

$$G(p, l, k) = \{q \in \mathbb{R}^2 \mid q = (x_p + il, y_p + jl), i, j \in \mathbb{Z}, -k \leq i, j \leq k\}.$$

Then  $G(p, l, k)$  is a grid of  $(2k + 1)^2$  points within a square of side length  $2kl$  centered at  $p$ , where the coordinates of each point are offset from the coordinates of  $p$  by a multiple of  $l$ . An example is shown in Figure 3.

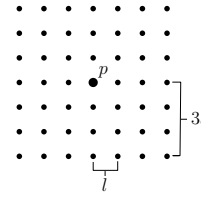


Figure 3: An example of  $G(p, l, 3)$ .

Now consider any point  $q$  whose Euclidean distance is no more than  $kl$  from  $p$ . This small distance forces point  $q$  to lie within the square convex hull of  $G(p, l, k)$ . This implies that there is a point of  $G(p, l, k)$  that is even closer to  $q$ :

**Lemma 3** Let  $i \in \{1, 2\}$ . Given two points  $p, q \in \mathbb{R}^2$ , if  $\|p - q\|_i \leq kl$ , then  $\|q - s\|_1 \leq l$  and  $\|q - s\|_2 \leq l/\sqrt{2}$ , where  $s$  is  $q$ 's closest neighbor in  $G(p, l, k)$ .

**A  $(1 + \epsilon)$ -Approximation Algorithm Under Translation and Rotation with Large Diameter.** Here, we achieve a  $(1 + \epsilon)$ -approximation ratio when we allow translations and rotations. Again, given two subsets of  $O$ ,  $P$  and  $B$ , with  $|P| = m$  and  $|B| = n$ , our goal is to minimize  $h_i(E(P), B)$  over all compositions  $E$  of one or more functions in  $\mathcal{T} \cup \mathcal{R}$ . We perform the following steps.

1. Run algorithm, `BaseTranslateRotateLarge`( $P, B$ ), from Section 3 to obtain an approximation  $h_{apr} \leq A \cdot h_{opt}$ , where  $A = A_1 + \epsilon$  or  $A = A_2 + \epsilon$ , for a constant  $\epsilon > 0$ .
2. For every  $b \in B$ , generate the grid of points  $G_b = G(b, \frac{h_{apr}\epsilon}{A^2 - A}, \lceil \frac{A^2 - A}{\epsilon} \rceil)$  for  $h_1$  or the grid  $G_b = G(b, \frac{\sqrt{2}h_{apr}\epsilon}{A^2 - A}, \lceil \frac{A^2 - A}{\sqrt{2}\epsilon} \rceil)$  for  $h_2$ . Let  $B'$  denote the resulting point set, which is of size  $O(A^4 n)$ , i.e.,  $|B'|$  is  $O(n)$  when  $A$  is a constant.
3. Run algorithm, `BaseTranslateRotateLarge`( $P, B'$ ), except use the original set,  $B$ , for nearest-neighbor queries in the query step.

It is easy to see that this simple algorithm runs in  $O(A^8 n^2 m \log n)$ , which is  $O(n^2 m \log n)$  when  $A$  is a constant (i.e., when the points in  $P$  have a large enough diameter).

**Theorem 4** *Let  $h_{opt}$  be  $h_i(E(P), B)$  where  $E$  is the composition of functions in  $\mathcal{T} \cup \mathcal{R}$  that attains the minimum of  $h_i$ . The algorithm above runs in time  $O(A^8 n^2 m \log n)$  and produces an approximation to  $h_{opt}$  that is at most  $(1 + \epsilon)h_{opt}$  for both  $h_1$  and  $h_2$ .*

**Base Algorithm Under Translation and Rotation with Small Diameter.** In this subsection, we present an alternative algorithm for solving the approximate oriented point-set pattern matching problem where we allow translations and rotations. Compared to the algorithm given in Section 3, this algorithm instead provides a good approximation ratio when the diameter of our pattern set is small. Once again, given two subsets of  $O$ ,  $P$  and  $B$ , with  $|P| = m$  and  $|B| = n$ , we wish to minimize  $h_i(E(P), B)$  over all compositions  $E$  of one or more functions in  $\mathcal{T} \cup \mathcal{R}$ . We perform the following algorithm (see Figure 4).

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**Algorithm** BaseTranslateRotateSmall( $P, B$ ):

Choose some  $p \in P$  arbitrarily.

**for** every points  $b \in B$  **do**

*Pin step:* Apply the translation,  $T_v \in \mathcal{T}$ , that takes  $p$  to  $b$ , and then apply the rotation,  $R_{p,\theta}$ , that makes  $p$  and  $b$  have the same orientation.

Let  $P'$  denote the transformed pattern set,  $P$ .

**for** every  $q \in P'$  **do**

*Query step:* Find a nearest-neighbor of  $q$  in  $B$  using the  $\mu_i$  metric, and update a candidate Hausdorff distance accordingly.

**end for**

**return** the smallest candidate Hausdorff distance found as the smallest distance,  $h_i(R_{p,\theta}(T_v(P)), B)$ .

**end for**

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**Theorem 5** *Let  $h_{opt}$  be  $h_i(E(P), B)$  where  $E$  is the composition of functions in  $\mathcal{T} \cup \mathcal{R}$  that attains the minimum of  $h_i$ . The algorithm above runs in time  $O(nm \log n)$  and produces an approximation to  $h_{opt}$  that is at most  $(A_i + \epsilon)h_{opt}$  for  $h_i$ , where  $i = \{1, 2\}$ ,  $\epsilon > 0$  is a fixed constant,  $A_1 = 2 + \sqrt{2}D$ , and  $A_2 = 2 + D$ .*

**A  $(1 + \epsilon)$ -Approximation Algorithm Under Translation and Rotation with Small Diameter.** In this subsection, we utilize the algorithm from Section 3 to achieve a  $(1 + \epsilon)$ -approximation ratio when we allow translations and rotations. Again, given two subsets of

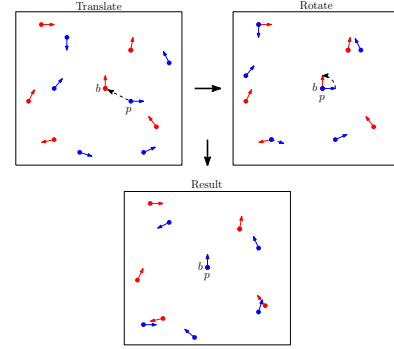


Figure 4: Illustration of the translation and rotation steps of the base approximation algorithm for translation and rotation in  $O$  when diameter is small.

$O$ ,  $P$  and  $B$ , with  $|P| = m$  and  $|B| = n$ , our goal is to minimize  $h_i(E(P), B)$  over all compositions  $E$  of one or more functions in  $\mathcal{T} \cup \mathcal{R}$ . We begin by describing another type of grid refinement we use in this case.

In particular, let us consider a set of points  $C(p, k) \subset O$  where  $p = (x_p, y_p, a_p)$  is some point in  $O$  and  $k$  is some positive integer. We define the set in the following way (see Figure 5):

$$C(p, k) = \{q \in O \mid$$

$$q = (x_p, y_p, a + 2\pi i/k \pmod{2\pi}), i \in \mathbb{Z}, 1 \leq i \leq k\}.$$

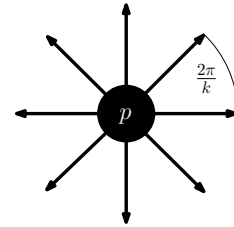


Figure 5: An example of  $C(p, 8)$ .

From this definition, we can see that  $C(p, k)$  is a set of points that share the same position as  $p$  but have different orientations that are equally spaced out, with each point's orientation being an angle of  $\frac{2\pi}{k}$  away from the previous point. Therefore, it is easy to see that, for any point  $q \in O$ , there is a point in  $C(p, k)$  whose orientation is at most an angle of  $\frac{\pi}{k}$  away from the orientation of  $q$ . Given this definition, our algorithm is as follows.

1. Run algorithm, BaseTranslateRotateSmall( $P, B$ ), from Section 3, to obtain  $h_{apr} \leq A \cdot h_{opt}$ .
2. For every  $b \in B$ , generate the point set

$$G_b = G \left( b, \frac{h_{apr}\epsilon}{2(A^2 - A)}, \left\lceil \frac{2(A^2 - A)}{\epsilon} \right\rceil \right)$$

for  $h_1$  or

$$G_b = G \left( b, \frac{h_{apr}\epsilon}{A^2 - A}, \left\lceil \frac{A^2 - A}{\epsilon} \right\rceil \right)$$

for  $h_2$ . Let  $B'$  denote the resulting set of points, i.e.,  $B' = \bigcup_{b \in B} G_b$ .

3. For every  $b' \in B'$ , generate the point set

$$C_{b'} = C \left( b', \frac{2(A^2 - A)}{\pi h_{apr}\epsilon} \right)$$

for  $h_1$  or

$$C_{b'} = C \left( b', \frac{\sqrt{2}(A^2 - A)}{\pi h_{apr}\epsilon} \right)$$

for  $h_2$ . Let  $B''$  denote the resulting set of points.

4. Run algorithm, BaseTranslateRotateSmall( $P, B''$ ), but continue to use the points in  $B$  for nearest-neighbor queries.

Intuitively, this algorithm uses the base algorithm to give us an indication of what the optimal solution might be. We then use this approximation to generate a larger set of points from which to derive transformations to test, but this time we also generate a number of different orientations for those points as well. We then use this point set in the base algorithm when deciding which transformations to iterate over, while still using  $B$  to compute nearest neighbors.

The first step of this algorithm runs in time  $O(nm \log n)$ , as we showed. The second step takes time proportional to the number of points which have to be generated, which is determined by  $n$ , our choice of the constant  $\epsilon$ , and the approximation ratio,  $A$ , of our base algorithm. The time needed to complete the second step is  $O(A^4n)$ . The third step generates even more points based on points generated in step two, which increases the size of  $B''$  to be  $O(A^6n)$ . The last step runs in time  $O(A^6nm \log n)$ , which is also the running time for the full algorithm.

**Theorem 6** *Let  $h_{opt}$  be  $h_i(E(P), B)$  where  $E$  is the composition of functions in  $\mathcal{T} \cup \mathcal{R}$  that attains the minimum of  $h_i$ . The algorithm above runs in time  $O(A^6nm \log n)$  and produces an approximation to  $h_{opt}$  that is at most  $(1 + \epsilon)h_{opt}$  for both  $h_1$  and  $h_2$ .*

**Combining the Algorithms for Large and Small Diameters.** For the two cases above, we provided two base algorithms that each had a corresponding  $(1 + \epsilon)$ -approximation algorithm. As mentioned above, we classified the two by whether the algorithm achieved a good approximation when the diameter of the pattern set was large or small. This is because the large diameter

base algorithm has an approximation ratio with terms that are inversely proportional to the diameter, and the small diameter base algorithm has an approximation ratio with terms that are directly proportional to the diameter.

Both of the resulting  $(1 + \epsilon)$ -approximation algorithms have running times which are affected by the approximation ratio of their base algorithm, meaning their run times are dependent upon the diameter of the pattern set. We can easily see, however, that the approximation ratios of the large and small diameter base algorithms intersect at some fixed constant diameter,  $D^*$ . For  $h_1$ , if we compare the expressions  $6 + \sqrt{2}\pi/D$  and  $2 + \sqrt{2}D$ , we find that the two expressions are equal at  $D^* = \sqrt{2} + \sqrt{2 + \pi} \approx 3.68$ . For  $h_2$ , we compare  $2 + \sqrt{2}(2 + \pi/D)$  and  $2 + D$  to find that they are equal at  $D^* = \sqrt{2} + \sqrt{2 + \sqrt{2}\pi} \approx 3.95$ . For diameters larger than  $D^*$ , the approximation ratio of the large diameter algorithm is smaller than at  $D^*$ , and for diameters smaller than  $D^*$ , the approximation ratio of the small diameter algorithm is smaller than at  $D^*$ . Thus, if we choose to use the small diameter algorithms when the diameter is less than  $D^*$  and we use the large diameter algorithms when the diameter is greater or equal to  $D^*$ , we ensure that the approximation ratio is no more than the constant value that depends on the constant  $D^*$ . Thus, based on the diameter of the pattern set, we can decide *a priori* if we should use our algorithms for large diameters or small diameters and just go with that set of algorithms. This implies that we are guaranteed that our approximation factor,  $A$ , in our base algorithm is always bounded above by a constant; hence, our running time for the translation-and-rotation case is  $O(n^2m \log n)$ .

## 4 Conclusion

We present distance metrics that can be used to measure the similarity between two point sets with orientations and we also provided fast algorithms that guarantee close approximations of an optimal transformation. In the appendices, we provide additional algorithms for other types of transformations and we also provide results of experiments.

## Acknowledgments

This work was supported in by the NSF under grants 1526631, 1618301, and 1616248, and by DARPA under agreement no. AFRL FA8750-15-2-0092. The views expressed are those of the authors and do not reflect the official policy or position of the Department of Defense or the U.S. Government.

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## A Postponed Proofs

In this appendix, we present proofs that were postponed from the body of our paper.

**Lemma 3.** *Let  $i \in \{1, 2\}$ . Given two points  $p, q \in \mathbb{R}^2$ , if  $\|p - q\|_i \leq kl$ , then  $\|q - s\|_1 \leq l$  and  $\|q - s\|_2 \leq l/\sqrt{2}$ , where  $s$  is  $q$ 's closest neighbor in  $G(p, l, k)$ .*

**Proof.** Because  $\|p - q\|_i \leq kl$ , we know that  $q$  exists within the square of side length  $2kl$  which encompasses  $G(p, l, k)$  (which we will refer to as  $G$  for the remainder of this proof). This square can be divided into  $(2k)^2$  non-overlapping squares of side length  $l$ . It is easy to see that the vertices of these squares are all points in  $G$  and that  $q$  exists within (or on the edge of) at least one of these squares. The point inside of a square that maximizes the distance to the square's closest vertex is the exact center of the square. If the side length is  $l$ , simple geometry shows us that at this point, the distance to any vertex is  $l$  with respect to the  $L_1$ -norm and  $l/\sqrt{2}$  with respect to the  $L_2$ -norm. Thus, because  $q$  exists within a square of side length  $l$  whose vertices are points in  $G$ , the furthest that  $q$  can be from its nearest neighbor in  $G$  is  $l$  for the  $L_1$ -norm and  $l/\sqrt{2}$  for the  $L_2$ -norm.  $\square$

**Lemma 1.** *Let  $P$  be a finite subset of  $O$ . Consider the rotation  $R_{c,\theta}$  in  $\mathcal{R}$ . Let  $q = (x_q, y_q, a_q)$  be the element in  $P$  such that  $\|(x_q, y_q) - (x_c, y_c)\|_2 = D$  is maximized. For any  $p = (x_p, y_p, a_p) \in P$ , denote  $R_{c,\theta}(x_p, y_p, a_p)$  as  $p' = (x_{p'}, y_{p'}, a_{p'})$ . Let  $i \in \{1, 2\}$ . Then for all  $p \in P$ ,  $\mu_i(p, p') \leq \|(x_q, y_q) - (x_{q'}, y_{q'})\|_i + \pi\|(x_q, y_q) - (x_{q'}, y_{q'})\|_2/(2D)$ .*

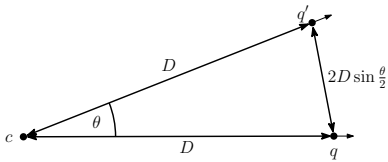


Figure 6: The rotation of  $q$  to  $q'$  about  $c$

**Proof.** After applying the rotation  $R_{c,\theta}$ , we know  $q$  has moved at least as far than any other point because it is the farthest from the center of rotation. Without loss of generality,  $0 \leq \theta \leq \pi$ . Then it is easily verifiable that  $\theta/\pi \leq \sin(\theta/2)$ . As  $2D \sin(\theta/2)$  is the Euclidean distance  $q$  moves under  $R_{c,\theta}$ , it follows that

$$\frac{2D\theta}{\pi} \leq 2D \sin(\theta/2) = \|(x_q, y_q) - (x_{q'}, y_{q'})\|_2.$$

This scenario is illustrated in Figure 6. Thus,  $\theta \leq (\pi\|(x_q, y_q) - (x_{q'}, y_{q'})\|_2)/(2D)$ , which implies that  $R_{c,\theta}$  moves the position of  $q$  by at most  $\|(x_q, y_q) - (x_{q'}, y_{q'})\|_i$

and changes the orientation of  $q$  by at most  $\pi\|(x_q, y_q) - (x_{q'}, y_{q'})\|_2/(2D)$ . Therefore, because  $q$  moves farther than any other point in  $P$ , any point  $p \in P$  has moved a distance of at most  $\|(x_q, y_q) - (x_{q'}, y_{q'})\|_i + \pi\|(x_q, y_q) - (x_{q'}, y_{q'})\|_2/(2D)$  with respect to the distance function  $\mu_i$ .  $\square$

**Theorem 2.** *Let  $h_{opt}$  be  $h_i(E(P), B)$  where  $E$  is the composition of functions in  $\mathcal{T} \cup \mathcal{R}$  that attains the minimum of  $h_i$ , for  $i \in \{1, 2\}$ . The algorithm above runs in time  $O(n^2 m \log n)$  and produces an approximation to  $h_{opt}$  that is at most  $(A_1 + \epsilon)h_{opt}$  for  $h_1$  and at most  $(A_2 + \epsilon)h_{opt}$  for  $h_2$ , where  $\epsilon > 0$  is a fixed constant,  $A_1 = 6 + \sqrt{2}\pi/D$ , and  $A_2 = 2 + \sqrt{2}(2 + \pi/D)$ .*

**Proof.** The additional  $\epsilon$  terms come entirely from using approximate nearest neighbor queries (defining BBD trees so they return  $(1 + \epsilon/A_i)$ -approximate nearest neighbors, for  $i \in \{1, 2\}$ ). So it is sufficient for us to prove approximation bounds that are  $A_i \cdot h_{opt}$ .

The first step is argued similarly to that of the proof of Theorem 8. Let  $E$  be the composition of functions in  $\mathcal{T} \cup \mathcal{R}$  that attains the minimum of  $h(E(P), B)$  and let  $P'$  be  $E(P)$ . Then for all  $p$  in  $P'$ , there exists  $b$  in  $B$  such that  $\mu_i(p, b) \leq h_{opt}$ . Let  $p', q' \in B$  be the closest background points to optimal positions of  $p$  and  $q$  respectively, where  $p$  and  $q$  are the diametric points we choose in the first step of the algorithm. Thus,

$$\|(x_p, y_p) - (x_{p'}, y_{p'})\|_i \leq \mu_i(p, p') \leq h_{opt}.$$

Apply the translation  $T_v$  on  $P'$  so that  $p$  coincides with  $p'$ , which is equivalent to moving every point  $\|(x_p, y_p) - (x_{p'}, y_{p'})\|_i$  with respect to position. Lemma 7, then, implies that all points have moved at most  $h_{opt}$ .

Next, apply the rotation  $R_{p,\theta}$  to  $P'$  that makes  $p, q$ , and  $q'$  co-linear. With respect to position,  $q$  moves at most a Euclidean distance of  $2D \sin(\theta/2)$  away from  $q'$  where  $D$  is the Euclidean distance between  $p$  and  $q$ . As all points were already at most  $2h_{opt}$  away from their original background point in  $B$ , this implies that  $2D \sin(\theta/2) \leq 2\sqrt{2}h_{opt}$ . Thus,  $\|(x_q, y_q) - (x_{q'}, y_{q'})\|_2$  is at most  $2\sqrt{2}h_{opt}$ . Then by Lemma 1, as  $q$  is the furthest point from  $p$ , the rotation moves all points at most  $2\sqrt{2}h_{opt} + \sqrt{2}\pi h_{opt}/D$  with respect to  $h_2$  and at most  $4h_{opt} + \sqrt{2}\pi h_{opt}/D$  for  $h_1$ .

Since each point in the pattern set started out at most  $h_{opt}$  away from a point in the background set, we combine this with the translation and rotation movements to find that every point ends up at most  $(6 + \sqrt{2}\pi/D)h_{opt}$  away from a background point for  $h_1$  and at most  $(2 + \sqrt{2}(2 + \pi/D))h_{opt}$  away from a background point for  $h_2$ . As our algorithm checks this combination of  $T_v$  and  $R_{p,\theta}$ , our algorithm guarantees at least this solution. Note that we assume  $p'$  and  $q'$  are not the same point. However if this is the case, then



we know that  $D \leq 2h_{\text{opt}}$  thus when we translate  $p$  to  $p'$  every point is within  $(\sqrt{5} + 2\pi/D)h_{\text{opt}}$  of  $p'$ , which is a better approximation than the case where  $p' \neq q'$  under our assumption that  $D$  is large.  $\square$

**Theorem 4.** *Let  $h_{\text{opt}}$  be  $h_i(E(P), B)$  where  $E$  is the composition of functions in  $\mathcal{T} \cup \mathcal{R}$  that attains the minimum of  $h_i$ . The algorithm above runs in time  $O(A^8 n^2 m \log n)$  and produces an approximation to  $h_{\text{opt}}$  that is at most  $(1 + \epsilon)h_{\text{opt}}$  for both  $h_1$  and  $h_2$ .*

**Proof.** Let  $E$  be the composition of functions in  $\mathcal{T} \cup \mathcal{R}$  that attains the minimum of  $h(E(P), B)$ . Let  $P'$  be  $E(P)$ . Then every point  $q \in P'$  is at most  $h_{\text{opt}}$  from the closest background point in  $B$ . By running the base algorithm, we find  $h_{\text{appr}} \leq A \cdot h_{\text{opt}}$ , where  $A$  is the approximation ratio of the base algorithm. Now consider the point  $b' \in B$  which is the closest background to some pattern point  $p \in P$ . The square which encompasses  $G_{b'}$  has a side length of  $2h_{\text{appr}}$ . This guarantees that  $p$ , which is at most  $h_{\text{opt}}$  away from  $b'$ , lies within this square. As we saw from Lemma 3, this means that  $p$  is at most  $\frac{\epsilon h_{\text{appr}}}{A^2 - A}$  away from its nearest neighbor in  $G_{b'}$ . Thus, if a transformation defined by the nearest points in  $B$  would move our pattern points at most  $(A - 1)h_{\text{opt}}$  from their optimal position, then using the nearest points in  $G_{b'}$  to define our transformation will move our points at most  $(A - 1)\frac{\epsilon h_{\text{appr}}}{A^2 - A} = \frac{\epsilon h_{\text{appr}}}{A} \leq \epsilon h_{\text{opt}}$ . Thus, the modified algorithm gives a solution that is at most  $(1 + \epsilon)h_{\text{opt}}$ .  $\square$

**Theorem 5.** *Let  $h_{\text{opt}}$  be  $h_i(E(P), B)$  where  $E$  is the composition of functions in  $\mathcal{T} \cup \mathcal{R}$  that attains the minimum of  $h_i$ . The algorithm above runs in time  $O(nm \log n)$  and produces an approximation to  $h_{\text{opt}}$  that is at most  $(A_i + \epsilon)h_{\text{opt}}$  for  $h_i$ , where  $i = \{1, 2\}$ ,  $\epsilon > 0$  is a fixed constant,  $A_1 = 2 + \sqrt{2}D$ , and  $A_2 = 2 + D$ .*

**Proof.** The additional  $\epsilon$  terms come entirely from using approximate nearest neighbor queries, so it is sufficient to prove approximations which do not include the  $\epsilon$  term using exact nearest neighbor queries (defining the BBD tree so that it returns points that are  $(1 + \epsilon/A_i)$ -approximate nearest neighbors). Particularly, we will prove a bound of  $(2 + \sqrt{2}D)h_{\text{opt}}$  for  $h_1$  and a bound of  $(2 + D)h_{\text{opt}}$  for  $h_2$ .

Let  $E$  be the composition of functions in  $\mathcal{T} \cup \mathcal{R}$  that attains the minimum of  $h(E(P), B)$ . Let  $P'$  be  $E(P)$ . Then every point  $p \in P'$  is at most  $h_{\text{opt}}$  from the closest background point in  $B$ . That is, for all  $p$  in  $P'$ , there exists  $b$  in  $B$  such that  $\mu_i(p, b) \leq h_{\text{opt}}$ . Let  $p' \in B$  be the closest background point to the optimal position of  $p$  where  $p$  is the point we chose in the first step of the algorithm. Thus,

$$\mu_i(p, p') \leq h_{\text{opt}}.$$

Apply the translation  $T_v$  and rotation  $R_{p, \theta}$  on  $P'$  so that  $p$  coincides with  $p'$  and both points have the same orientation. It is easy to see that  $p$  has moved from its optimal position by exactly  $\mu_i(p, p') \leq h_{\text{opt}}$ . Using Lemma 7 and the fact that a rotation on  $P$  causes the orientation of each point in  $P$  to change by the same amount, we find that every point  $q \in P$  has moved at most  $\mu_i(p, p') + d$  from its original position, where  $d$  is the change in the position of  $q$  caused by the rotation.

We know that the angle rotated,  $\theta$ , must be less than  $h_{\text{opt}}$  and, without loss of generality, we assume  $0 \leq \theta \leq \pi$ . Therefore it is easily verifiable that  $\sin(\theta/2) \leq \theta/2$ . If  $D$  is the diameter of  $P$ , then regardless of our choice of  $p$ , each point in  $P$  is displaced at most  $2D \sin(\theta/2)$  by the rotation. Thus each point is displaced at most  $D\theta \leq Dh_{\text{opt}}$ .

Since each point in the pattern set started out at most  $h_{\text{opt}}$  away from a point in the background set, we combine this with the translation and rotation movements to find that every point ends up at most  $(2 + \sqrt{2}D)h_{\text{opt}}$  away from a background point for  $h_1$  and at most  $(2 + D)h_{\text{opt}}$  away from a background point for  $h_2$ . As our algorithm checks this combination of  $T_v$  and  $R_{p, \theta}$ , our algorithm guarantees at least this solution.  $\square$

**Theorem 6.** *Let  $h_{\text{opt}}$  be  $h_i(E(P), B)$  where  $E$  is the composition of functions in  $\mathcal{T} \cup \mathcal{R}$  that attains the minimum of  $h_i$ . The algorithm above runs in time  $O(A^6 nm \log n)$  and produces an approximation to  $h_{\text{opt}}$  that is at most  $(1 + \epsilon)h_{\text{opt}}$  for both  $h_1$  and  $h_2$ .*

**Proof.** Let  $E$  be the composition of functions in  $\mathcal{T} \cup \mathcal{R}$  that attains the minimum of  $h_i(E(P), B)$ . Let  $P'$  be  $E(P)$ . Then every point  $q \in P'$  is at most  $h_{\text{opt}}$  from the closest background point in  $B$ . By running the base algorithm, we find  $h_{\text{appr}} \leq A \cdot h_{\text{opt}}$  where  $A$  is the approximation ratio of the base algorithm. Now consider the point  $b' \in B$  which is the closest background to some pattern point  $p \in P$ . The square which encompasses  $G_{b'}$  has a side length of  $2h_{\text{appr}}$ . This guarantees that  $p$ , which is at most  $h_{\text{opt}}$  away from  $b'$ , lies within this square. As we saw from Lemma 3, this means that  $p$  is at most  $\frac{\epsilon h_{\text{appr}}}{2(A^2 - A)}$  away from its nearest neighbor  $g$  in  $G_{b'}$  with respect to the  $L1$ -norm, and at most  $\frac{\epsilon h_{\text{appr}}}{\sqrt{2}(A^2 - A)}$  with respect to the  $L2$ -norm. For this point,  $g$ , there are a number of points in  $C_g$  which are at the same position but with different orientation. For some point  $c$  in  $C_g$ , the orientation of point  $p$  is within an angle of at most  $\frac{h_{\text{appr}} \epsilon}{2(A^2 - A)}$  for  $h_1$  and at most  $\frac{h_{\text{appr}} \epsilon}{\sqrt{2}(A^2 - A)}$  for  $h_2$ . If we combine together the maximum difference in position between  $p$  and  $c$ , and the maximum difference in orientation between  $p$  and  $c$ , then we see that for both  $\mu_1$  and  $\mu_2$ , the distance between  $p$  and  $c$  is at most  $\frac{h_{\text{appr}} \epsilon}{A^2 - A}$ . Thus, if a transformation defined by the nearest point in  $B$  would

move our pattern points at most  $(A - 1)h_{\text{opt}}$  from their optimal position, then using the nearest point in  $C_g$  to define our transformation will move our points at most  $(A - 1)\frac{\epsilon h_{\text{appr}}}{A^2 - A} = \frac{\epsilon h_{\text{appr}}}{A} \leq \epsilon h_{\text{opt}}$ . Thus, the modified algorithm gives a solution that is at most  $(1 + \epsilon)h_{\text{opt}}$ .  $\square$

## B Translations Only

In this section, we present our base algorithm and approximation algorithm for approximately solving the oriented point-set pattern matching problem where we allow only translations. In this way, we present the basic template and data structures that we will also use for the more interesting case of translations and rotations ( $\mathcal{T} \cup \mathcal{R}$ ).

Our methods for handling translations, rotations, and scaling is an adaptation of our methods for  $\mathcal{T} \cup \mathcal{R}$ ; hence, we give our methods for  $\mathcal{T} \cup \mathcal{R} \cup \mathcal{S}$  in an appendix.

Given two subsets of  $O$ ,  $P$  and  $B$ , with  $|P| = m$  and  $|B| = n$ , our goal here is to minimize  $h_i(E(P), B)$  where  $E$  is a transformation function in  $\mathcal{T}$ .

**Base Algorithm Under Translation Only.** Our base pin-and-query algorithm is as follows.

---

### Algorithm BaseTranslate( $P, B$ ):

Choose some  $p \in P$  arbitrarily.

**for** every  $b \in B$  **do**

*Pin step:* Apply the translation,  $T_v \in \mathcal{T}$ , that takes  $p$  to  $b$ .

**for** every  $q \in T_v(P)$  **do**

*Query step:* Find a nearest-neighbor of  $q$  in  $B$  using the  $\mu_i$  metric, and update a candidate Hausdorff distance for  $T_v$  accordingly.

**end for**

**return** the smallest candidate Hausdorff distance found as the smallest distance,  $h_i(T_v(P), B)$ .

**end for**

---

This algorithm uses a similar approach to an algorithm of Goodrich *et al.* [7], but it is, of course, different in how it computes nearest neighbors, since we must use an oriented distance metric rather than unoriented distance metric. One additional difference is that rather than find an exact nearest neighbor, as described above, we instead find an *approximate* nearest neighbor of each point,  $q$ , since we are ultimately designing an approximation algorithm anyway. This allows us to achieve a faster running time.

In particular, in the query step of the algorithm, for any point  $q \in T_v(P)$ , we find a neighbor,  $b \in B$ , whose distance to  $q$  is at most a  $(1 + \epsilon)$ -factor more than the distance from  $q$  to its true nearest neighbor. To achieve

this result, we adapt the balanced box-decomposition (BBD) tree of Arya *et al.* [2] to oriented point sets. Specifically, we insert into the BBD tree the following set of  $3n$  points in  $\mathbb{R}^3$ :

$$\begin{aligned} &\{b, b', b'' \mid b \in B, \\ &\quad b' = (x_p, y_b, a_b + 2\pi), \\ &\quad b'' = (x_b, y_b, a_b - 2\pi)\}. \end{aligned}$$

This takes  $O(n \log n)$  preprocessing and it allows the BBD tree to respond to nearest neighbor queries with an approximation factor of  $(1 + \epsilon)$  while using the  $L_1$ -norm or  $L_2$ -norm as the distance metric, since the BBD is effective as an approximate nearest-neighbor data structure for these metrics. Indeed, this is the main reason why we are using these norms as our concrete examples of  $\mu_i$  metrics. Each query takes  $O(\log n)$  time, so computing a candidate Hausdorff distance for a given transformation takes  $O(m \log n)$  time. Therefore, since we perform the pin step over  $n$  translations, the algorithm overall takes time  $O(nm \log n)$ . To analyze the correctness of this algorithm, we start with a simple observation that if we translate a point using a vector whose  $L_i$ -norm is  $d$ , then the distance between the translated point and its old position is  $d$ .

**Lemma 7** *Let  $(x, y, a)$  be an element of  $O$ . Consider a transformation  $T_v$  in  $\mathcal{T}$  where  $v$  is a translation vector. Let  $T_v(x, y, a) = (x', y', a)$ . If the  $L_i$ -norm of  $v$  is  $\|v\|_i = d$ , then  $\mu_i((x, y, a), (x', y', a)) = d$ , where  $i \in \{1, 2\}$ .*

**Proof.** First consider the case where  $i = 1$ . By definition of  $\mu_1$  and  $T_v$ ,

$$\begin{aligned} &\mu_1((x, y, a), (x', y', a)) \\ &= |x - x'| + |y - y'| + \min(|a - a|, 2\pi - |a - a|) \\ &= |v_x| + |v_y| \\ &= d. \end{aligned}$$

Now consider the case where  $i = 2$ :

$$\begin{aligned} &\mu_2((x, y, a), (x', y', a)) \\ &= \sqrt{(x - x')^2 + (y - y')^2 + \min(a - a, 2\pi - |a - a|)^2} \\ &= \sqrt{v_x^2 + v_y^2} \\ &= d. \end{aligned}$$

Thus, for either case, the lemma holds.  $\square$

**Theorem 8** *Let  $h_{\text{opt}}$  be  $h_i(E(P), B)$  where  $E$  is the translation in  $\mathcal{T}$  that attains the minimum of  $h_i$ . The algorithm above runs in time  $O(nm \log n)$  and produces an approximation to  $h_{\text{opt}}$  that is at most  $(2 + \epsilon)h_{\text{opt}}$ , for either  $h_1$  and  $h_2$ , for any fixed constant  $\epsilon > 0$ .*

**Proof.** The  $\epsilon$  term comes from the approximate nearest neighbor queries using the BBD tree, and expanding  $B$  to a set of size  $3n$  by making a copy of each point in  $B$  to have an angle that is  $2\pi$  greater and less than its original value. So it is sufficient to prove a 2-approximation using exact nearest neighbor queries (while building the BBD tree to return  $(1 + \epsilon/2)$ -approximate nearest neighbors). We prove this claim by a type of “backwards” analysis. Let  $E$  be a translation in  $\mathcal{T}$  that attains the minimum of  $h_i(E(P), B)$ , and let  $P' = E(P)$ . Then every point  $q \in P'$  is at most  $h_{\text{opt}}$  from its closest background point in  $B$ . That is, for all  $q$  in  $P'$ , there exists  $b$  in  $B$  such that  $\mu_i(q, b) \leq h_{\text{opt}}$ . Let  $b' \in B$  be the closest background point to the optimal position of  $p$ , where  $p$  is the point we choose in the first step of the algorithm. Thus,

$$\|(x_p, y_p) - (x_{b'}, y_{b'})\|_i \leq \mu_i(p, b') \leq h_{\text{opt}}.$$

Apply the translation  $T_v$  on  $P'$  so that  $p$  coincides with  $b'$ , which is equivalent to moving every point’s position by  $\|(x_p, y_p) - (x_{b'}, y_{b'})\|_i$ . Hence, by Lemma 7, all points have moved at most  $h_{\text{opt}}$ .

As all points in the pattern started at most  $h_{\text{opt}}$  away from a point in the background set and the translation  $T_v$  moves all points at most  $h_{\text{opt}}$ , all points in  $T_v(P')$  are at most  $2h_{\text{opt}}$  from a point in the background set  $B$ . Since our algorithm checks  $T_v$  as one of the translations in the second step of the algorithm, it will find a translation that is at least as good as  $T_v$ . Therefore, our algorithm guarantees an approximation of at most  $2h_{\text{opt}}$ , for either  $h_1$  and  $h_2$ .  $\square$

**A  $(1 + \epsilon)$ -Approximation Algorithm Under Translations Only.** In this subsection, we utilize the algorithm from Appendix B to achieve a  $(1 + \epsilon)$ -approximation when we only allow translations. Suppose, then, that we are given two subsets of  $O$ ,  $P$  and  $B$ , with  $|P| = m$  and  $|B| = n$ , and our goal is to minimize  $h_i(E(P), B)$  over translations  $E$  in  $\mathcal{T}$ . Our algorithm is as follows:

1. Run the base algorithm,  $\text{BaseTranslate}(P, B)$ , from Appendix B, to obtain an approximation,  $h_{\text{apr}} \leq A \cdot h_{\text{opt}}$ .
2. For every  $b \in B$ , generate the point set

$$G_b = G \left( b, \frac{\epsilon h_{\text{apr}}}{A^2 - A}, \left\lceil \frac{A^2 - A}{\epsilon} \right\rceil \right)$$

for  $h_1$  or

$$G_b = G \left( b, \frac{\epsilon \sqrt{2} h_{\text{apr}}}{A^2 - A}, \left\lceil \frac{A^2 - A}{\epsilon \sqrt{2}} \right\rceil \right)$$

for  $h_2$ . Let  $B'$  denote this expanded set of background points, i.e.,  $B' = \bigcup_{b \in B} G_b$ , and note that if  $A$  is a constant, then  $|B'|$  is  $O(n)$ .

3. Return the result from calling  $\text{BaseTranslate}(P, B')$ , but restricting the query step to finding nearest neighbors in  $B$  rather than in  $B'$ .

Intuitively, this algorithm uses the base algorithm to give us a first approximation for the optimal solution. We then use this approximation to generate a larger set of points from which to derive transformations to test. We then use this point set again in the base algorithm when deciding which transformations to iterate over, while still using  $B$  to compute nearest neighbors. The first step of this algorithm runs in time  $O(nm \log n)$ , as we showed. The second step takes time proportional to the number of points which have to be generated, which is determined by  $n$ , our choice of the constant  $\epsilon$ , and the approximation ratio of our base algorithm  $A$ , which we proved is the constant  $2 + \epsilon$ . The time needed to complete the second step is  $O(n)$ . In the last step, we essentially call the base algorithm again on sets of size  $m$  and  $O(n)$ , respectively; hence, this step requires  $O(nm \log n)$  time.

**Theorem 9** *Let  $h_{\text{opt}}$  be  $h_i(E(P), B)$  where  $E$  is the translation in  $\mathcal{T}$  that attains the minimum of  $h_i$ , for  $i \in \{1, 2\}$ . The algorithm above runs in time  $O(nm \log n)$  and produces an approximation to  $h_{\text{opt}}$  that is at most  $(1 + \epsilon)h_{\text{opt}}$ , for either  $h_1$  and  $h_2$ .*

**Proof.** Let  $E$  be the translation in  $\mathcal{T}$  that attains the minimum of  $h_i(E(P), B)$ . Let  $P'$  be  $E(P)$ . Then every point  $q \in P'$  is at most  $h_{\text{opt}}$  from the closest background point in  $B$ . By running the base algorithm the first time, we find  $h_{\text{apr}} \leq A \cdot h_{\text{opt}}$ , where  $A$  is the approximation ratio of the base algorithm. Now consider the point,  $b' \in B$ , that is the closest background to some pattern point  $p \in P$ . The square which encompasses  $G_{b'}$  has a side length of  $2h_{\text{apr}}$ . This guarantees that  $p$ , which is at most  $h_{\text{opt}}$  away from  $b'$ , lies within this square. As we saw from Lemma 3, this means that  $p$  is at most  $\frac{\epsilon h_{\text{apr}}}{A^2 - A}$  away from its nearest neighbor in  $G_{b'}$ . Thus, if a transformation defined by the nearest point in  $B$  would move our pattern points at most  $(A - 1)h_{\text{opt}}$  from their optimal position, then using the nearest point in  $G_{b'}$  to define our transformation will move our points at most  $(A - 1) \frac{\epsilon h_{\text{apr}}}{A^2 - A} = \frac{\epsilon h_{\text{apr}}}{A} \leq \epsilon h_{\text{opt}}$ . Therefore, our algorithm gives a solution that is at most  $(1 + \epsilon)h_{\text{opt}}$  from optimal.  $\square$

## C Translation, Rotation, and Scaling

In this appendix, we show how to adapt our algorithm for translations and rotations so that it works for translations, rotations, and scaling. The running times are the same as for the translation-and-rotation cases.

**Base Algorithm Under Translation, Rotation and Scaling with Large Diameter.** In this section we present an algorithm for solving the approximate oriented point-set pattern matching problem where we allow translations, rotations and scaling. This algorithm is an extension of the algorithm from Section 3 and similarly provides a good approximation ratio when the diameter of our pattern set is large. Given two subsets  $P$  and  $B$  of  $O$ , with  $|P| = m$  and  $|B| = n$ , we wish to minimize  $h_i(E(P), B)$  over all compositions  $E$  of one or more functions in  $\mathcal{T} \cup \mathcal{R} \cup \mathcal{S}$ . We perform the following algorithm:

---

**Algorithm** BaseTranslateRotateScaleLarge( $P, B$ ):

Find  $p$  and  $q$  in  $P$  having the maximum value of  $\|(x_p, y_p) - (x_q, y_q)\|_2$ .

**for** every pair of points  $b, b' \in B$  **do**

*Pin step:* Apply the translation,  $T_v \in \mathcal{T}$ , that takes  $p$  to  $b$ , and apply the rotation,  $R_{p,\theta}$ , that makes  $p, b'$ , and  $q$  collinear. Then apply the scaling,  $S_{p,s}$ , that makes  $q$  and  $b'$  share the same position.

Let  $P'$  denote the transformed pattern set,  $P$ .

**for** every  $q \in P'$  **do**

*Query step:* Find a nearest-neighbor of  $q$  in  $B$  using the  $\mu_i$  metric, and update a candidate Hausdorff distance accordingly.

**end for**

**return** the smallest candidate Hausdorff distance found as the smallest Hausdorff distance,  $h_i(S_{p,s}(R_{p,\theta}(T_v(P))), B)$ .

**end for**

---

This algorithm extends the algorithm presented in Section 3 so that after translating and rotating, we also scale the point set such that, after scaling,  $p$  and  $b$  have the same  $x$  and  $y$  coordinates, and  $q$  and  $b'$  have the same  $x$  and  $y$  coordinates. As with the algorithm presented in Section 3, this algorithm runs in  $O(n^2 m \log n)$  time.

**Theorem 10** Let  $h_{opt}$  be  $h_i(E(P), B)$  where  $E$  is the composition of functions in  $\mathcal{T} \cup \mathcal{R} \cup \mathcal{S}$  that attains the minimum of  $h_i$ . The algorithm above runs in time  $O(n^2 m \log n)$  and produces an approximation to  $h_{opt}$  that is at most  $(6 + \sqrt{2}(2 + \pi/D) + \epsilon)h_{opt}$  for  $h_1$  and at most  $(4 + \sqrt{2}(2 + \pi/D) + \epsilon)h_{opt}$  for  $h_2$ .

**Proof.** The additional  $\epsilon$  terms come entirely from using approximate nearest neighbor queries, so it is sufficient to prove approximations which do not include the  $\epsilon$  term using exact nearest neighbor queries. Particularly, we will prove a bound of  $(6 + \sqrt{2}(2 + \pi/D))h_{opt}$  for  $h_1$  and a bound of  $(4 + \sqrt{2}(2 + \pi/D))h_{opt}$  for  $h_2$ .

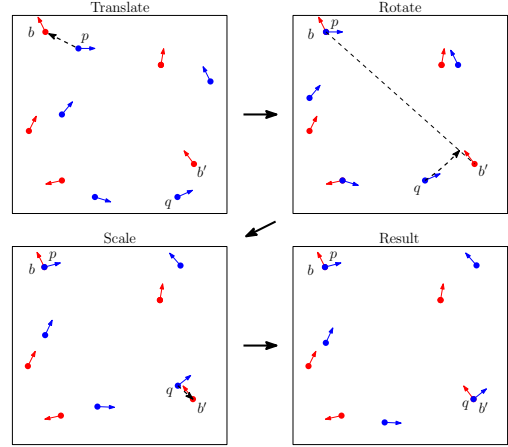


Figure 7: Illustration of the translation, rotation and scaling steps of the base approximation algorithm for translation, rotation and scaling in  $O$  when diameter is large.

Let  $E$  be the composition of functions in  $\mathcal{T} \cup \mathcal{R} \cup \mathcal{S}$  that attains the minimum of  $h_i(E(P), B)$ . Let  $P'$  be  $E(P)$ . Because this algorithm is only an extension of the algorithm in Section 3 we can follow the same logic as the proof of Theorem 2 to see that after the translation and rotation steps, each point  $p \in P'$  is at most  $Ah_{opt}$  away from a background point  $b \in B$  where  $A = 6 + \sqrt{2}\pi/D$  for  $h_1$  and  $A = 2 + \sqrt{2}(2 + \pi/D)$  for  $h_2$ . Now we need only look at how much scaling increases the distance our points have moved.

If  $p, q \in P'$  are our diametric points after translation and rotation, and  $p', q' \in B$  are the closest background points to the optimal position of  $p$  and  $q$  respectively, then let us define the point  $q_t$  as the position of  $q$  after translation, but prior to the rotation step. Now it is important to see that the points  $q, q'$  and  $q_t$  are three vertices of an isosceles trapezoid where the line segment  $q_t q'$  is a diagonal of the trapezoid and the line segment  $qq_t$  is a base of the trapezoid. This situation is depicted in Figure 8. The length of the line segment  $qq'$  is equal to the distance that  $q$  will move when we scale  $P'$  so that  $q$  and  $q'$  share the same position. Because  $qq'$  is a leg of the trapezoid, the length of that leg can be no more than the length of the diagonal  $q_t q'$ . In the proof of Theorem 2, we showed that  $q_t$  is at most  $2h_{opt}$  away from  $q'$  so this implies that the distance  $q$  moves from scaling is at most  $2h_{opt}$ .

Point  $q$  is the farthest point away from the point  $p$  that is the center for scaling. Thus, no point moved farther as a result of the scaling than  $q$  did, with respect to  $\mu_2$ . For  $\mu_1$  it is possible that, if  $q$  moved a distance  $d$ , another point could have moved up to a distance  $\sqrt{2}d$ . Thus, we find that after scaling, any point in  $P'$  is at most  $(A + 2\sqrt{2})h_{opt}$  and  $(A + 2)h_{opt}$  from its nearest background point for  $\mu_1$  and  $\mu_2$  respectively. Because

this is a transformation that the algorithm checks, we are guaranteed at least this solution. Note that we assume  $p'$  and  $q'$  are not the same point. However if this is the case, then we know that  $D \leq 2h_{\text{opt}}$  thus when we translate  $p$  to  $p'$  and scale  $q$  down to  $p'$  every point is within  $(2\pi/D)h_{\text{opt}}$  of  $p'$ , which is a better approximation than the case where  $p' \neq q'$  under our assumption that  $D$  is large.  $\square$

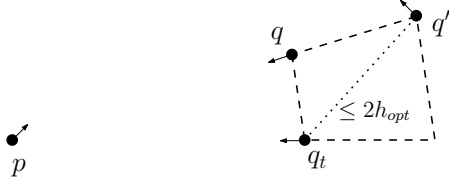


Figure 8: Illustration of the points  $q$ ,  $q'$ , and  $q_t$  forming three of the corners of an isosceles trapezoid, as described in the proof of Theorem 10

**A  $(1+\epsilon)$ -Approximation Algorithm Under Translation, Rotation and Scaling with Large Diameter.** In this subsection, we utilize the algorithm from Appendix C to achieve a  $(1 + \epsilon)$ -approximation ratio when we allow translations, rotations, and scaling. Again, given two subsets of  $O$ ,  $P$  and  $B$ , with  $|P| = m$  and  $|B| = n$ , our goal is to minimize  $h_i(E(P), B)$  over all compositions  $E$  of one or more functions in  $\mathcal{T} \cup \mathcal{R} \cup \mathcal{S}$ . We perform the following steps.

1. Run `BaseTranslateRotateScaleLarge( $P, B$ )`, from Appendix C, to obtain an approximation  $h_{\text{apr}} \leq A \cdot h_{\text{opt}}$ .
2. For every  $b \in B$ , generate the point set  $G_b = G(b, \frac{h_{\text{apr}}\epsilon}{A^2-A}, \lceil \frac{A^2-A}{\epsilon} \rceil)$  for  $h_1$  or  $G_b = G(b, \frac{\sqrt{2}h_{\text{apr}}\epsilon}{A^2-A}, \lceil \frac{A^2-A}{\sqrt{2}\epsilon} \rceil)$  for  $h_2$ . Let  $B'$  denote the resulting set.
3. Run `BaseTranslateRotateScaleLarge( $P, B'$ )`, from Appendix C, but use the set  $B$  for the nearest-neighbor queries.

This algorithm uses the base algorithm to give us an indication of what the optimal solution might be. We then use this approximation to generate a larger set of points from which to derive transformations to test. We next use this point set in the base algorithm when deciding which transformations to iterate over, while still using  $B$  to compute nearest neighbors. The running time is  $O(A^8 n^2 m \log n)$ , which is  $O(n^2 m \log n)$  for constant  $A$ .

**Theorem 11** *Let  $h_{\text{opt}}$  be  $h_i(E(P), B)$  where  $E$  is the composition of functions in  $\mathcal{T} \cup \mathcal{R} \cup \mathcal{S}$  that attains the*

*minimum of  $h_i$ . The algorithm above runs in time  $O(A^8 n^2 m \log n)$  and produces an approximation to  $h_{\text{opt}}$  that is at most  $(1 + \epsilon)h_{\text{opt}}$  for both  $h_1$  and  $h_2$ .*

**Proof.** Let  $E$  be the composition of functions in  $\mathcal{T} \cup \mathcal{R} \cup \mathcal{S}$  that attains the minimum of  $h_i(E(P), B)$ . Let  $P'$  be  $E(P)$ . Then every point  $q \in P'$  is at most  $h_{\text{opt}}$  from the closest background point in  $B$ . By running the base algorithm, we find  $h_{\text{apr}} \leq Ah_{\text{opt}}$  where  $A$  is the approximation ratio of the base algorithm. Now consider the point  $b' \in B$  which is the closest background to some pattern point  $p \in P$ . The square which encompasses  $G_{b'}$  has a side length of  $2h_{\text{apr}}$ . This guarantees that  $p$ , which is at most  $h_{\text{opt}}$  away from  $b'$ , lies within this square. As we saw from Lemma 3, this means that  $p$  is at most  $\frac{\epsilon h_{\text{apr}}}{A^2 - A}$  away from its nearest neighbor in  $G_{b'}$ . Thus, if a transformation defined by the nearest points in  $B$  would move our pattern points at most  $(A - 1)h_{\text{opt}}$  from their optimal position, then using the nearest points in  $G_{b'}$  to define our transformation will move our points at most

$$(A - 1) \frac{\epsilon h_{\text{apr}}}{A^2 - A} = \frac{\epsilon h_{\text{apr}}}{A} \leq \epsilon h_{\text{opt}}.$$

Thus, the modified algorithm gives a solution that is at most  $(1 + \epsilon)h_{\text{opt}}$ .  $\square$

**Base Algorithm Under Translation, Rotation and Scaling with Small Diameter.** In this subsection, we present an alternative algorithm for solving the approximate oriented point-set pattern matching problem where we allow translations, rotations and scaling. This algorithm is an extension of the algorithm from Section 3 and similarly provides a good approximation ratio when the diameter of our pattern set is small. Once again, given two subsets of  $O$ ,  $P$  and  $B$ , with  $|P| = m$  and  $|B| = n$ , we wish to minimize  $h_i(E(P), B)$  over all compositions  $E$  of one or more functions in  $\mathcal{T} \cup \mathcal{R}$ . We perform the following algorithm:

---

**Algorithm** `BaseTranslateRotateSmall( $P, B$ )`:

Find  $p$  and  $q$  in  $P$  having the maximum value of  $\|(x_p, y_p) - (x_q, y_q)\|_2$ .

**for** every point  $b \in B$  **do**

    1<sup>st</sup> *Pin*: Apply the translation,  $T_v \in \mathcal{T}$ , that takes  $p$  to  $b$ , and then apply the rotation,  $R_{p, \theta}$ , that makes  $p, b$  have the same orientation.

    Let  $P'$  denote the transformed pattern set,  $P$ .

**for** each point  $p$  in  $P'$  and each  $b' \in B$  **do**

        2<sup>nd</sup> *pin*: Apply the scaling,  $S_{p, s}$ , so that  $\|(x_p, y_p) - (x_q, y_q)\|_2 = \|(x_b, y_b) - (x_{b'}, y_{b'})\|_2$

        Let  $P''$  denote the transformed pattern set.

**for** every  $q \in P''$  **do**

*Query step:* Find a nearest-neighbor of  $q$  in  $B$  using the  $\mu_i$  metric, and update a candidate Hausdorff distance accordingly.

**end for**

**end for**

**return** the smallest candidate Hausdorff distance found as the smallest Hausdorff distance,

$h_i(S_{p,s}(R_{p,\theta}(T_v(P))), B)$ .

**end for**

This algorithm extends the algorithm from Section 3 by scaling the point set so that  $p$ ,  $q$ , and  $b'$  form the vertices of an isosceles triangle. This requires a factor of  $n$  more transformations to be computed. Thus, the running time of this algorithm is  $O(n^2 m \log n)$ .

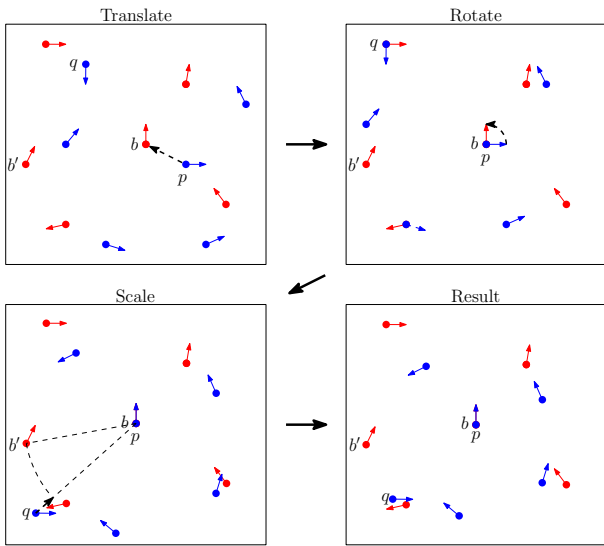


Figure 9: Illustration of the translation, rotation and scaling steps of the base approximation algorithm for translation, rotation and scaling in  $O$  when diameter is small.

**Theorem 12** Let  $h_{opt}$  be  $h_i(E(P), B)$  where  $E$  is the composition of functions in  $\mathcal{T} \cup \mathcal{R} \cup \mathcal{S}$  that attains the minimum of  $h_i$ . The algorithm above runs in time  $O(n^2 m \log n)$  and produces an approximation to  $h_{opt}$  that is at most  $((2 + 2\sqrt{2})(1 + D) + \epsilon)h_{opt}$  for  $h_1$  and at most  $(4 + 2D + \epsilon)h_{opt}$  for  $h_2$ .

**Proof.** The additional  $\epsilon$  terms come entirely from using approximate nearest neighbor queries, so it is sufficient to prove approximations which do not include the  $\epsilon$  term using exact nearest neighbor queries. Particularly, we will prove a bound of  $(6 + \sqrt{2}(2 + \pi/D))h_{opt}$  for  $h_1$  and a bound of  $(4 + \sqrt{2}(2 + \pi/D))h_{opt}$  for  $h_2$ .

Let  $E$  be the composition of functions in  $\mathcal{T} \cup \mathcal{R} \cup \mathcal{S}$  that attains the minimum of  $h_i(E(P), B)$ . Let  $P'$  be  $E(P)$ . Because this algorithm is only an extension

of the algorithm in Section 3 we can follow the same logic as the proof of Theorem 5 to see that after the translation and rotation steps, each point  $p \in P'$  is at most  $Ah_{opt}$  away from a background point  $b \in B$  where  $A = 2 + \sqrt{2}D$  for  $h_1$  and  $A = 2 + D$  for  $h_2$ . Now we need only look at how much scaling increases the distance our points have moved.

If  $p, q \in P'$  are our diametric points after translation and rotation, and  $p', q' \in B$  are the closest background points to the optimal position of  $p$  and  $q$  respectively, then let us define the point  $q_s$  as the position of  $q$  after scaling. The points  $q, q'$  and  $q_s$  are three vertices of an isosceles trapezoid where the line segment  $qq'$  is a diagonal of the trapezoid and the line segment  $q_sq'$  is a base of the trapezoid. The length of the line segment  $qq_s$  is equal to the distance that  $q$  will move when we scale  $P'$ . Because  $qq_s$  is a leg of the trapezoid, the length of that leg can be no more than the length of the diagonal  $qq'$ . In the proof of Theorem 5, we showed that  $q$  is at most  $Ah_{opt}$  away from  $q'$  so this implies that the distance  $q$  moves from scaling is at most  $Ah_{opt}$ .

Point  $q$  is the farthest point away from the point  $p$  which is the center of our scaling. Thus, no point moves farther as a result of the scaling than  $q$  does, with respect to  $\mu_2$ . For  $\mu_1$  it is possible that, if  $q$  moved a distance  $d$ , another point could have moved up to a distance  $\sqrt{2}d$ . Thus we find that after scaling, any point in  $P'$  is at most  $(1 + \sqrt{2})Ah_{opt}$  and  $2Ah_{opt}$  from its nearest background point for  $\mu_1$  and  $\mu_2$  respectively. Because this is a transformation that the algorithm checks, we are guaranteed at least this solution.  $\square$

### A $(1+\epsilon)$ -Approximation Algorithm Under Translation, Rotation and Scaling with Small Diameter.

In this subsection, we utilize the algorithm from Appendix C to achieve a  $(1 + \epsilon)$ -approximation ratio when we allow translations, rotations, and scalings. Again, given two subsets of  $O$ ,  $P$  and  $B$ , with  $|P| = m$  and  $|B| = n$ , our goal is to minimize  $h_i(E(P), B)$  over all compositions  $E$  of one or more functions in  $\mathcal{T} \cup \mathcal{R} \cup \mathcal{S}$ . We perform the following steps.

1. Run `BaseTranslateRotateScaleSmall( $P, B$ )`, from Appendix C to obtain an approximation  $h_{appr} \leq A \cdot h_{opt}$ .
2. For every  $b \in B$ , generate the point set  $G_b = G(b, \frac{h_{appr}\epsilon}{2(A^2-A)}, \lceil \frac{2(A^2-A)}{\epsilon} \rceil)$  for  $h_1$  or  $G_b = G(b, \frac{h_{appr}\epsilon}{A^2-A}, \lceil \frac{A^2-A}{\epsilon} \rceil)$  for  $h_2$ . Let  $B' = \bigcup_{b \in B} G_b$  denote the resulting set of points.
3. For every  $b' \in B'$ , generate the point set  $C_{b'} = C(b', \frac{2(A^2-A)}{\pi h_{appr}\epsilon})$  for  $h_1$  or  $C_{b'} = C(b', \frac{\sqrt{2}(A^2-A)}{\pi h_{appr}\epsilon})$  for  $h_2$ . Let  $B''$  denote the resulting set of points.

4. Run `BaseTranslateRotateScaleSmall( $P, B''$ )`, but use the points in  $B$  for nearest-neighbor queries.

This algorithm uses the base algorithm to give us an indication of what the optimal solution might be. We use this approximation to generate a larger set of points from which to derive transformations to test, but this time we also generate a number of different orientations for those points as well. We then use this point set in the base algorithm when deciding which transformations to iterate over, while still using  $B$  to compute nearest neighbors. The running time of this algorithm is  $O(A^{12}n^2m \log n)$ .

**Theorem 13** *Let  $h_{opt}$  be  $h_i(E(P), B)$  where  $E$  is the composition of functions in  $\mathcal{T} \cup \mathcal{R}$  that attains the minimum of  $h_i$ . The algorithm above runs in time  $O(A^{12}n^2m \log n)$  and produces an approximation to  $h_{opt}$  that is at most  $(1 + \epsilon)h_{opt}$  for both  $h_1$  and  $h_2$ .*

**Proof.** Let  $E$  be the composition of functions in  $\mathcal{T} \cup \mathcal{R} \cup \mathcal{S}$  that attains the minimum of  $h_i(E(P), B)$ . Let  $P'$  be  $E(P)$ . Then every point  $q \in P'$  is at most  $h_{opt}$  from the closest background point in  $B$ . By running the base algorithm, we find  $h_{appr} \leq Ah_{opt}$  where  $A$  is the approximation ratio of the base algorithm. Now consider the point  $b' \in B$  which is the closest background to some pattern point  $p \in P$ . The square which encompasses  $G_{b'}$  has a side length of  $2h_{appr}$ . This guarantees that  $p$ , which is at most  $h_{opt}$  away from  $b'$ , lies within this square. As we saw from Lemma 3, this means that  $p$  is at most  $\frac{\epsilon h_{appr}}{2(A^2 - A)}$  away from its nearest neighbor  $g$  in  $G_{b'}$  with respect to the  $L1$ -norm, and at most  $\frac{\epsilon h_{appr}}{\sqrt{2}(A^2 - A)}$  with respect to the  $L2$ -norm. For this point  $g$ , there are a number of points in  $C_g$  which are at the same position but with different orientation. For some point  $c$  in  $C_g$ , the orientation of point  $p$  is within an angle of at most  $\frac{h_{appr}\epsilon}{2(A^2 - A)}$  for  $h_1$  and at most  $\frac{h_{appr}\epsilon}{\sqrt{2}(A^2 - A)}$  for  $h_2$ . If we combine together the maximum difference in position between  $p$  and  $c$ , and the maximum difference in orientation between  $p$  and  $c$ , then we see that for both  $\mu_1$  and  $\mu_2$ , the distance between  $p$  and  $c$  is at most  $\frac{h_{appr}\epsilon}{A^2 - A}$ . As we explain at the beginning of this section, if a transformation defined by the nearest points in  $B$  would move our pattern points at most  $(A - 1)h_{opt}$  from their optimal position, then using the nearest points in  $C_g$  to define our transformation will move our points at most  $(A - 1)\frac{\epsilon h_{appr}}{A^2 - A} = \frac{\epsilon h_{appr}}{A} \leq \epsilon h_{opt}$ . Thus the modified algorithm gives a solution that is at most  $(1 + \epsilon)h_{opt}$ .  $\square$

As with our methods for translation and rotation, we can compute in advance whether we should run our algorithm for large diameter point sets or our algorithm for small diameter point sets. For  $h_1$ , we compare the expressions  $6 + \sqrt{2}(2 + \pi/D)$  and  $(2 + 2\sqrt{2})(1 + D)$ , and we find that the two expressions are equal at  $D^* \approx 1.46$ .

For  $h_2$ , we compare  $4 + \sqrt{2}(2 + \pi/D)$  and  $4 + 2D$  to find that they are equal at  $D^* \approx 2.36$ . Using  $D^*$  as the deciding value allows us to then find a transformation in  $\mathcal{T} \cup \mathcal{R} \cup \mathcal{S}$  that achieves a  $(1 + \epsilon)$ -approximation, for any constant  $\epsilon > 0$ , in  $O(n^2m \log n)$  time.

## D Experiments

In reporting the results of our experiments, we use the following labels for the algorithms:

- *GR*: the non-oriented translation and rotation algorithm from Goodrich *et al.* [7],
- *LD <sub>$h_1/h_2$</sub>* : the base version of the large diameter algorithm using either the  $h_1$  or  $h_2$  distance metric,
- *SD <sub>$h_1/h_2$</sub>* : the base version of the small diameter algorithm using either the  $h_1$  or  $h_2$  distance metric.

These algorithms were implemented in C++ (g++ version 4.8.5) and run on a Quad-core Intel Xeon 3.0GHz CPU E5450 with 32GB of RAM on 64-bit CentOS Linux 6.6.

**Accuracy Comparison.** We tested the ability of each algorithm to identify the original point set after it had been slightly perturbed. From set of randomly generated oriented background point sets, one was selected and a random subset of the points in the set were shifted and rotated by a small amount. Each algorithm was used to match this modified pattern against each of the background point sets and it was considered a success if the background set from which the pattern was derived had the smallest distance (as determined by each algorithm's distance metric). Figure 10 shows the results of this experiment under two variables: the number of background sets from which the algorithms could choose, and the size of the background sets. Each data point is the percentage of successes across 1000 different pattern sets.

In every case, the oriented algorithms are more successful at identifying the origin of the pattern than GR. LD was also more successful for each distance metric than SD.

**Performance Comparison.** We also compared the performance of the LD and SD algorithms against GR as we increased the pattern size and the background size. The most significant impact of increasing the background size is that the number of nearest neighbor queries increase, and thus the performance in this case is dictated by quality of the nearest neighbor data structure used. Therefore in Figure 11 we use the number of nearest neighbor queries as the basis for comparing performance. As the FD and GR algorithms only differ in how the nearest neighbor is calculated,

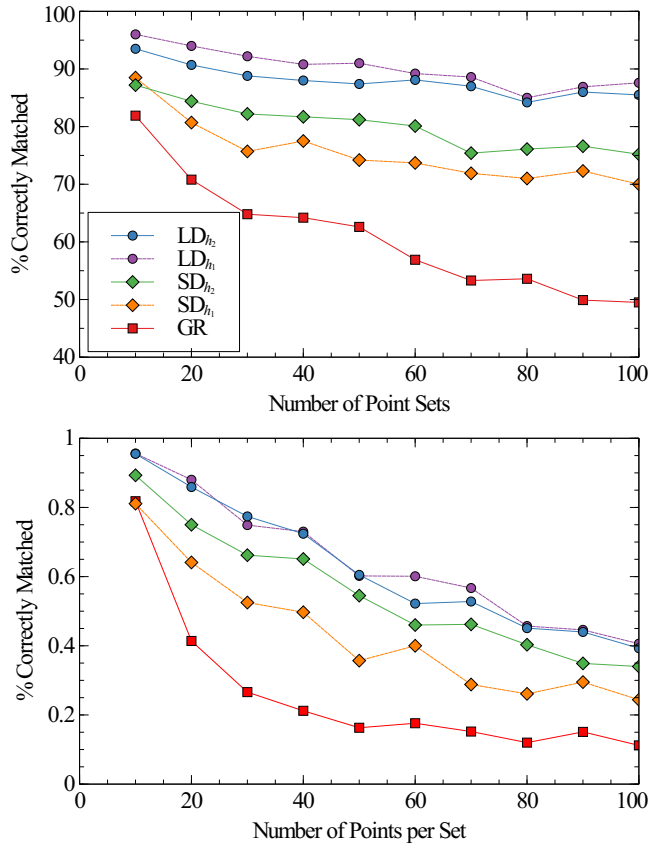


Figure 10: Results of Accuracy Comparison

they both perform the same number of queries while the SD algorithm performs significantly fewer nearest neighbor queries.

For pattern size, we compared running time and the results are shown in Figure 12. In this case, LD is slower than GR, while SD is significantly faster than either of the others.

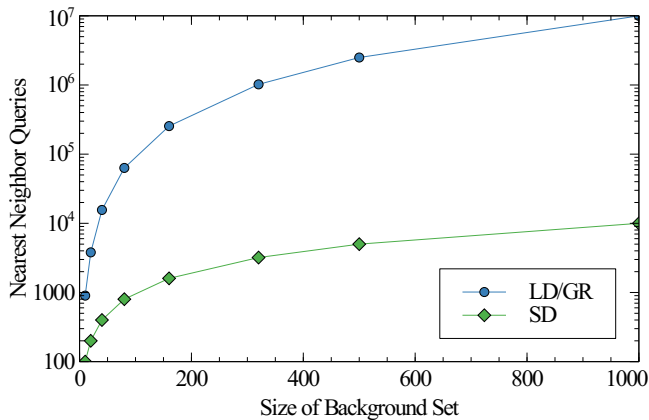


Figure 11: Comparison of nearest neighbor queries as function of background size

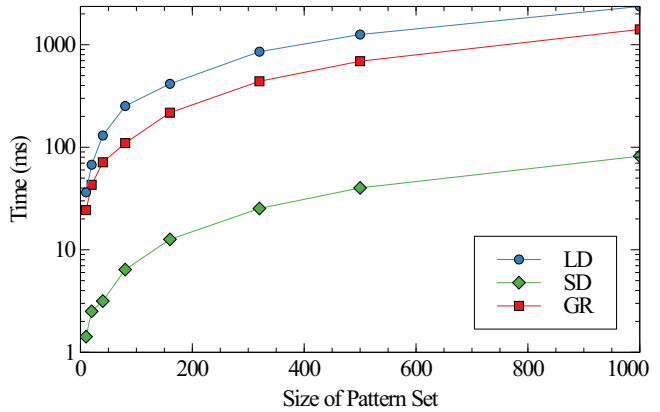


Figure 12: Comparison of running time as a function of pattern size