

C-Planarity Testing of Embedded Clustered Graphs with Bounded Dual Carving-Width

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Abstract

For a *clustered graph*, i.e., a graph whose vertex set is recursively partitioned into clusters, the C-PLANARITY TESTING problem asks whether it is possible to find a planar embedding of the graph and a representation of each cluster as a region homeomorphic to a closed disk such that **1.** the subgraph induced by each cluster is drawn in the interior of the corresponding disk, **2.** each edge intersects any disk at most once, and **3.** the nesting between clusters is reflected by the representation, i.e., child clusters are properly contained in their parent cluster. The computational complexity of this problem, whose study has been central to the theory of graph visualization since its introduction in 1995 [Feng, Cohen, and Eades, *Planarity for clustered graphs*, ESA'95], has only been recently settled [Fulek and Tóth, *Atomic Embeddability, Clustered Planarity, and Thickenability*, to appear at SODA'20]. Before such a breakthrough, the complexity question was still unsolved even when the graph has a prescribed planar embedding, i.e., for *embedded clustered graphs*.

We show that the C-PLANARITY TESTING problem admits a single-exponential single-parameter FPT algorithm for embedded clustered graphs, when parameterized by the carving-width of the dual graph of the input. This is the first FPT algorithm for this long-standing open problem with respect to a single notable graph-width parameter. Moreover, in the general case, the polynomial dependency of our FPT algorithm is smaller than the one of the algorithm by Fulek and Tóth. To further strengthen the relevance of this result, we show that the C-PLANARITY TESTING problem retains its computational complexity when parameterized by several other graph-width parameters, which may potentially lead to faster algorithms.

2012 ACM Subject Classification Human-centered computing → Graph drawings; Theory of computation → Fixed parameter tractability; Mathematics of computing → Graph theory

Keywords and phrases Clustered planarity, carving-width, non-crossing partitions, FPT

Digital Object Identifier 10.4230/LIPIcs.IPEC.2019.9

Related Version A full version of the paper [47] is available at <https://arxiv.org/abs/1910.02057>.

Funding *Giordano Da Lozzo*: Supported in part by H2020-MSCA-RISE project 734922 – “CONNECT”, by MIUR Project “AHeAD” under PRIN 20174LF3T8, by MIUR Project “MODE” under PRIN 20157EFM5C, and by MIUR-DAAD JMP N° 34120.

David Eppstein: Supported in part by the US NSF under grants CCF-1618301 and CCF-1616248.

Michael T. Goodrich: Supported in part by the US NSF under grant 1815073.

Siddharth Gupta: Supported in part by the Zuckerman STEM Leadership Program and by the Frankel Foundation.



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14th International Symposium on Parameterized and Exact Computation (IPEC 2019).

Editors: Bart M. P. Jansen and Jan Arne Telle; Article No. 9; pp. 9:1–9:17

Leibniz International Proceedings in Informatics



LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

1 Introduction

Many real-world data exhibit an intrinsic hierarchical structure that can be captured in the form of clustered graphs, i.e., graphs equipped with a recursive clustering of their vertices. This graph model has proved very powerful to represent information at different levels of abstraction and drawings of clustered networks appear in a wide variety of application domains, such as software visualization, knowledge representation, visual statistics, and data mining. More formally, a *clustered graph* (for short, *c-graph*) is a pair $\mathcal{C}(G, \mathcal{T})$, where G is the *underlying graph* and \mathcal{T} is the *inclusion tree* of \mathcal{C} , i.e., a rooted tree whose leaves are the vertices of G . Each non-leaf node μ of \mathcal{T} corresponds to a cluster containing the subset V_μ of the vertices of G that are the leaves of the subtree of \mathcal{T} rooted at μ . Edges between vertices of the same cluster (resp., of different clusters) are *intra-cluster edges* (resp., *inter-cluster edges*).

A natural and well-established criterion for a readable visualization of a c-graph has been derived from the classical notion of graph planarity. A *c-planar drawing* of a c-graph $\mathcal{C}(G, \mathcal{T})$ (see Fig. 4c) is a planar drawing of G together with a representation of each cluster μ in \mathcal{T} as a region $D(\mu)$ homeomorphic to a closed disc such that: (i) for each cluster μ in \mathcal{T} , region $D(\mu)$ contains the drawing of the subgraph $G[V_\mu]$ of G induced by V_μ ; (ii) for every two clusters μ and η in \mathcal{T} , it holds $D(\eta) \subseteq D(\mu)$ if and only if η is a descendant of μ in \mathcal{T} ; (iii) each edge crosses the boundary of any cluster disk at most once; and (iv) the boundaries of no two cluster disks intersect. A c-graph is *c-planar* if it admits a c-planar drawing.

The C-PLANARITY TESTING problem, introduced by Feng, Cohen, and Eades more than two decades ago [31], asks for the existence of a c-planar drawing of a c-graph. Despite several algorithms having been presented in the literature to construct c-planar drawings of c-planar c-graphs with nice aesthetic features [9, 28, 42, 48], determining the computational complexity of the C-PLANARITY TESTING problem has been one of the most challenging quests in the graph drawing research area [16, 21, 53]. To shed light on the complexity of the problem, several researchers have tried to highlight its connections with other notoriously difficult problems in the area [3, 53], as well as to consider relaxations [6, 7, 10, 30, 25, 56] and more constrained versions [2, 4, 5, 8, 23, 32, 33] of the classical notion of c-planarity. Algebraic approaches have also been considered [34, 41]. Only recently, Fulek and Tóth settled the question by giving a polynomial-time algorithm for a generalization of the C-PLANARITY TESTING problem called Atomic Embeddability [36].

A cluster μ is *connected* if $G[V_\mu]$ is connected, and it is *disconnected* otherwise. A c-graph is *c-connected* if every cluster is connected. Efficient algorithms for the c-connected case have been known since the early stages of the research on the problem [22, 27, 31]. Afterwards, polynomial-time algorithms have also been conceived for c-graphs satisfying other, weaker, connectivity requirements [20, 38, 40]. A c-graph is *flat* if each leaf-to-root path in \mathcal{T} consists of exactly two edges, that is, the clustering determines a partition of the vertex set; see, e.g., Fig. 4a. For flat c-graphs polynomial-time algorithms are known for several restricted cases [2, 11, 13, 17, 29, 35, 33, 43, 45, 46].

Motivations and contributions. In this paper, we consider the parameterized complexity of the C-PLANARITY TESTING problem for *embedded c-graphs*, i.e., c-graphs with a prescribed combinatorial embedding; see also [13, 18, 26, 46] for previous work in this direction.

In Section 2, we show that the C-PLANARITY TESTING problem retains its complexity when restricted to instances of bounded path-width and to connected instances of bounded tree-width. Such a result implies that the goal of devising an algorithm parameterized by graph-width parameters that are within a constant factor from tree-width (e.g., branch-

width [50]) or that are bounded by path-width (e.g., tree-width, rank-width [49], boolean-width [1], and clique-width [19]) and with a dependency on the input size which improves upon the one in [36] has to be regarded as a major algorithmic challenge.

Remarkably, before the results presented in [36], the computational complexity of the problem was still unsolved even for instances with faces of bounded size, and polynomial-time algorithms were known only for “small” faces and in the flat scenario. Namely, Jelinkova et al. [46] presented a quadratic-time algorithm for 3-connected flat c -graphs with faces of size at most 4. Subsequently, Di Battista and Frati [29] presented a linear-time/linear-space algorithm for embedded flat c -graphs with faces of size at most 5.

Motivated by the discussion above and by the results in Section 2, we focus our attention on embedded c -graphs $\mathcal{C}(G, \mathcal{T})$ whose underlying graph G has *both* bounded tree-width and bounded face size, i.e., instances such that the *carving-width* of the dual $\delta(G)$ of G is bounded. In Section 3, we present an FPT algorithm based on a dynamic-programming approach on a bond-carving decomposition to solve the problem for embedded flat c -graphs, which is ultimately based on maintaining a succinct description of the internal cluster connectivity via non-crossing partitions. We remark that, to the best of the authors’ knowledge, this is the first FPT algorithm for the C-PLANARITY TESTING problem, with respect to a *single* graph-width parameter. More formally, we prove the following.

► **Theorem 1.** C-PLANARITY TESTING can be solved in $O(2^{4\omega+\log\omega}n + n^2)$ time for any n -vertex embedded flat c -graph $\mathcal{C}(G, \mathcal{T})$, where ω is the carving-width of $\delta(G)$, if a carving decomposition of $\delta(G)$ of width ω is provided, and in $O(2^{4\omega+\log\omega}n + n^3)$ time, otherwise.

It is well known that the carving-width $\text{cw}(\delta(H))$ of the dual graph $\delta(H)$ of a plane graph H with maximum face size $\ell(H)$ and tree-width $\text{tw}(H)$ satisfies the relationship $\text{cw}(\delta(H)) \leq \ell(H)(\text{tw}(H) + 2)$ [12, 15]. Therefore, Theorem 1 provides the first¹ polynomial-time algorithm for instances of bounded face size and bounded tree-width, which answers an open question posed by Di Battista and Frati [29, Open Problem (ii)] for instances of bounded tree-width; also, since any n -vertex planar graph has tree-width in $O(\sqrt{n})$, it provides an $2^{O(\sqrt{n})}$ subexponential-time algorithm for instances of bounded face size, which improves the previous $2^{O(\sqrt{n} \log n)}$ time bound presented in [26] for such instances.

Further implications of Theorem 1 for instances of bounded embedded-width and of bounded dual cut-width are discussed in Section 4. Moreover, we extend Theorem 1 to get an FPT algorithm for general non-flat embedded c -graphs, whose running time is $O(4^{4\omega+\log\omega}n + n^2)$ if a carving decomposition of $\delta(G)$ of width ω is provided, and is $O(4^{4\omega+\log\omega}n + n^3)$, otherwise. The details for such an extension can be found in the full version [47].

2 Definitions and Preliminaries

In this section, we give definitions and preliminaries that will be useful throughout.

Graphs and connectivity. A *graph* $G = (V, E)$ is a pair, where V is the set of *vertices* of G and E is the set of *edges* of G , i.e., pairs of vertices in V . A *multigraph* is a generalization of a graph that allows the existence of multiple copies of the same edge. The *degree* of a vertex is the number of its incident edges. We denote the *maximum degree* of G by $\Delta(G)$. Also, for any $S \subseteq V$, we denote by $G[S]$ the subgraph of G induced by the vertices in S .

¹ The results in this paper were accepted for publication before the contribution of Fulek and Tóth in [36].

A graph is *connected* if it contains a path between any two vertices. A *cutvertex* is a vertex whose removal disconnects the graph. A connected graph containing no cutvertices is *2-connected*. The *blocks* of a graph are its maximal 2-connected subgraphs. In this paper, we only deal with connected graphs, unless stated otherwise.

Planar graphs and embeddings. A drawing of a graph is *planar* if it contains no edge crossings. A graph is *planar* if it admits a planar drawing. Two planar drawings of the same graph are *equivalent* if they determine the same *rotation* at each vertex, i.e, the same circular orderings for the edges around each vertex. A *combinatorial embedding* (for short, *embedding*) is an equivalence class of planar drawings. A planar drawing partitions the plane into topologically connected regions, called *faces*. The bounded faces are the *inner faces*, while the unbounded face is the *outer face*. A combinatorial embedding together with a choice for the outer face defines a *planar embedding*. An *embedded graph* (resp. *plane graph*) G is a planar graph with a fixed combinatorial embedding (resp. fixed planar embedding). The *length* of a face f of G is the number of occurrences of the edges of G encountered in a traversal of the boundary of f . The *maximum face size* $\ell(G)$ of G is the maximum length over all faces of G .

C-Planarity. An *embedded c-graph* $\mathcal{C}(G, \mathcal{T})$ is a c-graph whose underlying graph G has a prescribed combinatorial embedding, and it is *c-planar* if it admits a c-planar drawing that preserves the given embedding. Since we only deal with embedded c-graphs, in the remainder of the paper we will refer to them simply as c-graphs. Also, when G and \mathcal{T} are clear from the context, we simply denote $\mathcal{C}(G, \mathcal{T})$ as \mathcal{C} . A *candidate saturating edge* of \mathcal{C} is an edge not in G between two vertices of the same cluster in \mathcal{T} that are incident to the same face of G ; refer to Fig. 4b. A c-graph $\mathcal{C}'(G', \mathcal{T}')$ with $\mathcal{T}' = \mathcal{T}$ obtained by adding to \mathcal{C} a subset E^+ of its candidate saturating edges is a *super c-graph* of \mathcal{C} ; also, set E^+ is a *planar saturation* if G' is planar. Further, c-graph \mathcal{C} is *hole-free* if there exists a face f in G such that when f is chosen as the outer face for G no cycle composed of vertices of the same cluster encloses a vertex of a different cluster in its interior. Finally, two c-graphs are *equivalent* if and only if they are both c-planar or they are both not c-planar.

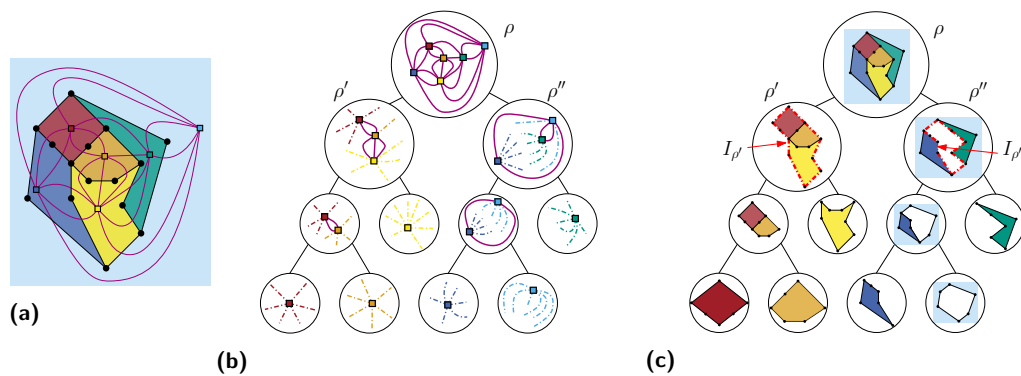
► **Remark 2.** In this paper, we only consider c-graphs whose underlying graph is connected, unless stated otherwise.

We will exploit the following characterization presented by Di Battista and Frati [29], which holds true also for non-flat c-graphs although originally only proved for flat c-graphs.

► **Theorem 3** ([29], Theorem 1). *A c-graph $\mathcal{C}(G, \mathcal{T})$ is c-planar if and only if:*

- (i) G is planar,
- (ii) \mathcal{C} is hole-free, and
- (iii) there exists a super c-graph $\mathcal{C}^*(G^*, \mathcal{T}^*)$ of \mathcal{C} such that G^* is planar and \mathcal{C}^* is c-connected.

Condition i of Theorem 3 can be tested using any of the known linear-time planarity-testing algorithms. Condition ii of Theorem 3 can be verified in linear time as described by Di Battista and Frati [29, Lemma 7], by exploiting the linear-time algorithm for checking if an embedded, possibly non-flat, c-graph is hole-free presented by Dahlhaus [27]. Therefore, in the following we will assume that any c-graph satisfies these conditions and thus view the C-PLANARITY TESTING problem as one of testing Condition iii.



■ **Figure 1** (a) Running example: An embedded graph G and its dual $\delta(G)$. (b) A bond-carving decomposition (D, γ) of the dual $\delta(G)$ of the graph G in Fig. 1a. (c) The decomposition (D, γ) where the vertices of $\delta(G)$ are replaced by the corresponding faces of G .

Tree-width. A *tree decomposition* of a graph G is a tree T whose nodes, called *bags*, are labeled by subsets of vertices of G . For each vertex v the bags containing v must form a nonempty contiguous subtree of T , and for each edge (u, v) of G at least one bag of T must contain both u and v . The *width* of the decomposition is one less than the maximum cardinality of any bag. The *tree-width* $\text{tw}(G)$ of G is the minimum width of any of its tree decompositions.

Cut-sets and duality. Let $G = (V, E)$ be a connected graph and let S be a subset of V . The partition $\{S, V \setminus S\}$ of V is a *cut* of G and the set $(S, V \setminus S)$ of edges with an endpoint in S and an endpoint in $V \setminus S$ is a *cut-set* of G . Also, cut-set $(S, V \setminus S)$ is a *bond* if $G[S]$ and $G[V \setminus S]$ are both non-null and connected.

For an embedded graph, the *dual* $\delta(G)$ of G is the planar multigraph that has a vertex v_f , for each face f of G , and an edge (v_f, v_g) , for each edge e shared by faces f and g . The edge e is the *dual edge* of (v_f, v_g) , and vice versa. Also, $\delta(G)$ is 2-connected if and only if G is 2-connected. Fig. 1a shows a plane graph G (black edges) and its dual $\delta(G)$ (purple edges); we will use these graphs as running examples throughout the paper. The following duality is well known.

► **Lemma 4** ([37], Theorem 14.3.1). *If G is an embedded graph, then a set of edges is a cycle of G if and only if their dual edges form a bond in $\delta(G)$.*

Carving-width. A *carving decomposition* of a graph $G = (V, E)$ is a pair (D, γ) , where D is a rooted binary tree whose leaves are the vertices of G , and γ is a function that maps the non-root nodes of D , called *bags*, to cut-sets of G as follows. For any non-root bag ν , let D_ν be the subtree of D rooted at ν and let \mathcal{L}_ν be the set of leaves of D_ν . Then, $\gamma(\nu) = (\mathcal{L}_\nu, V \setminus \mathcal{L}_\nu)$. The *width* of a carving decomposition (D, γ) is the maximum of $|\gamma(\nu)|$ over all bags ν in D . The *carving-width* $\text{cw}(G)$ of G is the minimum width over all carving decompositions of G . The *dual carving-width* is the carving-width of the dual of G . A *bond-carving decomposition* is a special kind of carving decomposition in which each cut-set is a bond of the graph; i.e., in a bond-carving decomposition every cut-set separates the graph into two connected components [51, 54].

In this paper, we view a bond-carving decomposition of the vertices of the dual $\delta(G)$ of an embedded graph G as a decomposition of the faces of G ; see Fig. 1c. A similar approach was followed in [12]. In particular, due to the duality expressed by Lemma 4, the cut-sets $\gamma(\nu)$ of

the bags ν of D correspond to cycles that can be used to *recursively partition* the faces of the primal graph, where these cycles are formed by the edges of the primal that are dual to those in each cut-set.

Partitions. Let $\mathcal{Q} = \{q_1, q_2, \dots, q_n\}$ be a ground set. A *partition* of \mathcal{Q} is a set $\{Q_1, \dots, Q_k\}$ of non-empty subsets Q_i 's of \mathcal{Q} , called *parts*, such that $\mathcal{Q} = \bigcup_{i=1}^k Q_i$ and $Q_i \cap Q_j = \emptyset$, with $1 \leq i < j \leq k$. Observe that $k \leq |\mathcal{Q}|$. Let now $\mathcal{S} = (s_1, s_2, \dots, s_n)$ be a *cyclically-ordered set*, i.e., a set equipped with a circular ordering. Let a, b , and c be three elements of \mathcal{S} such that b appears after a and before c in the circular ordering of \mathcal{S} ; we write $a \prec_b c$. A partition P of \mathcal{S} is *crossing*, if there exist elements $a, c \in S_i$ and $b, d \in S_j$, with $S_i, S_j \in P$ and $i \neq j$, such that $a \prec_b c$ and $c \prec_d a$; and, it is *non-crossing*, otherwise. We denote the set of all the non-crossing partitions of \mathcal{S} by $\mathcal{NC}(\mathcal{S})$. Note that, $|\mathcal{NC}(\mathcal{S})|$ coincides with the *Catalan number* $\mathcal{CAT}(n)$ of n , which satisfies $\mathcal{CAT}(n) \leq 2^{2n}$.

2.1 Relationship between Graph-Width Parameters and Connectivity

In this section, we present reductions that shed light on the effect that the interplay between some notable graph-width parameters and the connectivity of the underlying graph have on the computational complexity of the C-PLANARITY TESTING problem.

We will exploit recent results by Cortese and Patrignani, who proved the following:

- (a) Any n -vertex non-flat c-graph $\mathcal{C}(G, \mathcal{T})$ can be transformed into an equivalent $O(n \cdot h)$ -vertex flat c-graph in quadratic time [24, Theorem 1], where h is the height of \mathcal{T} .
- (b) Any n -vertex flat c-graph can be turned into an equivalent $O(n)$ -vertex *independent flat c-graph*, i.e., a flat c-graph such that each non-root cluster induces an independent set, in linear time [24, Theorem 2].

We remark that the reductions from [24] preserve the connectivity of the underlying graph.

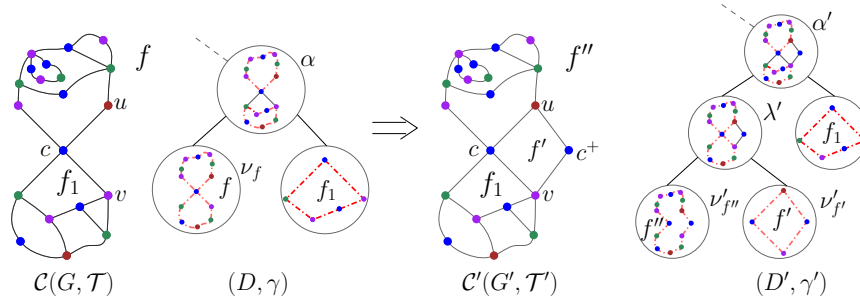
► **Theorem 5.** *Let $\mathcal{C}(G, \mathcal{T})$ be an n -vertex (flat) c-graph and let h be the height of \mathcal{T} . In $O(n^2)$ time (in $O(n)$ time), it is possible to construct an $O(n \cdot h)$ -vertex ($O(n)$ -vertex) independent flat c-graph $\mathcal{C}'(G', \mathcal{T}')$ that is equivalent to \mathcal{C} such that:*

- (i) G' is a collection of stars or
- (ii) G' is a tree.

Sketch. Let $\mathcal{C}(G, \mathcal{T})$ be an n -vertex (flat) c-graph. By the results a and b above, we can construct an $O(n \cdot h)$ -vertex ($O(n)$ -vertex) independent flat c-graph $\mathcal{C}^+(G^+, \mathcal{T}^+)$ equivalent to \mathcal{C} in $O(n^2)$ time (in $O(n)$ time). Note that, G^+ only contains inter-cluster edges.

Let $e = (u, v)$ be an edge of G^+ . Consider a c-graph $\mathcal{C}^1(G^1, \mathcal{T}^1)$ obtained from \mathcal{C}^+ as follows. First, subdivide the edge e with two dummy vertices u_e and v_e to create edges (u, u_e) , (u_e, v_e) , and (v_e, v) . Then, delete the edge (u_e, v_e) . Note that, the rotation scheme at u and v is same in G^1 as in G , considering $u = v_e$ and $v = u_e$ in the cyclic ordering of the neighbors of v and of u respectively in G . Finally, assign u_e and v_e to a new cluster μ_e , and add μ_e as a child of the root of the tree \mathcal{T}^+ . To construct \mathcal{C}^1 in case (i), we perform the above transformation for all the edges of G^+ . To construct \mathcal{C}^1 in case (ii), as long as the graph contains a cycle, we perform the above transformation for an edge e of such a cycle. Since the construction of \mathcal{C}^1 from \mathcal{C}^+ can be done in linear time both in case (i) and (ii), the running time follows. The equivalence can be proved by performing the above transformation, and its reverse, in c-connected super c-graphs of \mathcal{C}^+ and of \mathcal{C}^1 , respectively. ◀

We point out that by applying the reduction in the above proof without enforcing a specific embedding, Theorem 5 also holds for general instances of the C-PLANARITY TESTING problem, i.e., non-embedded c-graphs. Moreover, since the reduction given in [24]



■ **Figure 2** Reduction of Lemma 6 focused on cutvertex c . The transformation of (D, γ) into (D', γ') is shown. The red dashed edges are dual to those in the cut-set of each bag.

also works for disconnected instances, applying the reduction of Theorem 5 for case (i) to a general disconnected instance $\mathcal{C}(G, \mathcal{T})$ would result in an equivalent independent flat c-graph $\mathcal{C}'(G', \mathcal{T}')$ such that G' is a collection of stars. An immediate, yet important, consequence of this discussion is that an algorithm with running time in $O(r(n))$ for flat instances whose underlying graph is a collection of stars would result in an algorithm with running time in $O(r(n))$ for flat instances and in $O(r(n^2) + n^2)$ for general, possibly non-flat, instances.

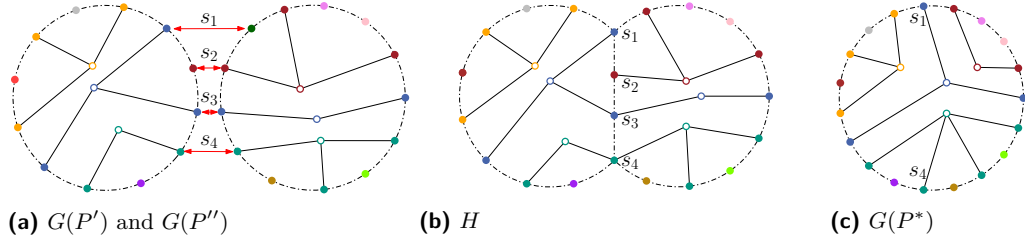
The proof of the following lemma, which will turn useful in the following sections, is based on the duality expressed by Lemma 4.

► **Lemma 6.** *Given an n -vertex c-graph $\mathcal{C}(G, \mathcal{T})$ and a carving decomposition (D, γ) of $\delta(G)$ of width ω , in $O(n)$ time, it is possible to construct an $O(n)$ -vertex c-graph $\mathcal{C}'(G', \mathcal{T}')$ that is equivalent to \mathcal{C} such that G' is 2-connected, and a carving decomposition (D', γ') of $\delta(G')$ of width $\omega' = \max(\omega, 4)$.*

Proof. We construct $\mathcal{C}'(G', \mathcal{T}')$ as follows. Let $\beta(G) < n$ be the number of blocks of G .

Let c be a cutvertex of G , let μ be the cluster that is the parent of c in \mathcal{T} , and let (u, c) and (v, c) be two edges belonging to different blocks of G that are incident to the same face f of G . Consider the c-graph $\mathcal{C}^+(G^+, \mathcal{T}^+)$ obtained from \mathcal{C} by embedding a path (u, c^+, v) inside f , where c^+ is a new vertex that we add as a child of μ ; see Fig. 2. We denote by f' the face of G^+ bounded by the cycle (u, c^+, v, c) and by f'' the other face of G^+ incident to c^+ . Clearly, this augmentation can be done in $O(1)$ time and G^+ contains $\beta(G) - 1$ blocks. Also, \mathcal{C} and \mathcal{C}^+ are equivalent. This is due to the fact that any saturating edge (c, x) incident to c and lying in f (of a c-connected c-planar super c-graph of \mathcal{C}) can be replaced by two saturating edges (x, c^+) lying in f'' and (c^+, c) lying in f' (of a c-connected c-planar super c-graph of \mathcal{C}^+), and vice versa.

We now show how to modify the carving decomposition (D, γ) of $\delta(G)$ of width ω into a carving decomposition (D^+, γ^+) of $\delta(G^+)$ of width $\omega^+ = \max(\omega, 4)$ in $O(1)$ time. Consider the leaf bag ν_f of D corresponding to face f and let α be the parent of ν_f in D . We construct D^+ from D as follows. First, we initialize $(D^+, \gamma^+) = (D, \gamma)$; in the following, we denote by ν' the bag of D^+ corresponding to the bag ν of D . We remove ν_f from D^+ , add a new non-leaf bag λ' as a child of α' and two leaf bags $\nu'_{f'}$, corresponding to face f' , and $\nu'_{f''}$, corresponding to face f'' , as children of λ' ; refer to Fig. 2. Further, we have $\gamma^+(\nu'_{f'}) = \{(u, c), (u, c^+), (v, c), (v, c^+)\}$, $\gamma^+(\nu'_{f''}) = \gamma(\nu_f) \setminus \{(u, c), (v, c)\} \cup \{(u, c^+), (v, c^+)\}$, $\gamma^+(\lambda') = \gamma(\nu_f)$, and $\gamma^+(\nu') = \gamma(\nu)$, for any other bag ν belonging to both D^+ and D . In particular, the size of the edge-cuts defined by all the bags different from $\nu'_{f'}$ stays the same, while the size of the edge-cut of $\nu'_{f'}$ is 4. Therefore, (D^+, γ^+) is a carving decomposition of $\delta(G^+)$ of width $\omega^+ = \max(\omega, 4)$.



■ **Figure 3** Illustrations for the definition of bubble merge.

Repeating the above procedure, eventually yields a 2-connected c-graph $\mathcal{C}'(G', \mathcal{T}')$, with $|V(G')| = n + \beta(G) - 1 = O(n)$, that is equivalent to \mathcal{C} and a carving decomposition (D', γ') of $\delta(G')$ of width $\omega' = \max(\omega, 4)$. Since each execution of the above procedure takes $O(1)$ time and since the cutvertices and the blocks of G can be computed in $O(n)$ time [44], we have that \mathcal{C}' and (D', γ') can be constructed in $O(n)$ time. This concludes the proof. ◀

3 A Dynamic-Programming Algorithm for Flat Instances

In this section, we present an FPT algorithm for the C-PLANARITY TESTING problem of flat c-graphs parameterized by the dual carving-width. We first describe a dynamic-programming algorithm to test whether a 2-connected flat c-graph \mathcal{C} is c-planar, by verifying whether \mathcal{C} satisfies Condition iii of Theorem 3. Then, by combining this result and Lemma 6, we extend the algorithm to simply-connected instances.

Basic operations. Let $\mathcal{C}(G, \mathcal{T})$ be a flat c-graph. A partition $\{S_1, \dots, S_k\}$ of $V' \subseteq V(G)$ is *good* if, for each part S_i , there exists a non-root cluster μ such that all the vertices in S_i belong to μ ; also, we say that the part S_i *belongs* to the cluster μ . Further, a partition of a cyclically-ordered set $\mathcal{S} \subseteq V(G)$ is *admissible* if it is both good and non-crossing. We define the binary operator \uplus , called *generalized union*, that given two good partitions P' and P'' of ground sets \mathcal{Q}' and \mathcal{Q}'' , respectively, returns a good partition $P^* = P' \uplus P''$ of $\mathcal{Q}' \cup \mathcal{Q}''$ obtained as follows. Initialize $P^* = P' \cup P''$. Then, as long as there exist $Q_i, Q_j \in P^*$ such that $Q_i \cap Q_j \neq \emptyset$, replace sets Q_i and Q_j with their union $Q_i \cup Q_j$ in P^* . We have the following.

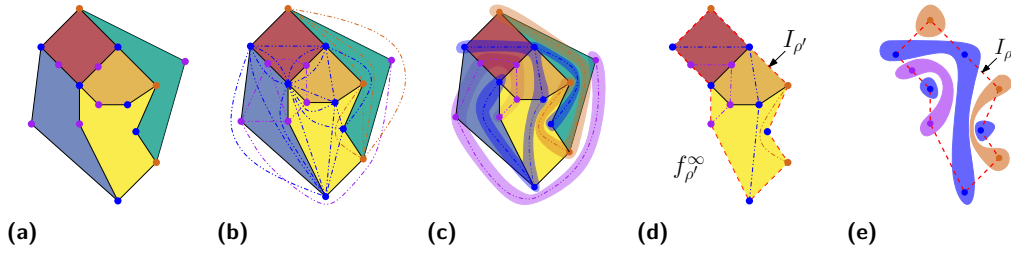
► **Lemma 7.** $P^* = P' \uplus P''$ can be computed in $O(|\mathcal{Q}'| + |\mathcal{Q}''|)$ time.

Let P be a good partition of the ground set \mathcal{Q} and let $\mathcal{Q}' \subset \mathcal{Q}$. The *projection* of P onto \mathcal{Q}' , denoted as $P|_{\mathcal{Q}'}$, is the good partition of \mathcal{Q}' obtained from P by first replacing each part $S_i \in P$ with $S_i \cap \mathcal{Q}'$ and then removing empty parts, if any.

An admissible partition P of a cyclically-ordered set \mathcal{S} can be naturally associated with a 2-connected plane graph $G(P)$ as follows. The outer face of $G(P)$ is a cycle $C(P)$ whose vertices are the elements in \mathcal{S} and the clockwise order in which they appear along $C(P)$ is the same as in \mathcal{S} . Also, for each part $S_i \in P$ such that $|S_i| \geq 2$, graph $G(P)$ contains a vertex v_i in the interior of $C(P)$ that is adjacent to all the elements in S_i , i.e., removing all the edges of $C(P)$ yields a collection of stars, whose central vertices are the v_i 's, and isolated vertices. We say that $G(P)$ is the *cycle-star* associated with P ; see, e.g., Fig. 3c.

We also extend the definitions of *generalized union* and *projection* to admissible partitions by regarding the corresponding cyclically-ordered sets as unordered.

Let P' and P'' be two admissible partitions of cyclically-ordered sets \mathcal{S}' and \mathcal{S}'' , respectively, with the following properties (where $\mathcal{S}_\cap = \{s_1, s_2, \dots, s_k\}$ denotes the set of elements that are common to \mathcal{S}' and \mathcal{S}''): (i) $|\mathcal{S}_\cap| \geq 2$ and $\mathcal{S}' \cup \mathcal{S}'' \setminus \mathcal{S}_\cap \neq \emptyset$, (ii) the elements of \mathcal{S}_\cap



■ **Figure 4** (a) A flat c-graph $\mathcal{C}(G, \mathcal{T})$. (b) A super c-graph of \mathcal{C} containing all the candidate saturating edges. (c) A c-planar drawing of \mathcal{C} and the corresponding planar saturation. (d) A planar saturation of a c-graph, whose underlying graph is the graph $G_{\rho'}$ of the decomposition in Fig. 1, where no saturating edge lies in the interior of $f_{\rho'}^{\infty}$. (e) The admissible partition P determined by the planar saturation in (d); sets of vertices of $I_{\rho'}$ belonging to the same cluster and connected by saturating edges in (d) form distinct parts in P (enclosed by shaded regions).

appear consecutively both in \mathcal{S}' and \mathcal{S}'' , and (iii) the cyclic ordering of the elements in \mathcal{S}_{\cap} determined by \mathcal{S}' is the reverse of the cyclic order of these elements determined by \mathcal{S}'' . We define the binary operator \mathcal{B} , called *bubble merge*, that returns an admissible partition $P^* = P' \mathcal{B} P''$ obtained as follows. Consider the cycle-stars $G(P')$ and $G(P'')$ associated with P' and P'' , respectively. First, we identify the vertices corresponding to the same element of \mathcal{S}_{\cap} in both $C(P')$ and $C(P'')$ (see Fig. 3a) to obtain a new plane graph H (see Fig. 3b). Observe that H is 2-connected since $G(P')$ and $G(P'')$ are 2-connected and since $|\mathcal{S}_{\cap}| \geq 2$; therefore, the outer face f_H of H is a simple cycle. Second, we traverse f_H clockwise to construct a cyclically-ordered set $\mathcal{S}^* \subseteq \mathcal{S}' \cup \mathcal{S}''$ on the vertices of f_H . Finally, we set $P^* = (P' \uplus P'')|_{\mathcal{S}^*}$. P^* is good by the definition of generalized union. The fact that P^* is a non-crossing partition of \mathcal{S}^* follows immediately by the planarity of H (see Fig. 3c). Lemma 7 and the fact that the graph H can be easily constructed from P' and P'' in linear time imply the following.

► **Lemma 8.** $P^* = P' \mathcal{B} P''$ can be computed in $O(|\mathcal{S}'| + |\mathcal{S}''|)$ time.

Algorithm. Let $\mathcal{C}(G, \mathcal{T})$ be a 2-connected flat c-graph. Let (D, γ) be a bond-carving decomposition of $\delta(G)$ of width at most ω and let ν be a non-root bag of D . We denote by F_{ν} the set of faces of G that are dual to the vertices of $\delta(G)$ that are leaves of the subtree D_{ν} of D rooted at ν . Also, let G_{ν} be the embedded subgraph of G induced by the edges of the faces in F_{ν} . The *interface graph* I_{ν} of ν is the subgraph of G_{ν} induced by the edges that are incident to a face of G_{ν} not in F_{ν} . The *boundary* B_{ν} of ν is the vertex set of I_{ν} . Note that, the edges of I_{ν} are dual to those in $\gamma(\nu)$. By Lemma 4 and by the definition of bond-carving decomposition, we derive the next observation about I_{ν} .

► **Observation 1.** The interface graph I_{ν} of ν is a cycle of length at most ω .

Since G is 2-connected, by Observation 1, the vertices in B_{ν} have a natural (clockwise) circular ordering defined by cycle I_{ν} , and I_{ν} bounds the *unique* face f_{ν}^{∞} of G_{ν} not in F_{ν} . Therefore, from now on, we regard B_{ν} as a cyclically-ordered set.

Let $P \in \mathcal{NC}(B_{\nu})$ be an admissible partition and let $\mathcal{C}_{\nu}(G_{\nu}, \mathcal{T}_{\nu})$ be the flat c-graph obtained by restricting \mathcal{C} to G_{ν} . Also, let $\mathcal{C}_{\nu}^{\diamond}(G_{\nu}^{\diamond}, \mathcal{T}_{\nu}^{\diamond})$ be a super c-graph of \mathcal{C}_{ν} containing no saturating edges in the interior of f_{ν}^{∞} and such that G_{ν}^{\diamond} is planar.

► **Definition 9.** The c-graph \mathcal{C}_ν^\diamond realizes P if (refer to Fig. 4):

- (a) for every two vertices $u, v \in B_\nu$, we have that u and v belong to the same part $S_i \in P$ if and only if they are connected in G_ν^\diamond by paths of intra-cluster edges of the cluster the part S_i belongs to,
- (b) for each cluster μ in \mathcal{T} such that $V_\mu \cap B_\nu \neq \emptyset$, all the vertices of μ in G_ν are connected to some vertex of μ in B_ν by paths of intra-cluster edges of μ in G_ν^\diamond , and
- (c) for each cluster μ in \mathcal{T} such that $V_\mu \subseteq V(G_\nu) \setminus B_\nu$, all the vertices of μ are in G_ν and are connected by paths of intra-cluster edges of μ in G_ν^\diamond .

A vertex v of G_ν is *dominated* by some part S_i of P , if either $v \in S_i$ or v is connected to some vertex of S_i by paths of intra-cluster edges in \mathcal{C}_ν^\diamond . Note that, by Conditions a and b of Definition 9, for each part S_i of P , the set of vertices of G_ν dominated by S_i induces a connected subgraph of G_ν^\diamond . Also, by Condition c, cycle I_ν does not form a *cluster separator*, that is, a cycle of G such that the vertices of some cluster μ appear both in its interior and in its exterior, but not in it; note that, in fact, this is a necessary condition for the existence of a c-planar drawing of \mathcal{C} . Thus, partition P “represents” the internal-cluster connectivity in \mathcal{C}_ν^\diamond of the clusters whose vertices appear in B_ν in a *potentially positive instance*. Also, P is *realizable* by \mathcal{C}_ν if there exists a super c-graph $\mathcal{C}'_\nu(G'_\nu, \mathcal{T}'_\nu)$ of \mathcal{C}_ν that realizes P containing no saturating edges in the interior of f_ν^∞ and such that G'_ν is planar. From Theorem 3 and the definition of realizable partition we have the following.

► **Lemma 10 (Necessity).** Let ν be a non-root bag of D . Then, \mathcal{C} is c-planar only if there exists an admissible partition P of B_ν that is realizable by \mathcal{C}_ν .

We are going to exploit the next lemma, which holds for any bond-carving decomposition.

► **Lemma 11.** Let ρ' and ρ'' be the two children of the root ρ of D . Then, $I_{\rho'} = I_{\rho''}$.

Lemmas 10 and 11 allow us to derive the following useful characterization.

► **Theorem 12 (Characterization).** The 2-connected flat c-graph $\mathcal{C}(G, \mathcal{T})$ is c-planar if and only if there exist admissible partitions $P' \in \mathcal{NC}(B_{\rho'})$, $P'' \in \mathcal{NC}(B_{\rho''})$ such that:

- (i) P' and P'' are realizable by $\mathcal{C}_{\rho'}(G_{\rho'}, \mathcal{T}_{\rho'})$ and by $\mathcal{C}_{\rho''}(G_{\rho''}, \mathcal{T}_{\rho''})$, respectively, and
- (ii) no two distinct parts $S_i, S_j \in P^*$, with $P^* = P' \uplus P''$, belong to the same cluster of \mathcal{T} .

Proof. We first prove the *only if* part. The necessity of Condition i follows from Lemma 10. For the necessity of Condition ii, suppose for a contradiction, that for any two realizable partitions $P' \in \mathcal{NC}(B_{\rho'})$ and $P'' \in \mathcal{NC}(B_{\rho''})$, it holds that $P^* = P' \uplus P''$ does not satisfy such a condition. Then, there is no set of saturating edges of $\mathcal{C}_{\rho'}$ and $\mathcal{C}_{\rho''}$, where none of these edges lies in the interior of $f_{\rho'}^\infty$ and of $f_{\rho''}^\infty$, respectively, that when added to \mathcal{C} yields a c-connected c-graph $\mathcal{C}^\diamond(G^\diamond, \mathcal{T}^\diamond)$ with G^\diamond planar. Thus, by Theorem 3, \mathcal{C} is not c-planar, a contradiction.

We now prove the *if part*. By Lemma 11, it holds $G = G_{\rho'} \cup G_{\rho''}$ and $I_{\rho'} = I_{\rho''} = G_{\rho'} \cap G_{\rho''}$. Let $\mathcal{C}_{\rho'}^\diamond$ be a super c-graph of $\mathcal{C}_{\rho'}$ realizing P' and let $\mathcal{C}_{\rho''}^\diamond$ be a super c-graph of $\mathcal{C}_{\rho''}$ realizing P'' ; these c-graphs exist since Condition i holds. Let \mathcal{C}^\diamond be the super c-graph of \mathcal{C} obtained by augmenting \mathcal{C} with the saturating edges of both $\mathcal{C}_{\rho'}^\diamond$ and $\mathcal{C}_{\rho''}^\diamond$. Note that, G^\diamond is planar.

We show that every cluster μ is connected in \mathcal{C}^\diamond , provided that Condition ii holds. This proves that \mathcal{C}^\diamond is a c-connected super c-graph of \mathcal{C} , thus by Condition iii of Theorem 3, c-graph \mathcal{C} is c-planar. We distinguish two cases, based on whether some vertices of μ appear along cycle $I_{\rho'} = I_{\rho''}$ or not. Let $B = B_{\rho'} = B_{\rho''}$.

Consider first a cluster μ containing vertices in B . By Condition b of Definition 9, we have that every vertex in μ is dominated by at least a part of P' or of P'' , i.e., they either belong to B or they are connected by paths of intra-cluster edges in either $\mathcal{C}_{\rho'}^\diamond$ or $\mathcal{C}_{\rho''}^\diamond$ to a

vertex in B . Since, by Condition ii of the statement, there exists only one part $S_\mu \in P^*$ that contains vertices of cluster μ , we have that the different parts of P' and of P'' containing vertices of μ are joined together by the vertices of μ in B . Therefore, the cluster μ is connected in \mathcal{C}^\diamond . Finally, consider a cluster μ such that no vertex of μ belongs to B . Then, all the vertices of cluster μ only belong to either $\mathcal{C}_{\rho'}$ or $\mathcal{C}_{\rho''}$, by Condition c of Definition 9. Suppose that μ only belongs to $\mathcal{C}_{\rho'}$, the case when μ only belongs to $\mathcal{C}_{\rho''}$ is analogous. Since $\mathcal{C}_{\rho'}$ realizes P' , by Condition c of Definition 9, all the vertices of μ in G'_ρ are connected by paths of intra-cluster edges. Thus, cluster μ is connected in \mathcal{C}^\diamond , since it is connected in $\mathcal{C}_{\rho'}$. This concludes the proof. \blacktriangleleft

We now present our main algorithmic tool.

Algorithm 1. Let (D, γ) be a bond-carving decomposition of $\delta(G)$ of width ω . Let ν be a non-root bag of D , we denote by R_ν the set of all the admissible partitions of B_ν that are realizable by \mathcal{C}_ν . We process the bags of D bottom-up and compute the following *relevant information*, for each non-root bag ν of D : **1.** the set R_ν , and **2.** for each admissible partition $P \in R_\nu$ and for each part $S_i \in P$, the number $count(S_i)$ of vertices of cluster μ belonging to G_ν that are dominated by S_i , where μ is the cluster S_i belongs to.

- If ν is a leaf bag of D , then $G_\nu = I_\nu$ consists of the vertices and edges of a single face of G . Further, by Observation 1, graph G_ν is a cycle of length at most ω . In this case, R_ν simply coincides with the set of all the admissible partitions of B_ν . Therefore, we can construct R_ν by enumerating all the possible at most $\mathcal{CAT}(\omega) \leq 2^{2\omega}$ non-crossing partitions of B_ν and by testing whether each such partition is good in $O(\omega)$ time. Further, for each $P \in R_\nu$, we can compute all counters $count(S_i)$ for every $S_i \in P$, in total $O(\omega)$ time, by visiting cycle I_ν .
- If ν is a non-leaf non-root bag of D , we have already computed the relevant information for the two children ν' and ν'' of ν . In the following way, we either detect that \mathcal{C} does not satisfy Condition iii of Theorem 3 or construct the relevant information for ν :
 - (1) Initialize $R_\nu = \emptyset$;
 - (2) For every pair of realizable admissible partitions $P' \in \mathcal{R}_{\nu'}$ and $P'' \in \mathcal{R}_{\nu''}$, perform the following operations:
 - (2a) Compute $P^* = P' \uplus P''$ and compute the counters $count(S_i)$, for each $S_i \in P^*$, from the counters of the parts in $P' \cup P''$ whose union is S_i .
 - (2b) If there exists some $S_i \in P^*$ such that $S_i \cap B_\nu = \emptyset$ and $count(S_i)$ is smaller than the number of vertices in the cluster S_i belongs to, then **reject** the instance.
 - (2c) Compute $P = P' \uplus P''$ and add P to R_ν .

► **Remark 13.** Algorithm 1 rejects the instance at step (2b), if I_μ forms a cluster separator. This property is independent of the specific generalized union P^* considered at this step and implies that no P^* (and, thus, no P at step (2c)) can satisfy Condition c of Definition 9.

As the total number of pairs of partitions at step (2) is at most $(\mathcal{CAT}(\omega))^2$ and as P^* and P can be computed in $O(\omega)$ time, by Lemmas 7 and 8, we get the following.

► **Lemma 14.** For each non-root bag ν of D , Algorithm 1 computes the relevant information for ν in $O(2^{4\omega + \log \omega})$ time, given the relevant information for its children.

Proof. Let ν' and ν'' be the two children of ν in D . We will first show the correctness of the algorithm and then argue about the running time.

Let R_ν^* be the set of all the admissible partitions of B_ν that are realizable by \mathcal{C}_ν and let R_ν be the set of all the admissible partitions of B_ν computed by Algorithm 1. We show $R_\nu = R_\nu^*$.

We first prove $R_\nu^* \subseteq R_\nu$. Let P_ν be a realizable admissible partition in R_ν^* . Since P_ν is realizable by \mathcal{C}_ν , there exists a super c-graph $\mathcal{C}_\nu^*(G_\nu^*, \mathcal{T}_\nu^*)$ of \mathcal{C}_ν that realizes P_ν containing no saturating edges in the interior of f_ν^∞ and such that G_ν^* is planar. Let $\mathcal{C}_{\nu'}^*(G_{\nu'}^*, \mathcal{T}_{\nu'}^*)$ (resp. $\mathcal{C}_{\nu''}^*(G_{\nu''}^*, \mathcal{T}_{\nu''}^*)$) be the super c-graph of $\mathcal{C}_{\nu'}(G_{\nu'}, \mathcal{T}_{\nu'})$ (resp. of $\mathcal{C}_{\nu''}(G_{\nu''}, \mathcal{T}_{\nu''})$) obtained by adding to $\mathcal{C}_{\nu'}$ (resp. to $\mathcal{C}_{\nu''}$) all the saturating edges in G_ν^* laying in the interior of the faces of $G_{\nu'}$ (resp. of $G_{\nu''}$) that are also faces of G_ν . Clearly, c-graph $\mathcal{C}_{\nu'}^*(G_{\nu'}^*, \mathcal{T}_{\nu'}^*)$ (resp. c-graph $\mathcal{C}_{\nu''}^*(G_{\nu''}^*, \mathcal{T}_{\nu''}^*)$) contains no saturating edges in the interior of $f_{\nu'}^\infty$ (resp. in the interior of $f_{\nu''}^\infty$), since such a face does not belong to G_ν . Let P' and P'' be the admissible partitions of $B_{\nu'}$ and of $B_{\nu''}$ realized by $\mathcal{C}_{\nu'}^*(G_{\nu'}^*, \mathcal{T}_{\nu'}^*)$ and by $\mathcal{C}_{\nu''}^*(G_{\nu''}^*, \mathcal{T}_{\nu''}^*)$, respectively. By hypothesis, we have $P' \in R_{\nu'}$ and $P'' \in R_{\nu''}$. We show that when step (2) of Algorithm 1 considers partitions P' and P'' , it successfully adds P_ν to the set R_ν . It is clear by the construction of P' and of P'' that $P_\nu = P' \uplus P''$. Therefore, we only need to show that when the algorithm considers the pair (P', P'') , it does not reject the instance at step (2b), and thus P_ν is added to R_ν at step (2c). Let $P^* = P' \uplus P''$, which is constructed at step (2a) of the algorithm. Suppose, for a contradiction, that \mathcal{C} is rejected at step (2b). Then, there exists a part S_i of P^* such that $S_i \cap B_\nu = \emptyset$ and $\text{count}(S_i)$ is smaller than the number of vertices in the cluster μ the part S_i belongs to. Therefore, the cluster μ contains vertices that belong to $G \setminus G_\nu$, which implies that P_ν cannot satisfy Condition c of Definition 9, a contradiction. This concludes the proof of this direction.

We now prove $R_\nu \subseteq R_\nu^*$. Let P be a partition in R_ν obtained from the partitions $P' \in R_{\nu'}$ and $P'' \in R_{\nu''}$ (selected at step (2) of the algorithm). We show that P is realizable by \mathcal{C}_ν .

By the definition of realizable partition, there exists a super c-graph $\mathcal{C}_{\nu'}^*(G_{\nu'}^*, \mathcal{T}_{\nu'}^*)$ (resp. $\mathcal{C}_{\nu''}^*(G_{\nu''}^*, \mathcal{T}_{\nu''}^*)$) of $\mathcal{C}_{\nu'}(G_{\nu'}, \mathcal{T}_{\nu'})$ (resp. of $\mathcal{C}_{\nu''}(G_{\nu''}, \mathcal{T}_{\nu''})$) that realizes P' (resp. P'') containing no saturating edges in the interior of $f_{\nu'}^\infty$ (resp. of $f_{\nu''}^\infty$) and such that $G_{\nu'}^*$ (resp. $G_{\nu''}^*$) is planar. Let $\mathcal{C}_\nu^*(G_\nu^*, \mathcal{T}_\nu^*)$ be the super c-graph of $\mathcal{C}_\nu(G_\nu, \mathcal{T}_\nu)$ constructed by adding to \mathcal{C}_ν the saturating edges in $\mathcal{C}_{\nu'}^*$ and $\mathcal{C}_{\nu''}^*$. We show that the c-graph $\mathcal{C}_\nu^*(G_\nu^*, \mathcal{T}_\nu^*)$ realizes P , contains no saturating edges in the interior of f_ν^∞ , and G_ν^* is planar.

First, we have that G_ν^* is planar, since $G_{\nu'}^*$ and $G_{\nu''}^*$ are planar and do not contain saturating edges in the interior of $f_{\nu'}^\infty$ and of $f_{\nu''}^\infty$, respectively. By the previous arguments, we also have that f_ν^∞ contains no saturating edges.

We show that Condition a of Definition 9 holds. Recall that $P = P' \uplus P''$. Let S_i be a part of P that also belongs to P' or to P'' . Then, since $\mathcal{C}_{\nu'}^*$ and $\mathcal{C}_{\nu''}^*$ realize P' and P'' , respectively, the vertices of S_i are connected by paths of intra-cluster edges in \mathcal{C}_ν^* as they are connected by paths of intra-cluster edges in either $\mathcal{C}_{\nu'}^*$ or $\mathcal{C}_{\nu''}^*$, by Condition a of Definition 9. Otherwise, let S_i be a part of P that does not belong to either P' or P'' . Then, by the definition of bubble merge, the part S_i is obtained by projecting onto B_ν the generalized union $P^* = P' \uplus P''$. Thus, S_i is a subset of a part S_i^* of P^* . Also, the vertices in each of the parts of P' and of P'' contributing to the creation of S_i^* are connected by paths of intra-cluster edges in $\mathcal{C}_{\nu'}^*$ and $\mathcal{C}_{\nu''}^*$, respectively, by Condition a of Definition 9. Therefore, we have that the connectivity of such sets implies the connectivity of the elements of S_i by paths of intra-cluster edges that connect at their shared vertices in $B_{\nu'} \cap B_{\nu''}$. We show that Condition b of Definition 9 holds. Suppose, for a contradiction, that there exists some cluster μ whose vertices appear in B_ν such that there is at least a vertex of μ in G_ν that is not connected by a path of intra-cluster edges to some vertex of μ in B_ν . Then, consider the part $S_i \in P^*$ that dominates this vertex, which exists since P' and P'' are realizable by $\mathcal{C}_{\nu'}$ and by $\mathcal{C}_{\nu''}$, respectively. We have that $S_i \cap B_\nu = \emptyset$ and that $\text{count}(S_i)$ is smaller than the

number of vertices of μ . Thus, step (2b) would reject the instance, and thus P would not be added to R_ν , a contradiction. Finally, we show that Condition c of Definition 9 holds. Suppose, for a contradiction, that there exists some cluster μ whose vertices only belong to $V(G_\nu) \setminus B_\nu$ and that there exist two vertices u and v of μ in G_ν that are not connected by a path of intra-cluster edges in G_ν^* . Then, consider the part $S_i \in P^*$ that dominates u . Observe that, S_i does not dominate v . Similarly to the proof of Condition b, we have that $S_i \cap B_\nu = \emptyset$ and that $\text{count}(S_i)$ is smaller than the number of vertices of μ . Thus, step (2b) would reject the instance, and thus P would not be added to R_ν , a contradiction.

We conclude by analyzing the running time. Step (2a) can be performed in linear time in the sum of the sizes of $B_{\nu'}$ and $B_{\nu''}$, since the generalized union P^* can be computed in $O(|B_{\nu'}| + |B_{\nu''}|)$ time, by Lemma 7, and since the size of P^* , and thus the number of counters to be updated, is in $O(|B_{\nu'}| + |B_{\nu''}|)$. Step (2b) can also be done in linear time by the previous argument. Step (2c) can be performed in $O(|B_{\nu'}| + |B_{\nu''}|)$ time, by Lemma 8. Further, the number of pairs of realizable partitions considered at step (2) is bounded by $|\mathcal{NC}(B_{\nu'})| \cdot |\mathcal{NC}(B_{\nu''})|$, which is bounded by $2^{2(|B_{\nu'}| + |B_{\nu''}|)}$. Finally, $|B_{\nu'}| \leq \omega$ and $|B_{\nu''}| \leq \omega$. Thus, Algorithm 1 runs in $O(2^{4\omega}\omega) = O(2^{4\omega + \log \omega})$ time. ◀

By Lemma 14 and since D contains $O(n)$ bags, we have the following.

► **Lemma 15.** *Sets $R_{\rho'}$ and $R_{\rho''}$ can be computed in $O(2^{4\omega + \log \omega} n)$ time.*

We obtain the next theorem by combining Lemma 15 and Theorem 12, where the additive $O(n^2)$ factor in the running time derives from the time needed to convert a carving decomposition of $\delta(G)$ into a bond-carving decomposition of the same width [54].

► **Theorem 16.** *C-PLANARITY TESTING can be solved in $O(2^{4\omega + \log \omega} n + n^2)$ time for any 2-connected n -vertex flat c -graph $\mathcal{C}(G, \mathcal{T})$, if a carving decomposition of $\delta(G)$ of width ω is provided.*

We are finally ready to prove our main result.

Proof of Theorem 1. Let (D, γ) be a carving decomposition of $\delta(G)$ of optimal width $\omega = \text{cw}(\delta(G))$. First, we apply Lemma 6 to \mathcal{C} to obtain, in $O(n)$ time, a 2-connected flat c -graph $\mathcal{C}'(G', \mathcal{T}')$ equivalent to \mathcal{C} and a corresponding carving decomposition (D', γ') of width $\omega' \leq \max(\omega, 4)$. Then, we apply Theorem 16 to test whether \mathcal{C}' (and thus \mathcal{C}) is c -planar. The running time follows from the running time of Theorem 16, from the fact that $\omega' = O(\omega)$, $|V(G')| \in O(n)$, and that a carving decomposition of $\delta(G)$ of optimal width can be computed in $O(n^3)$ time [39, 55]. This concludes the proof of the theorem.

We remark that in the recent reduction presented by Patrignani and Cortese to convert any non-flat c -graph $\mathcal{C}(G, \mathcal{T})$ into an equivalent independent flat c -graph $\mathcal{C}'(G', \mathcal{T}')$, the carving-width of $\delta(G')$ is within an $O(h)$ multiplicative factor from the carving-width of $\delta(G)$, where h is the height of \mathcal{T} . This is due to the fact that, by [24, Lemma 10], G' is a subdivision of G (which implies that $\text{tw}(G') = \text{tw}(G)$) and that each inter-cluster edge of G is replaced by a path of length at most $4h - 4$ in G' (which implies that $\ell(G') = \ell(G)(4h - 4)$). Therefore, by [24] and by the results presented in this section, we immediately derive an FPT algorithm for the non-flat case parameterized by h and the dual carving-width of G . In the full version [47], we show how to drop the dependency on h , by suitably adapting the relevant concepts defined for the flat case so that Algorithm 1 can also be applied to the non-flat case.

4 Graph-Width Parameters Related to the Dual Carving-Width

In this section, we discuss implications of our algorithm for instances of bounded embedded-width and of bounded dual cut-width.

Embedded-width. A tree decomposition of an embedded graph G *respects* the embedding of G if, for every face f of G , at least one bag contains all the vertices of f [14]. The *embedded-width* $\text{emw}(G)$ of G is the minimum width of any of its tree decompositions that respect the embedding of G . For consistency with other graph-width parameters, in the original definition of this width measure [14] the vertices of the outer face are not required to be in some bag. Here, we adopt the variant presented in [26], where the tree decomposition must also include a bag containing the outer face. In the full version [47], we prove the following.

► **Lemma 17.** *Let G be an embedded graph. Then, $\text{cw}(\delta(G)) \leq \text{emw}^2(G) + 2\text{emw}(G)$.*

Cut-width. Let π be a linear order of the vertex set of a graph $G = (V, E)$. By splitting π into two linear orders π_1 and π_2 such that π is the concatenation of π_1 and π_2 , we define a *cut* of π . The *width* of this cut is the number of edges between a vertex in π_1 and a vertex in π_2 . The *width* of π is the maximum width over all its possible cuts. Finally, the *cut-width* of G is the minimum width over all the possible linear orders of V . The *dual cut-width* is the cut-width of the dual of G .

The following relationship between cut-width and carving-width has been proved in [52].

► **Theorem 18** (Theorem 4.3, [52]). *The carving-width of G is at most twice its cut-width.*

By Lemma 17 and Theorem 18, we have that single-parameter FPT algorithms also exist with respect to the embedded-width and to the dual cut-width of the underlying graph.

5 Conclusions

In this paper, we studied the C-PLANARITY TESTING problem for c-graphs with a prescribed combinatorial embedding. We showed that the problem is polynomial-time solvable when the dual carving-width of the underlying graph of the input c-graph is bounded. In particular, this addresses a question we posed in [26], regarding the existence of notable graph-width parameters such that the C-PLANARITY TESTING problem is fixed-parameter tractable with respect to a single one of them. Namely, we answer this question in the affirmative when the parameters are the embedded-width of the underlying graph, and the carving-width and cut-width of its planar dual.

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