Computing the Arrangement of Curve Segments: Divide-and-Conquer Algorithms via Sampling

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Abstract

We describe two deterministic algorithms for constructing the arrangement determined by a set of (algebraic) curve segments in the plane. They both use a divide-and-conquer approach based on derandomized geometric sampling and achieve the optimal running time $O(n \log n + k)$, where n is the number of segments and k is the number of intersections. The first algorithm, a simplified version of one presented in [1], generates a structure of size $O(n \log \log n + k)$ and its parallel implementation runs in time $O(\log^2 n)$. The second algorithm is better in that the decomposition of the arrangement constructed has optimal size $O(n + k)$ and it has a parallel implementation in the EREW PRAM model that runs in time $O(\log^{3/2} n)$. The improvements in the second algorithm are achieved by means of an approach that adds some degree of globality to the divide-and-conquer approach based on random sampling. The approach extends previous work by Dehne *et al.*[7], Deng and Zhu [8] and Kühn [9], that use small separators for planar graphs in the design of randomized geometric algorithms for coarse grained multicomputers. The approach simplifies other previous geometric algorithms [1, 2], and also has the potential of providing efficient deterministic algorithms for the external memory model.

1 Problem and Previous Work

We consider a classical problem in computational geometry: computing the arrangement determined by a set of curve segments in the plane. There has been a considerable amount of work on this problem in the computational geometry community, particularly for line segments. Starting with a first efficient algorithm by Bentley and Ottman [4], optimal output sensitive algorithms algorithms were obtained using a deterministic approach by Chazelle and Edelsbrunner [5] and using randomized approaches by Clarkson and Shor [6] and by Mumuley [10]. These optimal algorithms perform $O(n \log n + k)$ work, where n is the number of segments and k is the number of pairwise intersections. They can be adapted so that they are output sensitive even when multiple intersection points are allowed (a point where many segments intersect is counted only once). On the other hand, unlike its randomized counterparts in [6, 10], the deterministic algorithm in [5] can only handle line segments. An alternative deterministic algorithm by Amato *et al.*[1], which follows a divide-and-conquer approach based on derandomization of geometric sampling, has the advantage of being parallelizable. However, it can only handle line segments and pairwise intersection points, and the decomposition of the arrangement that it constructs has size $O(n \log \log n + k)$, as opposed to the optimal $O(n+k)$. One more variation on the problem is to report all the intersections while using only a linear amount of work space. The solutions in [6] and [1] can be adapted to achieve this. Alternatively, Balaban [3] proposed an elementary deterministic algorithm to achieve this; however, it does not construct the arrangement, it does not seem to parallelize, and it cannot handle multiple intersection points.

The algorithm in [1] uses an approach based on random sampling that refines iteratively by using small samples to divide the problem. This divide-and-conquer approach and also the well-known random incremental construction (RIC) approach date from work by Clarkson and Shor [6]. Unfortunately, unlike the RIC approach, divide-and-conquer most often leads to non-optimal algorithms, at least as far as the most basic analysis can tell, because the dividing step creates spurious boundaries that increase the complexity of the constructed decomposition of the arrangement. In fact, the literature is plagued with running times that are a factor n^ϵ or $\log^c n$ away from optimal. Some techniques have been used to correct this and obtain optimal algorithms: *sparse cuttings*, *pruning* and *biased sampling*. In particular, the algorithm in [1] achieves optimality through

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the use of a complicated pruning step that limits the total size of the decomposition.

2 New Results

We describe two deterministic algorithms for constructing the arrangement determined by a set of (algebraic) curve segments in the plane. They both use a divide-and-conquer approach based on derandomized geometric sampling and achieve the optimal running time $O(n \log n + k)$, where n is the number of segments and k is the number of intersections.

Our first algorithm (`Alg1`) is a simplified version of the algorithm in [1]. It follows a plain divide-and-conquer approach without the use of the *pruning* step used in [1]. As a result, it works for curve segments in addition to straight lines. The key is a more careful analysis that shows that the increase of size due to spurious boundaries is not too large. As the approach is given to parallelization and derandomization, this leads to a new work-optimal parallel algorithm that is simpler than the one in [1].

Our second algorithm (`Alg2`) introduces an approach that combines the advantages of the random incremental approach, which maintains a canonical decomposition of the complete arrangement, and of the divide-and-conquer approach, which recurses on each subproblem independently. We apply the approach to the segment intersection problem through the use of small separators for planar graphs. The resulting algorithm achieves optimal storage space $O(n + k)$, in contrast to `Alg1` and the previous algorithm in [1] which uses $O(n \log\log n + k)$ space. The approach is motivated by the use of graph separators in the design of some geometric algorithms for coarse grained multicomputers by Dehne *et al.*[7] (3-d convex hulls), by Kühn [9] (2-d Voronoi diagrams), and by Deng and Zhu [8] (Voronoi diagrams of line segments). The idea of the approach is to cluster subproblems (using a graph decomposition induced by small size separators) obtained through random sampling, so that the number of spurious boundaries is minimized. As opposed to the previous algorithms, [7, 8, 9], we combine the benefit of this clustering with the global information that can be deduced from the analysis of random sampling (while they use only a local bound, namely, that each subproblem is at most of certain size; we also make use of a global bound, namely, a bound on the sum of the sizes of all the subproblems).

The graph separator approach of `Alg2` also yields a parallel algorithm that is faster than previous ones: we achieve a running time $O(\log^{3/2} n)$ in the EREW PRAM model, in contrast to `Alg1`, and to the algorithm in [1], which have a running times of $O(\log^2 n)$. It also provides an efficient deterministic algorithm for computing a $(1/r)$-cutting of optimal size for an arrangement of segments.

The divide-and-conquer with partial clean-up approach also simplifies other algorithms (3-d convex hulls, 2-d abstract Voronoi diagrams, 3-d diameter, single face in an arrangement of segments [1, 2]), and leads to the same time speed-up for the corresponding parallel algorithms. These results will appear in a companion paper. We expect that the approach will find further applications. Specifically, in the design of deterministic geometric algorithms in the external memory model.

A complete version of this paper is available from the authors' web sites.

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