Matching Points to a Convex Polygonal Boundary

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Abstract

In this paper we give a new solution to a constrained polygon annulus placement problem for offset polygons, providing a new efficient primitive operation for computational metrology and dimensional tolerancing. Given a polygon P and a planar point set S, the goal is to find the smallest annulus region of P containing S. The constraint we are given is that inner polygon defining the annulus is fixed; we shrink the annulus by finding a minimum outer offset such that all the points are contained in the annulus region. We provide a solution to this problem that is both simpler and asymptotically faster than previous solutions.

1 Introduction

The research areas of computational metrology and dimensional tolerancing are focused on developing repertories of basic tests, such as for "roundness," "flatness," and angle conformity, so as to build a systematic collection of ways for determining if manufactured parts conform to their design specifications [Req, SV]. After a part is manufactured, its surface is sampled by a device known as a coordinate measuring machine (CMM) and these sample points are then tested against various design constraints to see if this part is conforming or not. The collection of tests that can be done simply and efficiently is therefore a limiting factor on the richness and sophistication of the constraints that designers can specify with confidence that their designs will be faithfully tested. Efficient methods for several computational metrology primitives have been presented in the algorithms and computational geometry literatures (e.g., see [AAS, Ch, BBBR, DP, DGR, LL, Lee, MSY, Ram, SSTY, SY, SJ, SLW]).

Computing optimal placements of annulus regions is a fundamental aspect of many computational metrology tests for quality control in manufacturing. For example, the width of the thinnest circular annulus containing a set of points is the measure used by the American National Standards Institute (see [F, pp. 40–42]) and by the International Standards Office for testing "roundness." The usual goal is to find, for a certain type of annulus region, a placement of the annulus that contains a given set or subset of points. Optimality of the placement can be measured either by *minimizing* the size of the annulus region necessary to contain all (or a certain number) of the points, or by *maximizing* the number of points contained in a fixed-size annulus. In addition to the tolerancing applications, these problems also arise in pattern matching and robot localization [GMR]. Thus we are interested in extending the collection of simple and efficient tolerancing tests to include new kinds of minimum or maximum annulus placement constraints.

One set of such problems studied recently by Barequet *et al.* in [BBDG] and [BBD] involves the optimal placement of *polygonal* annulus regions. Barequet *et al.* noted that the polygonal annulus can be defined as the difference region between two *offset* copies of P. In many applications, including those dealing with manufacturing processes, defining distance in terms of an *offset* from a polygon (either inward or outward) is more natural than scaling (which some have proposed as an alternate metric). This preference for offsetting is motivated from the fact that the relative error of a production tool (milling head, laser beam, etc.) is independent of the location of the produced feature relative to some artificial reference point (the origin). Thus, a tool is more likely to allow (and expect) local errors bounded by some tolerance, rather than scaled

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errors relative to some (arbitrary) center. For this reason, a study of the polygon offset operation, of the related distance function and its Voronoi diagram, and of annulus-placement problems for offset-polygons, are particularly interesting. Theoretical aspects of this distance function and Voronoi diagram were studied in [BDG] and [BBD], and were used in these papers as well as in [BBDG] in solutions to the offset versions of several problems involving one or the other definition of optimization given in the previous paragraph.

A constrained version of polygon annulus placement problem arises when either one of the two annulus boundaries (inner or outer) is fixed. Fixing the size of the inner or outer shape of an annulus is itself an important aspect of quality control. Consider, for example, manufacturing a part (like a cylinder) that must fit inside a sleeve. For the part that must fit inside the sleeve, the outer polygon defining the annulus has an absolute maximum which is fixed. For the sleeve, however, it is the polygon defining the inner boundary that is crucial and must be fixed. This leads to several problem definitions, including the following:

Problem 1: *Given a convex polygon P and a set of points S, find a translation τ that minimizes an outer offset O of P, such that all points in S lie in the annulus region between P and O.*

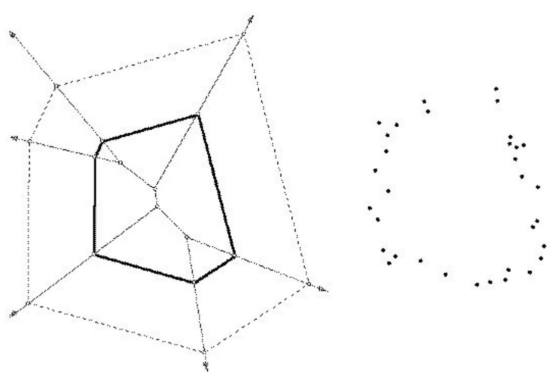

Figure 1: A sample offset polygon P and set of points to be tested.

In Figure 1, we show a sample polygon P (solid), an outer offset of P (dashed), and a set S of points (to the right). For reference, the straight-skeleton of P is also shown in light lines. In Figure 2, we see (in larger scale) a solution to Problem 1 for the given P and S of Figure 1. That is, we see an annulus region containing all the points of S, whose inner boundary is the fixed P, and whose outer boundary is the smallest offset of P such that S can be contained in the annulus.

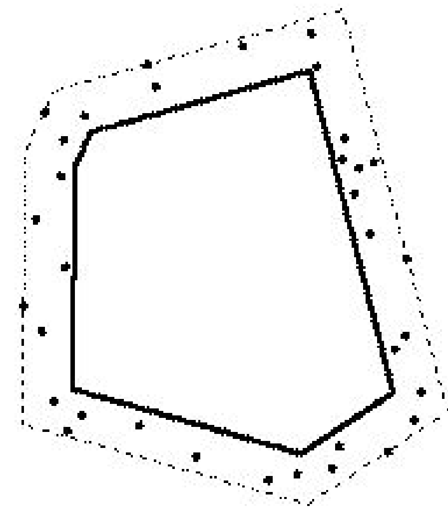

Figure 2: An optimal offset and placement for the polygon P for given set of points of Figure 1.

For circular annuli, it was shown by de Berg *et al.* in [BBBR] that when either the inner circular boundary or the outer circular boundary is of fixed size, the annulus minimization placement problem can be solved more efficiently than for the minimization problem when neither boundary is fixed. In [BBD], Barequet *et al.* extended those results to the problem of polygonal annuli for convex polygons. They gave several competing algorithms for both constrained (inner and outer) annulus problems, for annuli defined by either scaling or offsetting. When the annulus is defined by the offset operation, their algorithm requires $O(n(\log n + \log^2 m) + m(\log n + \log m))$ time for the problem when the outer polygon is fixed and the inner polygon is maximized, and $O(nm \log(nm))$ time for the problem when the inner polygon is fixed and the outer polygon is minimized (Problem 1 above). They also gave solutions to Problem 2 below, which can be considered a special case of Problem 1 when the inner polygon is empty, and is also an offset polygon version of the famous "smallest enclosing circle" problem:

Problem 2: *Given a convex polygon P and a set of points S, find the smallest offset translated copy* *of P containing all the points in S.*

In this paper we present an algorithm to solve Problem 1 in $O(m + n \log n \log m)$ time. In addition to improving on the results of [BBD] by almost a linear factor, our algorithm is also considerably simpler in that it does not use the offset distance function Voronoi diagram of [BDG]. Our algorithm uses some of the ideas developed by Barequet *et al.* for Problem 2, but does so in a way that extends to Problem 1 in a more efficient way.

1.1 Other Related Work

The polygon-offset operation was studied by Aicholzer *et al.* [AA, AAAG] in the context of a novel polygon skeleton, called the *straight skeleton*. They discussed the straight skeleton for both convex and simple polygons. Barequet *et al.* [BDG] also studied the polygon-offset operation in a different context: that of a new distance function and the related Voronoi diagram. They give efficient algorithms for computing compact representations of the nearest- and furthest-neighbor diagrams. Polygon offsets were also used in the solution to various annulus placement problems [BBD, BBDG]. Some of the approaches of [BBD] were extensions of the work of de Berg, *et al.* [BBBR] from circular to polygonal annuli, and also used the idea presented in [BBBR] of "feasible" regions of placement.

In this paper we adopt the terminology of [BBDG] and refer the reader to Section 1.3 of that paper for a more precise definition of the offset operation. Likewise, for a formal definition of the corresponding distance function, the reader is referred to [BDG]. We give here only a brief definition and description.

2 Offset Distance Functions: Preliminary Observations

The *outer* δ-offset of a convex polygon P is obtained by translating each edge $e \in P$ by δ in a direction orthogonal to e and by extending it until it meets the translations of the edges neighboring to e. In other words, the new edge e' lies on a line parallel to that containing e and at distance δ from it on the outside of P. The edge e' is trimmed by the lines parallel and at distance δ (outside of P) of the neighboring edges of e. We denote by $O_{P,\delta}$ the version of P offset outward by δ.

2.1 Feasible Translations

We now present both some terminology and some additional properties of the offset-polygon distance function. We first define what we mean by a *translation* of a polygon to a point. Throughout the paper we assume polygon P has a fixed reference point. For a given point q in the plane, by a translation of P to q we mean a translation of P and its reference point so that the reference point coincides with q. Similarly, when we speak of the *reflection* of P, we mean the rotation of P by π around its reference point. The translation of a reflection of P to a point q translates the polygon so that the reference of the reflected copy is mapped to q.

Note that for offset-polygons, the natural reference is the offsetting center, which is the point to which the inner polygon collapses when the polygon is offset inward. (In general, this point is the center of the medial axis of the polygon [BDG]. In the degenerate case, the offset center is a segment, so we arbitrarily select a point of it as the center, say, the median of the segment.)

We say that a polygon P *contains* point q if q lies on the boundary or in the interior of P. We say that a polygon P *properly contains* point q if q lies strictly in the interior of P (e.g., not on its boundary). The annulus region for a polygon P and a positive distance δ is defined as the set of points contained in the outer offset $O_{P,\delta}$ but not properly contained in P.

The following observations, reworded from [BBD], are fundamental in our algorithm:

Observation 1: *Given a polygon P and two points p and q, a translation of P to p contains q if and only if a translation of the reflection of P to q contains p.*

This observation follows from simple vector arithmetic, and leads to the following generalizations:

Observation 2: *A translation of a polygon P to a point q contains all the points of a set S if and only the intersection of the n copies of the reflection of P translated to the points of S contains q.*

Observation 3: *A translation of a polygon P to a point q properly contains none the points of a set S if and only q is not properly contained in the union of n copies of the reflection of P translated to the points of S.*

From the previous observations, we see that to solve Problem 1 we are interested in finding a translation τ of P and some $O_{P,\delta}$ (for a δ to be determined) such that τ lies in (e.g. maps the origin to) the intersection of n reflected copies of $O_{P,\delta}$, but not properly in the union of n reflected copies of P.

2.2 Efficiency of Some Substeps

The intersection of any number of copies of a convex m-gon has complexity $O(m)$. In [BBD], several algorithms were given to compute the intersection of n copies of a convex m-gon. Asymptotically, the most efficient of these required only $O(n \log h + m)$ time where h is the number of points on the convex hull of S.

Since translated copies of the same convex polygon together define a set of pseudo-disks, the union of n translated copies of a convex m-gon has complexity $O(nm)$, where "complexity" refers to the total number of edges (and vertices) possible on its boundary [KLPS]. The union may be stored more compactly, however, using an implicit representation. Consider the boundary of a convex polygon $P = (e_0, e_1, \ldots, e_{m-1})$ as being defined by a set of m edges listed in counter-clockwise order. We can represent any continuous portion of the boundary of P as (p, i, q, j), where p is the starting endpoint of the polygonal "arc," e_i is the edge containing p, q is the terminating endpoint of the polygonal "arc," and e_j is the edge containing q. If we consistently represent maximal continuous portions of copies of a convex polygon P in this way, then the bound of Kedem *et al.* [KLPS] regarding pseudo-disks implies that such a *compact representation* of the union of n translated copies of a convex m-gon can be stored in $O(n + m)$ space. Moreover, by a simple divide-and-conquer algorithm, we can construct such a compact representation in $O(m + n \log n \log m)$ time.

3 The Algorithm

We now present our algorithm for Problem 1. The idea is to find some δ large enough to guarantee a containing annulus translation. Based on this δ, we compute the intersection n reflected of $O_{P,\delta}$ translated to the points of S. We call this intersection $I_{P,\delta,S}$, or just I when P,S, and δ are clear from context. We also compute the union of n reflected copies of P translated to the points of S, which we denote $U_{P,S}$, or just U when P and S are clear from context.

Figure 3 shows a sample polygon P (solid) and an outer offset $O_{P,\delta}$ (dashed). Figure 4 shows (as a shaded simple polygon with a solid boundary) the union U of several copies of a reflection of P , and also (as a lighter grey polygon with its straight-skeleton) the intersection I of several reflected copies of the larger offset. (Note that the union polygon U is not necessarily a single polygon, but may be a collection of polygons with holes.)

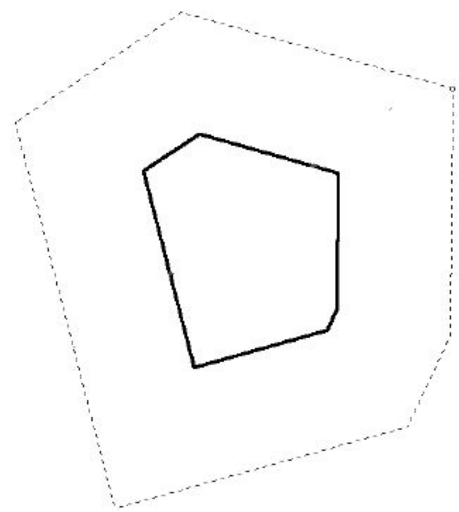

Figure 3: A sample polygon P and its offset $O_{P,\delta}$.

If q be any point that is contained in I but not in properly contained in U, then a translation to q of the original P and $O_{P,\delta}$ gives a containing placement of the annulus region for set S. However it is not yet a solution to Problem 1 because δ is not minimized. What we want to do is to shrink δ back down to the smallest value such that I has a nonempty intersection with the boundary or exterior of U.

This leads to the following sketch of an algorithm:

Algorithm 1

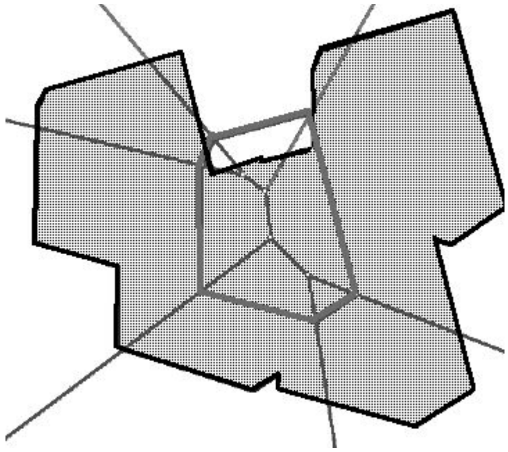

Figure 4: The union $U_{P,S}$ (shaded polygon with a solid boundary) and intersection $I_{P,\delta,S}$ (lighter grey polygon with its straight-skeleton) for a polygon P.

1. Compute an outer offset $O_{P,\delta}$ of P for some $\delta > 0$ large enough that there exists a placement of the annulus region between P and $O_{P,\delta}$ containing S.

2. Compute the intersection $I_{P,\delta,S}$ of the n reflected copies of $O_{P,\delta}$ translated to the points of S.

3. Compute the union U of n reflected copies of P translated to the points of S.

4. Find the minimum value of $\delta_{\min}$ such that $I_{P,\delta_{\min},S}$ contains a point exterior to or on the boundary of U.

Before giving more precise detail about the algorithm, we make a few further observations about the first and final steps. In the first step we need to compute a δ large enough that $I_{P,\delta,S}$ is not empty–that is, a δ that guarantees an annulus region large enough to contain S. (We can't shrink the annulus down if it isn't big enough to start with.) To this end we note that at any vertex v of P, there is a semi-circle of radius δ centered at v that lies in the annulus region between P and $O_{P,\delta}$. So let w be the width of some boundary square whose sides are parallel to the $x-$ and $y-$ axes and that contains all of S. Then for $\delta = w\sqrt{5}/2$ there is a semi-circle centered at the leftmost vertex of P that lies in the annulus region between P and $O_{P,\delta}$ and which is large enough to contain the bounding square around S.

Now we consider the final step. If we offset $I_{P,\delta,S}$ inward by some amount, say α, the resulting polygon is simply $I_{P,S,(\delta-\alpha)}$, the intersection polygon that would have resulted if the original outer offset had been $\delta - \alpha$ instead of δ. So in order to compute the minimal outer offset $\delta_{\min}$, we really need only compute the value of α that determines how far inward the polygon $I_{P,\delta,S}$ can be offset.

This leads to a further observation. Equivalent to offsetting I inward until it no longer contains a point that is not properly contained in U, we could compute the straight-line skeleton of I and consider offsetting I outward from its center until it contacts the first point that is not properly contained in U. This leads to the following:

Lemma 4: *Let the center c of $SLL(I)$ be inside U. Consider an expanding offset of I that begins at c and grows outward. Then there is some point x such that x is a first point on the boundary of U hit by this expanding offset, and x is either a reflex vertex of U or is on the intersection of $SLL(I)$ and the boundary of U.*

Proof. We prove by contradiction. Let x be a first point on the boundary of U that is hit by the expanding offset of I. Suppose that x is neither a reflex vertex of U, nor a point on $SLL(I)$. It follows that x falls on some edge e_i of the expanding I but not on a vertex of I (since the vertices of I move outward along $SLL(I)$). Suppose x is on a vertex of U. By our assumption, it is not a reflex vertex, and so it is a convex vertex with respect to the interior of U. Thus at least one edge of U adjacent to x is interior to I at the moment the expanding e_i contacts x, but this would mean that e_i intersected that edge before intersecting x which is a contradiction of our assumption that x is an initial contact.

Suppose instead, then, that x is on an edge e_u of U, but not on a vertex of U. If e_u is not parallel to e_i, then one direction along e_u is closer to the inside of I and therefore e_i will intersect e_u before it reaches x, which is a contradiction. However if e_i and e_u are parallel, then the initial point of contact is a segment one of whose endpoints is either a vertex of e_i (and thus on $SLL(I)$ which is a contradiction) or is a convex vertex of e_u which we assumed was not the case. **End Proof.**

We now present a more detailed version of Step 4 of Algorithm 1, enhancing the details of the final step.

Step 4: Expanded

4. (a) Compute $SLL(I)$, the straight-line skeleton of I. Let c be the center of $SLL(I)$.

   (b) Determine whether c is properly contained in U. If it is not, then we are done. We let α be the amount by which we offset I inward until it degenerates to a point c. Then our $\delta-\alpha$ is the width of the smallest annulus, and c is the translation of P and $O_{P,\delta-\alpha}$ that contains S.

   (c) If c is not properly contained in U, then we compute (using Lemma 4) the smallest inner offset α of I that contains a point x not properly contained in U. Our optimal annulus region is $\delta - \alpha$ and its containing translation is given by x.

3.1 Analysis

Step 1 requires $O(n+m)$ time: $O(n)$ time is required to compute a bounding square of S and $O(m)$ time to offset P by this much. In step 2, we can compute the intersection of n copies of a convex polygon in $O(n \log n + m)$ time, or $O(n \log h + m)$ time where h is the size of the convex hull of S [BBD].

In step 3, the union of n copies of a convex polygon has complexity $O(nm)$ and an explicit version can be computed in time $O(nm \log(nm))$ using a divide-and-conquer strategy. The straight-line skeleton of I in step 4 can be computed in time $O(m)$ using the technique of [AGSS].

The last two parts of step 4 are the most complex. We can perform point location of c in time $O(log(nm))$. We then use ray-shooting for each of the m edges of $SLL(I)$ to determine where it intersects U. Conversely, we test each of the n reflex vertices of U to determine which region of $SLL(I)$ it falls in and then compute the offset at which the edge sweeps through it.

3.2 Improving the Algorithm with a Compact Diagram

The run time of Algorithm 1 can be improved by almost a linear factor (in the case when m and n are both large) by using a compact representation of $U_{P,S}$: the union of the n copies of the reflection of P. Note that $U_{P,S}$ has complexity $O(nm)$, but only $O(n)$ of those vertices are reflex vertices representing the intersection of the boundaries of two reflected copies of P, since the copies of P form a family of n pseudo-disks [KLPS]. Furthermore, all reflex vertices of $U_{P,S}$ are of these $O(n)$ intersection-type vertices. The rest of the vertices are from some copy of the reflection of P.

We want to compute a representation of $U_{P,S}$ that explicitly stores only these intersection vertices. As noted in Section 2.2, the portions of $U_{P,S}$ in between these intersection vertices are just parts of chains of a copy of a reflection of P and are stored implicitly with two points that specify what portion of a chain of which copy. This compact diagram U^* can be computed in $O(m + n \log n \log m)$ time, by using a simple divide-and-conquer strategy (where in the merge step we compute crossing points between two copies of P using appropriate "binary search" strategies). The reflex vertices needed in Step (4c) (see Lemma 4) are explicitly stored in U^*. It is only slightly more complex to compute the intersection of $SLL(I)$ with $U_{P,S}$. We do a $O(\log n)$ time ray-shooting query on U^* to determine which portion of a polygon the ray from $SLL(I)$ passes through, and then a second $O(\log m)$ time ray-shooting query on that particular portion of a polygon. (Note that these are not nested steps, in that we don't need to do the second ray shooting query until we know which region of U^* we are searching.)

Thus, we have shown the following:

Theorem 5: *Given a convex m-gon P and a set S of n points in the plane, we can determine the translation for the minimum outer offset of P that contains all the points of S in $O(m + n \log n \log m)$ time.*

4 Summary and Open Problems

We have given an $O(m + n \log n \log m)$ time algorithm for Problem 1, finding the smallest constrained annulus containing a set S of n points, where the annulus is defined by a convex m-gon P and the offset operation, and the inner boundary of the annulus is fixed. This algorithms is both simpler than previous approaches (not requiring the complex offset-distance function Voronoi diagram) and asymptotically faster.

Several open problems remain:

Problem 3: *OPEN Give a theoretical lower bound on the asymptotic run time required to solve Problem 1.*

Problem 4: *OPEN Give efficient solutions for these annulus placement problems when the annulus is defined by a simple polygon (not necessarily convex).*

Problem 5: *OPEN Give efficient solutions for Problem 1 for polyhedra in 3-space.*

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