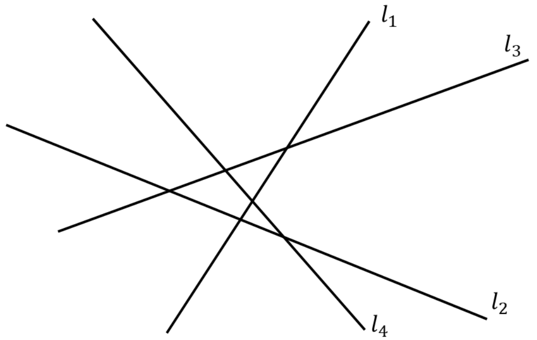
Computational Geometry



Line Arrangements

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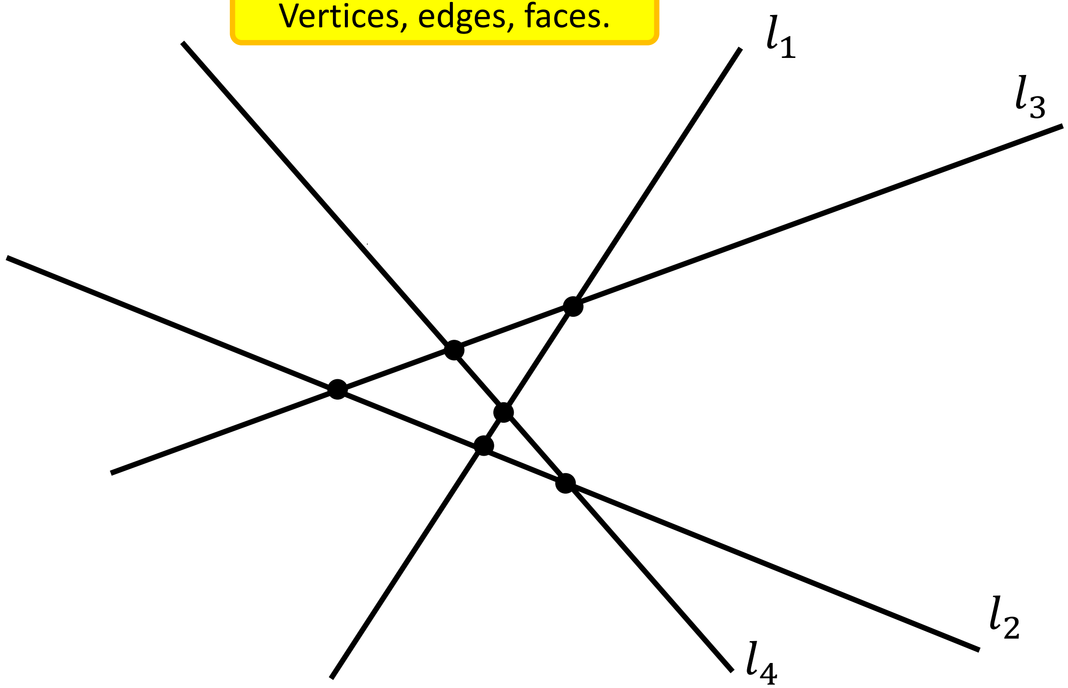
with slides from Carola Wenk

Arrangement of Lines

Lines. Not line segments.

Let $\mathrm{L} = \{l_1, \dots, l_n\}$ be a set of n lines in $\mathbb{R}^2$. Then $\mathcal{A}(L)$ is called the arrangement of L. It is defined as the planar subdivision induced by all lines in L.

Vertices, edges, faces.



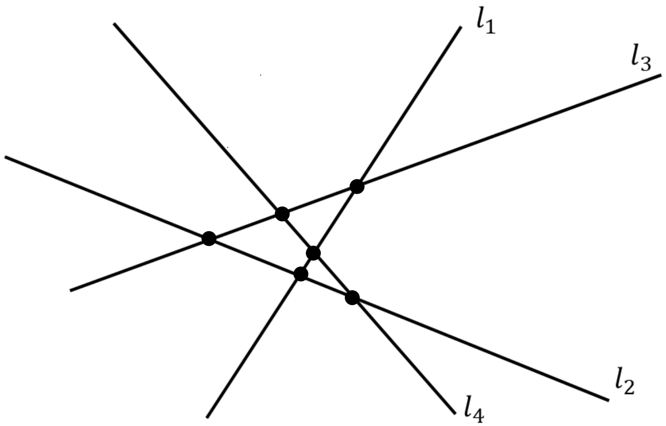
$\mathcal{A}(L)$ is simple if no three lines meet in one point, and no two lines are parallel.

Arrangement Complexity

Because it's a planar subdivision.

The complexity of $\mathcal{A}(L)$ is its #vertices + #edges + #faces.

$\mathrm{L} = \{l_1, \ldots, l_n\}$



#vertices $\leq \binom{n}{2} = \frac{n(n-1)}{2}$

We have n lines, and in the worst case every pair intersects (vertex).

#edges $\leq n^2$

On one line we can have at most $n-1$ vertices, so at most n edges total.

#faces $\leq \frac{n^2+n+2}{2}$

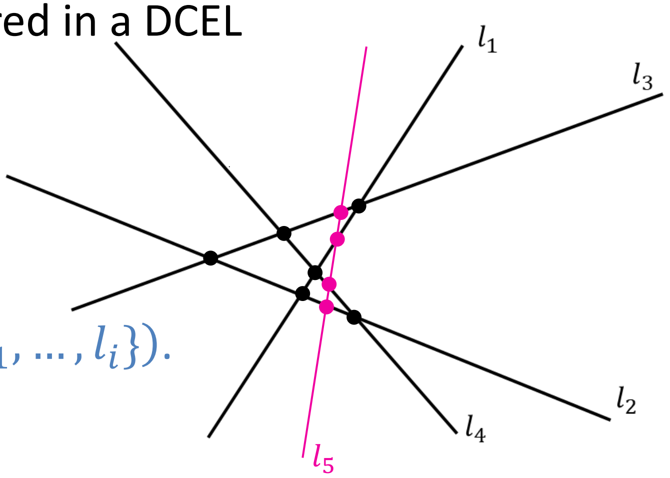
Consider an incremental construction, just for counting purposes right now. Let $\mathcal{A}_i = \mathcal{A}(\{l_1, \ldots, l_i\})$. Line l_i splits a face of $\mathcal{A}_{i-1}$ in two. This creates i additional faces (since l_i has at most i edges in $\mathcal{A}_i$ see above).

$\Rightarrow$ #faces in$\mathcal{A}(L) = \mathcal{A}(\{l_1, \ldots, l_n\}) = 1 + \sum_{i=1}^{n} i = 1 + \frac{n(n+1)}{2}$

Arrangement Construction

Input: A set of n lines $\mathrm{L} = \{l_1, \ldots, l_n\}$ in $\mathbb{R}^2$

Output: The arrangement $\mathcal{A}_n = \mathcal{A}(\{l_1, \ldots, l_n\})$ stored in a DCEL



1) Sweep-line construction:

Takes $O(n^2 \log n)$ time.

2) Incremental construction:

Insert one line after the other. Again let $\mathcal{A}_i = \mathcal{A}(\{l_1, \ldots, l_i\})$.

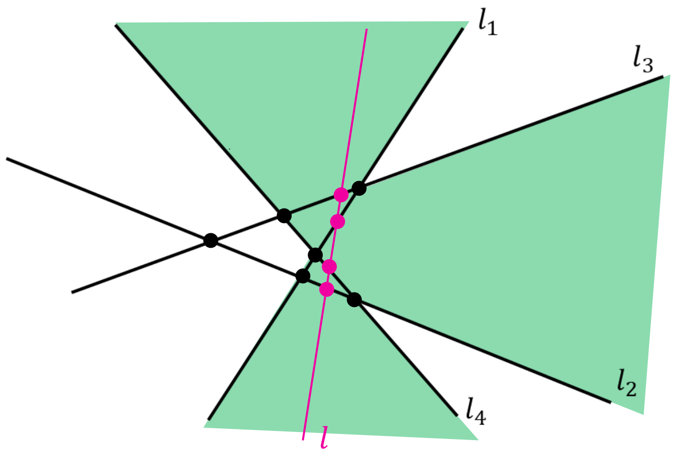
Construct_arrangement($\mathrm{L} = \{l_1, \ldots, l_n\}$){
    $\mathcal{A}_0$ = whole plane
    for i=1 to n{
$O(i)$ $\mathcal{A}_i$ = insert l_i into $\mathcal{A}_{i-1}$ by threading l_i through $\mathcal{A}_{i-1}$ face by face and splitting edges and faces accordingly (using the DCEL!).

Using "zone theorem"

Runtime: $O\left(\sum_{i=1}^{n} i\right) = O(n^2)$

Zone Theorem

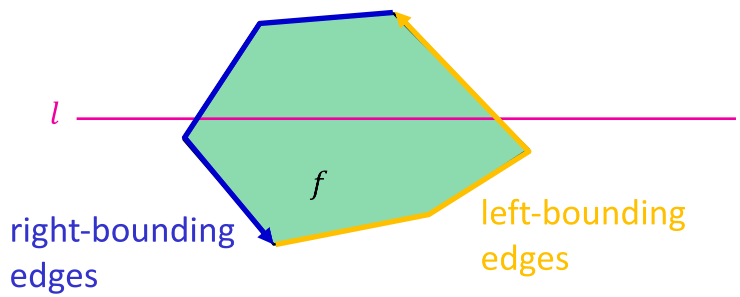
Zone Theorem: Let $\mathcal{A}$ be an arrangement of n lines and let l be another line. The *zone* of l in $\mathcal{A}$ is the planar subdivision consisting of all faces, edges, and vertices intersected by l. The complexity of the zone of l in $\mathcal{A}$ is $O(n)$.



How can the zone have complexity $O(n)$ if $\mathcal{A}$ has complexity $O(n^2)$?

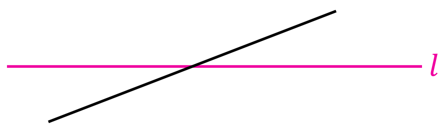
Zone Theorem Proof

Assume l is horizontal. Also assume $\mathcal{A}$ is simple and as no horizontal edges

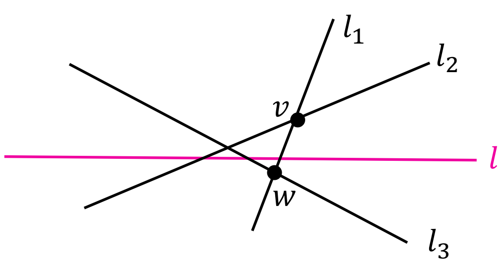


Goal: Prove that # left-bounding edges in the zone is $\leq 3n$, using induction.

- Base: $n = 1$



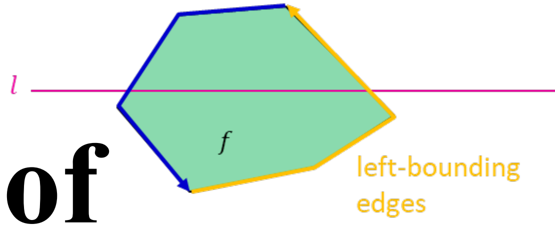
- Step: $n - 1 \rightarrow n$
  Let L be a set of n lines and $\mathcal{A}(L)$ its arrangement.



l_1 = line that has the **rightmost** intersection with l

v = vertex on l_1 **above** l, closest to l

w = vertex on l_1 **below** l, closest to l



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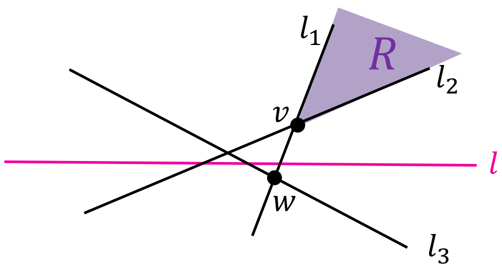

l_1 = line that has the **rightmost** intersection with l
v = vertex on l_1 **above** l, closest to l
w = vertex on l_1 **below** l, closest to l

Think about $\mathcal{A}(L \setminus \{l_1\})$.
$\Rightarrow$ # left-bounding edges in $\mathcal{A}(L \setminus \{l_1\})$ is $\leq 3(n-1)$ by inductive hypothesis.

Now insert l_1 into $\mathcal{A}(L \setminus \{l_1\})$:

$\Rightarrow$ $\overline{vw}$ is a new edge, and two edges were split into two.
$\Rightarrow$ 3 new edges
$\Rightarrow$ No more new edges: Region R is not in $zone(l)$ but is the only part of l_1 above v that could contribute with left-bounding edges.
$\Rightarrow$ In total, the zone has $3(n-1) + 3$ edges

For l in $\mathcal{A}(L \setminus \{l_1\})$