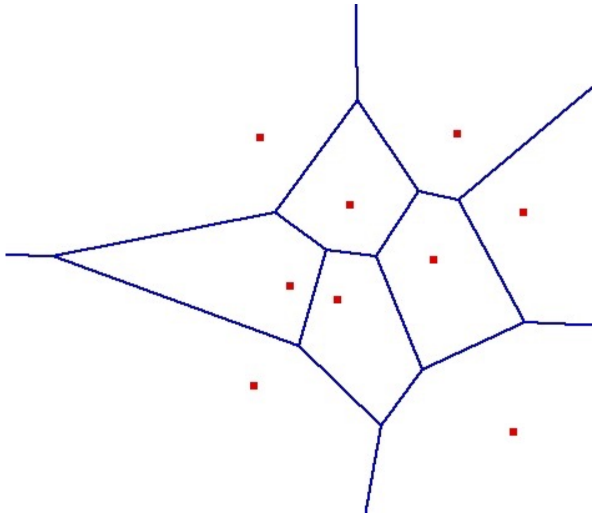
Computational Geometry



Voronoi Diagrams and Their Applications

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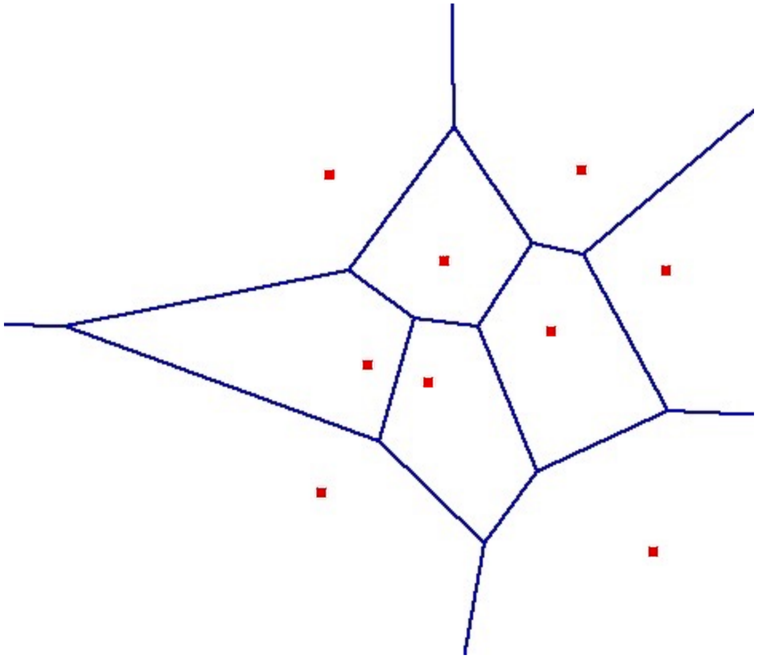
with slides from Carola Wenk and Rodrigo I. Silveira

Voronoi Diagram
(Dirichlet Tesselation)

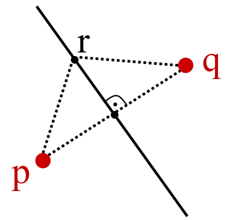
- **Given:** A set of point sites $P = \{p_1, \ldots, p_n\} \subseteq R^2$
- **Voronoi cell:** Partition R^2 into Voronoi cells
  $V(p_i) = \{q \in R^2 | \; d(p_i, q) < d\big(p_j, q\big)$ for all $j \neq i\}$
- **Voronoi diagram:** The planar subdivision obtained by removing all Voronoi cells from R^2.



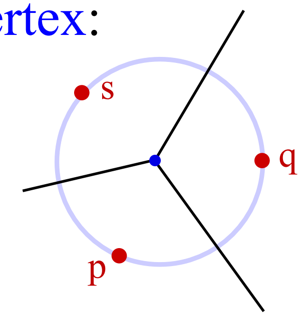
Bisectors

- Voronoi edges are portions of bisectors
- For two points p, q, the bisector $b(p,q)$ is defined as

$$b(p,q) = \{r \in R^2 \mid d(p,r) = d(q,r)\}$$



- Voronoi vertex:

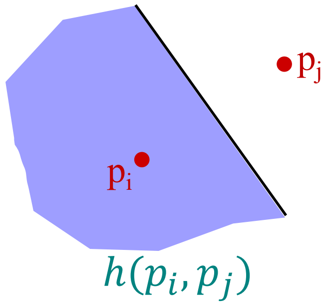


Voronoi cell

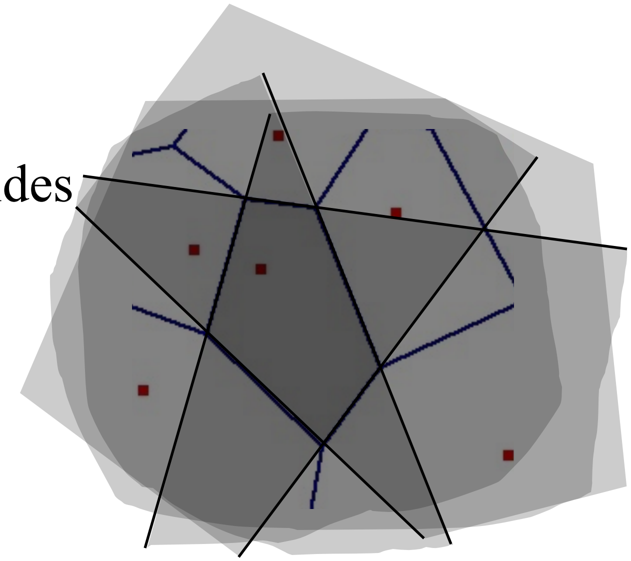
- Each Voronoi cell $V(p_i)$ is convex and

$$V(p_i) = \bigcap_{\substack{p_j \in P \\ j \neq i}} h(p_i, p_j),$$

where $h(p_i, p_j)$ is the halfspace that is defined by bisector $b(p_i, p_j)$ and that contains p_i



$\Rightarrow$ A Voronoi cell has at most $n-1$ sides



Voronoi Diagram

The Voronoi diagram is a planar, embedded, connected graph with vertices, edges (possibly infinite), and faces (possibly infinite).

- **Theorem:** Let $P = \{p_1, \ldots, p_n\} \subseteq R^2$. Let n_v be the number of vertices in $VD(P)$ and let n_e be the number of edges in $VD(P)$. Then

  $n_v \leq 2n - 5$, and

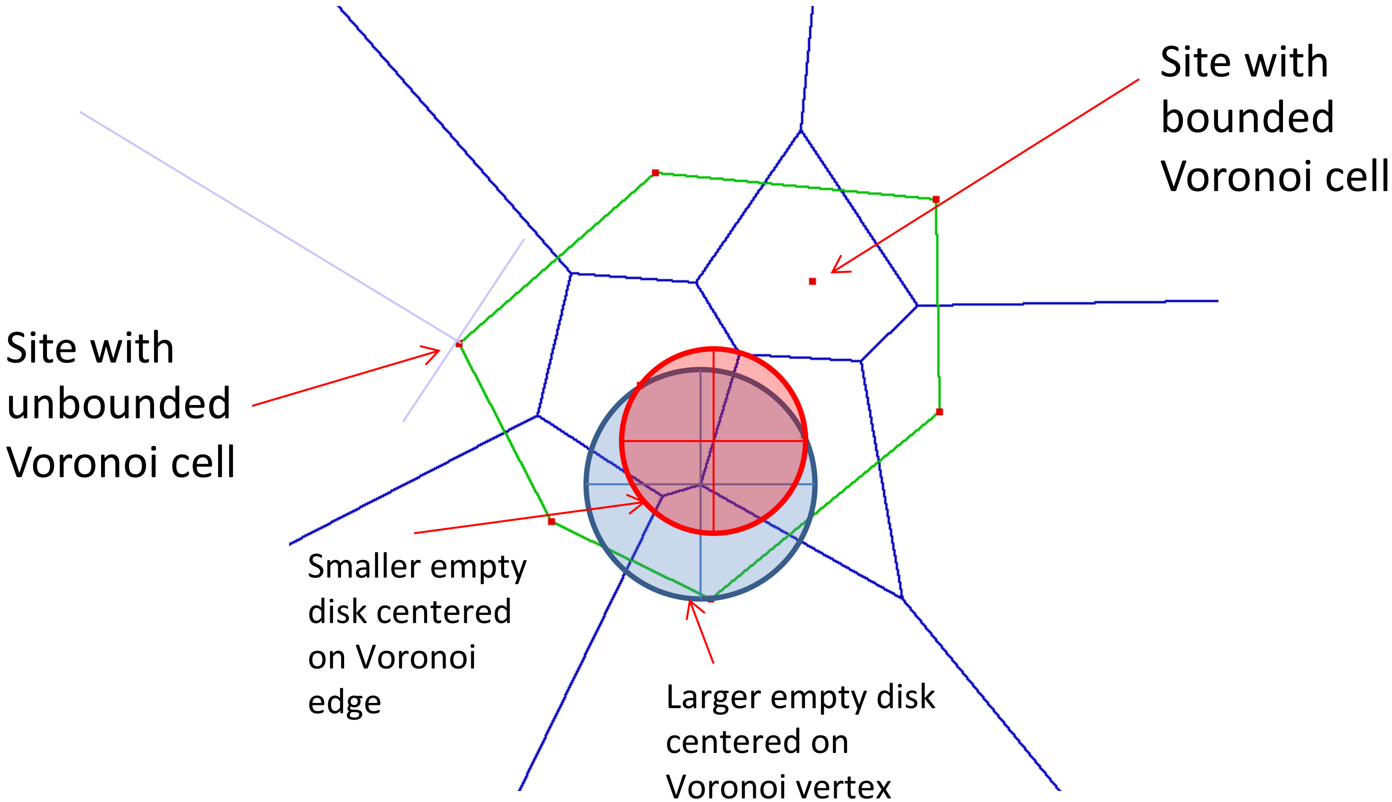
  $n_e \leq 3n - 6$

Add vertex at infinity

Proof idea: An application of Euler's formula $n_v + 1 - n_e + n = 2$ with "=" because the planar graph is connected, and $2n_e = \sum_{v \in V} \deg(v) \geq 3(n_v + 1)$.

Properties

1. A Voronoi cell $V(p_i)$ is unbounded iff p_i is on the convex hull of the sites.
2. v is a Voronoi vertex iff it is the center of an empty circle (disk) that passes through three sites.



Applications of Voronoi Diagrams

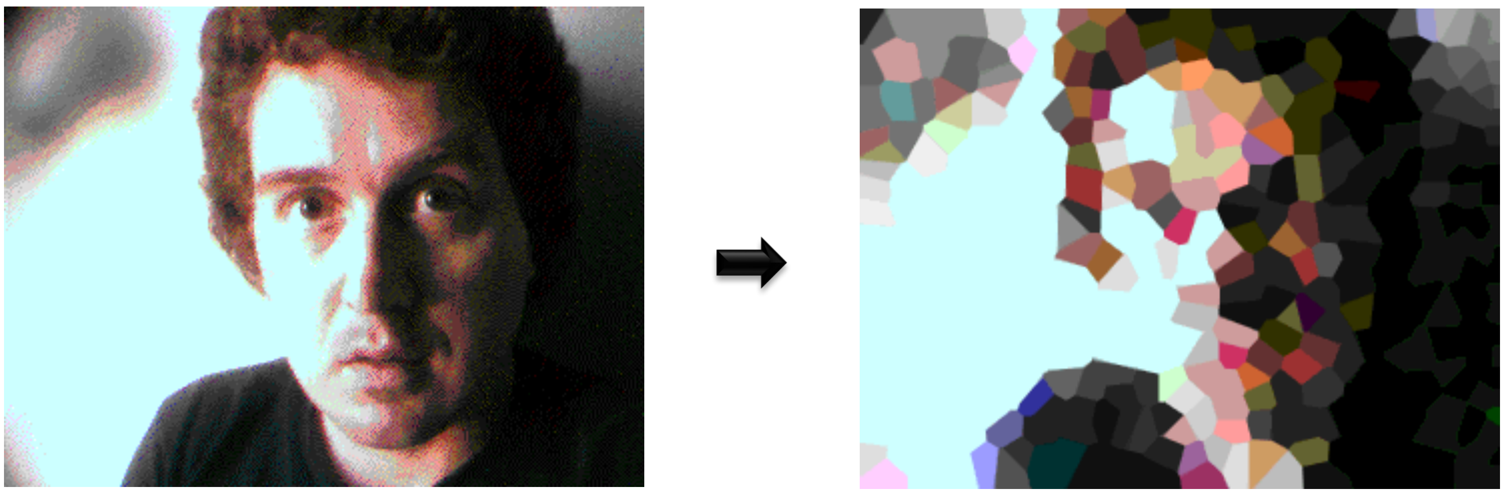
- Nearest neighbor queries:
  - Sites are post offices, restaurants, gas stations
  - For a given query point, locate the nearest point site in $O(\log n)$ time
    $\rightarrow$ point location

- Closest pair computation:
  - Naïve $O(n^2)$ algorithm; sweep line algorithm in $O(n \log n)$ time
  - Each site and the closest site to it share a Voronoi edge
    $\rightarrow$ Check all Voronoi edges (in $O(n)$ time)

- Facility location: Build a new gas station (site) where it has minimal interference with other gas stations
  - Find largest empty disk and locate new gas station at center
  - If center is restricted to lie within $CH(P)$ then the center has to be on a Voronoi edge

What can you do with a Voronoi diagram?

- All sort of things!

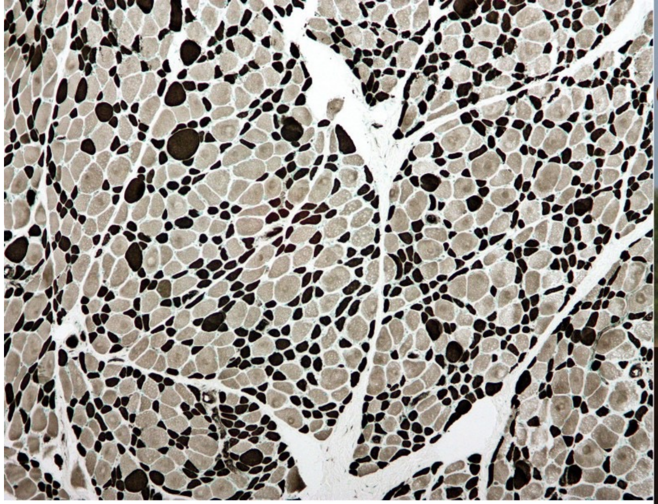


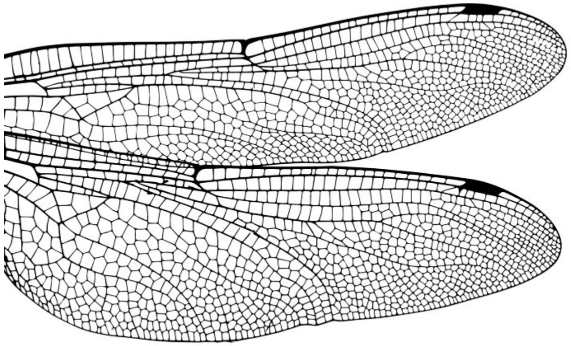
Source: http://www.ics.uci.edu/~eppstein/vorpic.html

Voronoi Diagrams in Nature

- Models for territories, spreading, or growth
- From top-left to bottom-right:
  - muscle cross-section
  - giraffes coat patterns
  - wings of a dragonfly
  - garlic bulb
  - corn cob
  - jackfruits

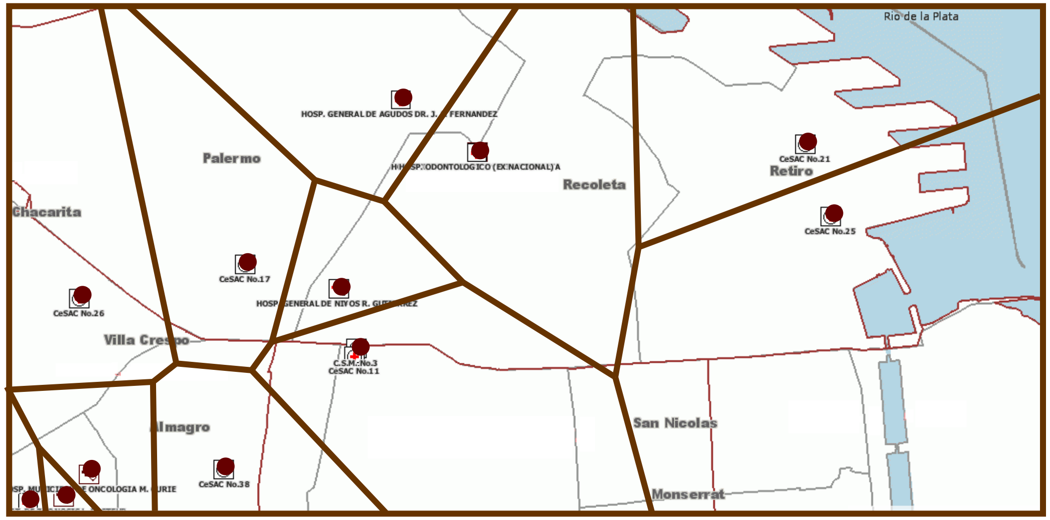
Image from https://towardsdatascience.com/the-fascinating-world-of-voronoi-diagrams-da8fc700fa1b





What other useful things can you do with a Voronoi diagram?

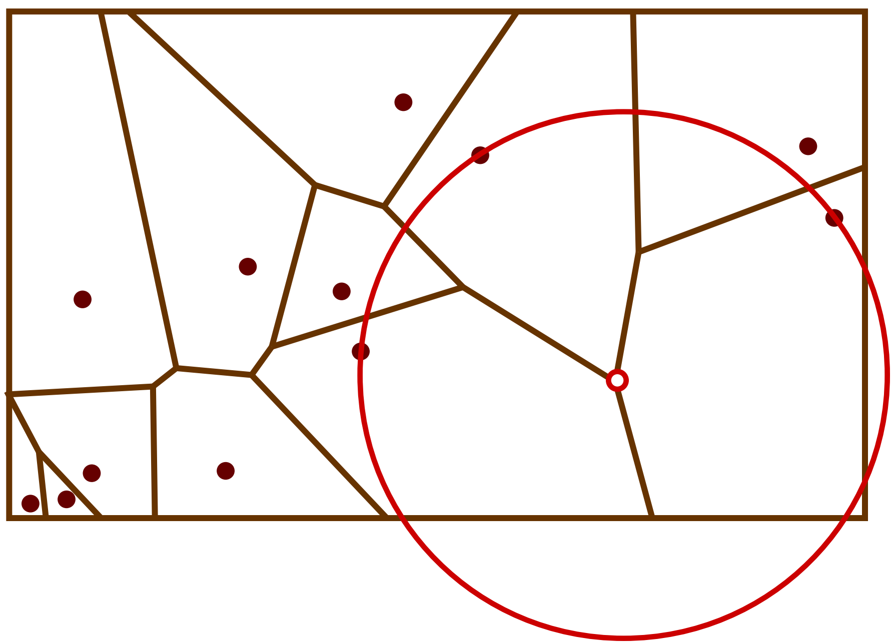
- Already mentioned a few applications
  - Find nearest… hospital, restaurant, gas station,...



Facility location

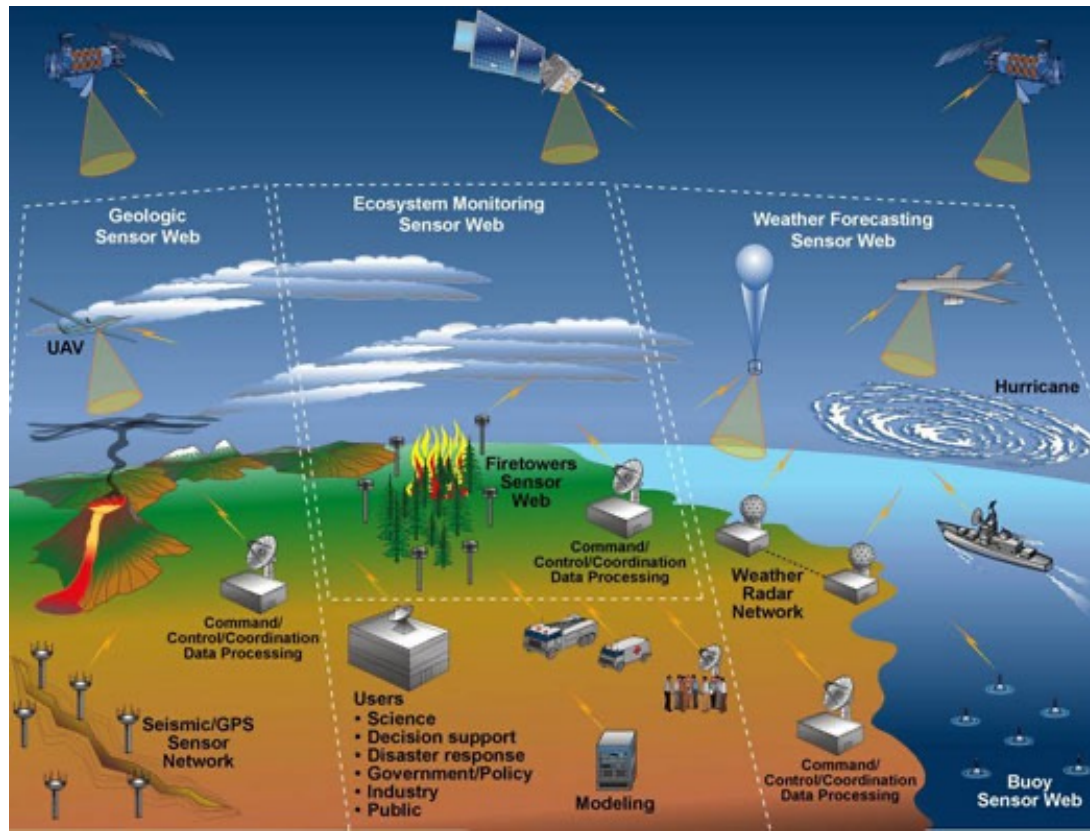
- Determine a location to maximize distance to its “competition”
- Find **largest empty circle**

- Must be centered at a vertex of the Voronoi
- diagram



Coverage in sensor networks

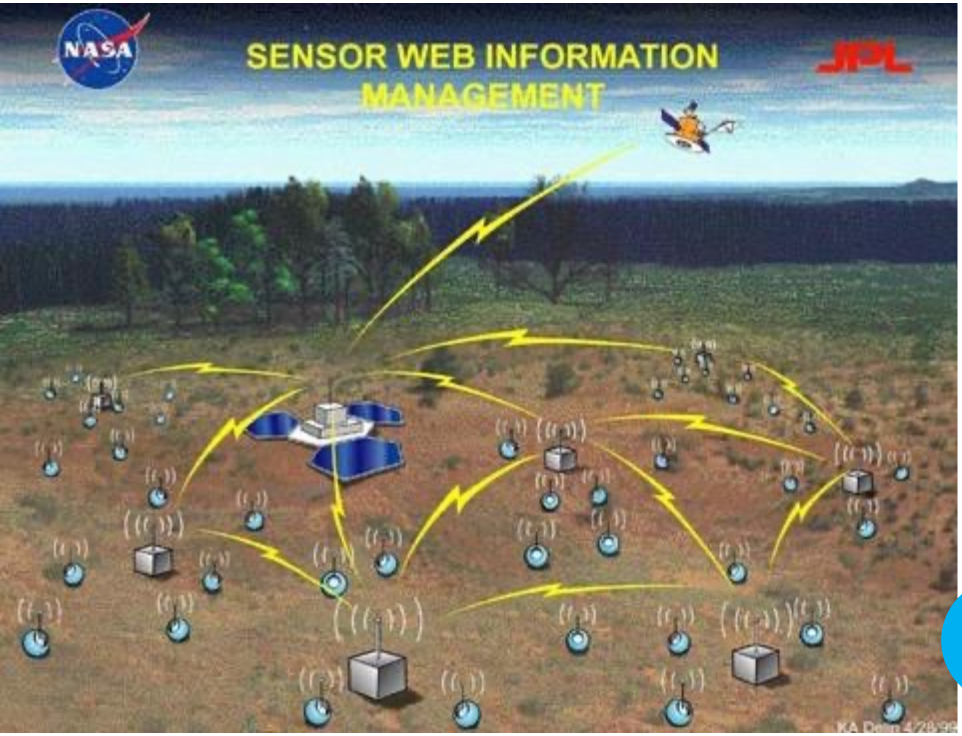
- Sensor network
  - Sensors distributed in an area to monitor some condition



Source: http://seamonster.jun.alaska.edu/lemon/pages/tech_sensorweb.html

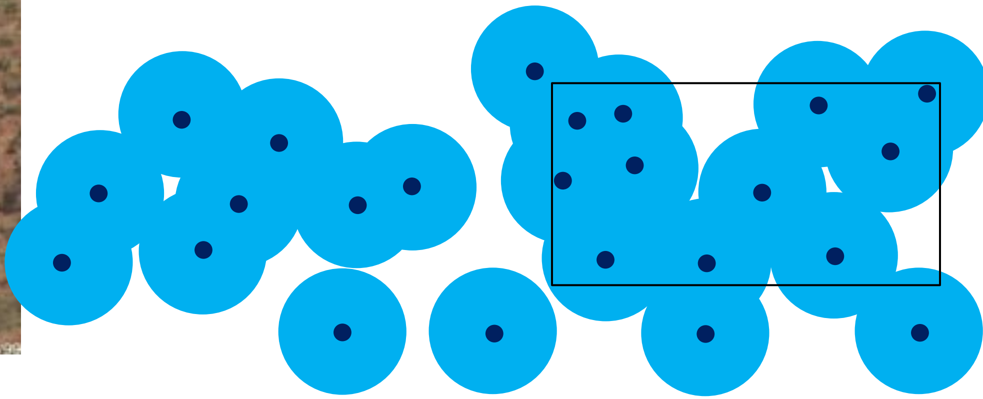
Coverage in sensor networks

- Given: locations of sensors
- Problem: Do they cover the whole area?



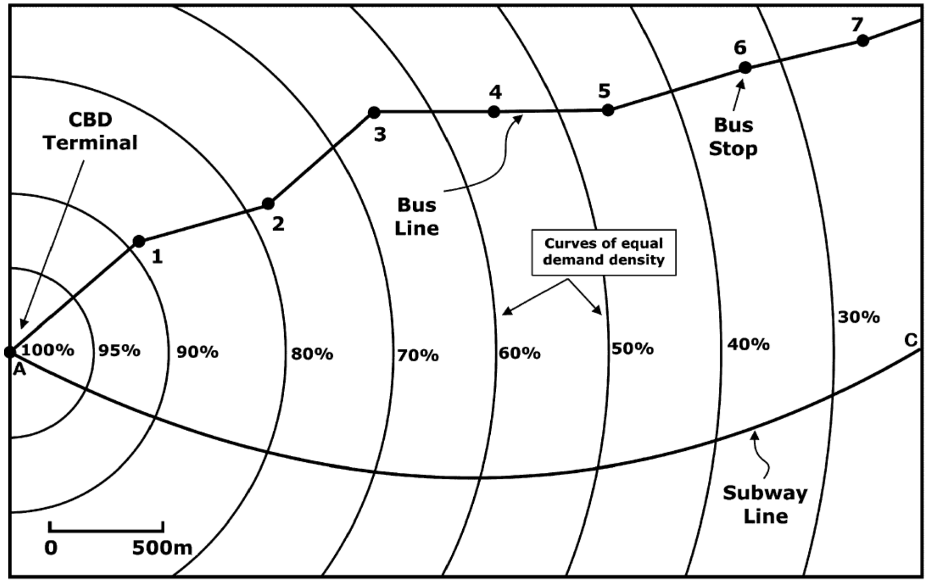
Assume sensors have a fixed coverage range

Solution: Look for largest empty disk, check its radius



Building metro stations

- Where to place stations for metro line?
  - People commuting to CBD terminal

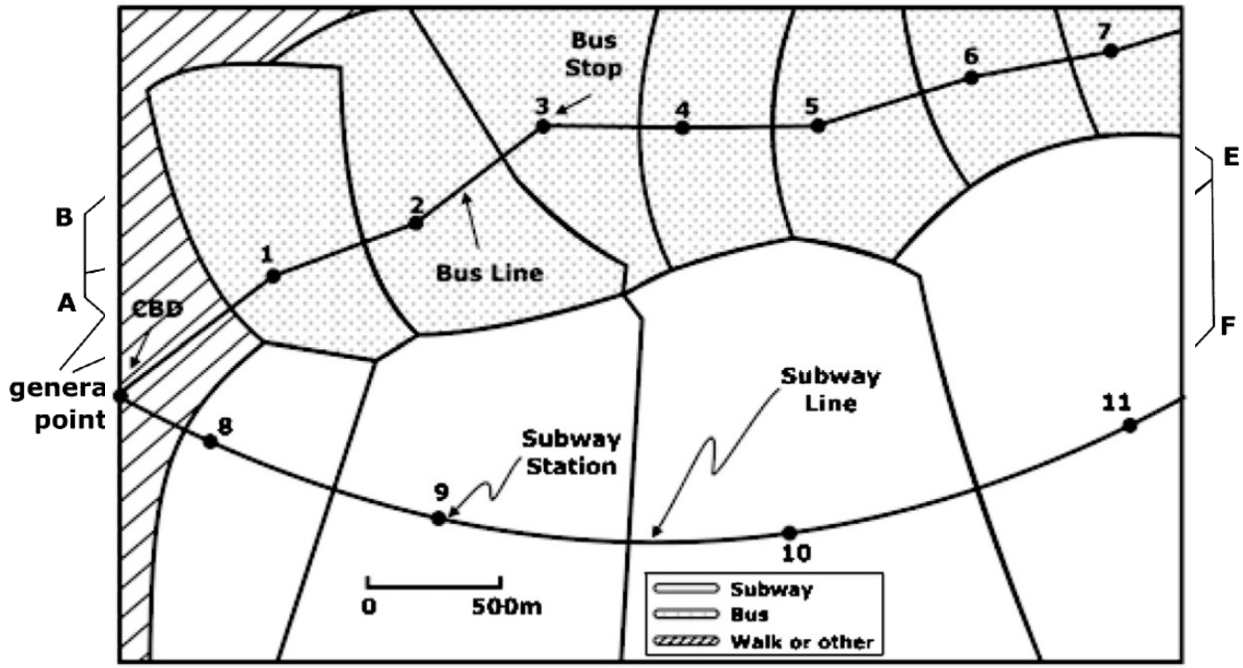


Source: Novaes et al (2009). DOI:10.1016/j.cor.2007.07.004

- People can also
  - Walk
    - 4.4 km/h +
    - 35% correction
  - Take bus
    - Some avg speed

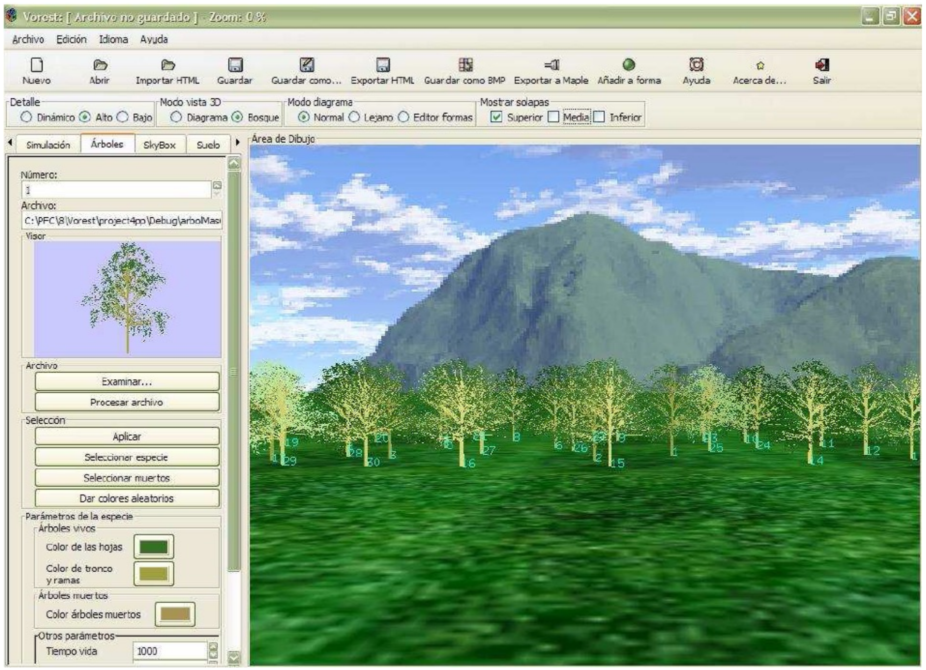
Building metro stations

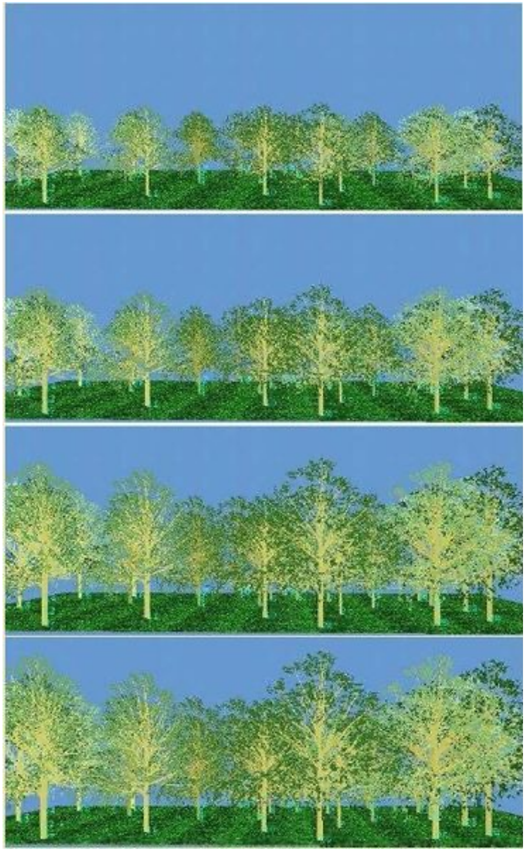
- Weighted Voronoi Diagram
  - Distance function is not Euclidean anymore
  - $dist_w(p,site)=(1/w)\ dist(p,s)$



Forestal applications

- VOREST: Simulating how trees grow

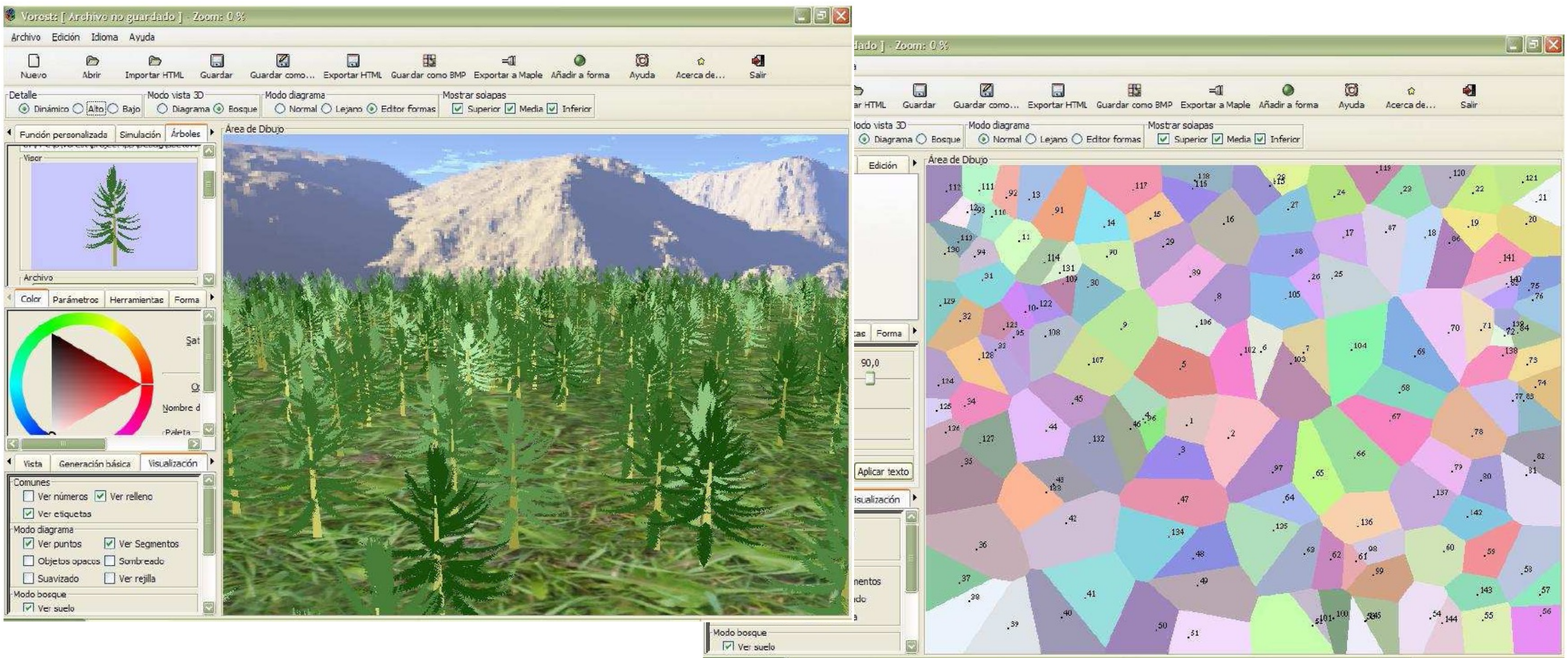




More info: http://www.dma.fi.upm.es/mabellanas/VOREST/

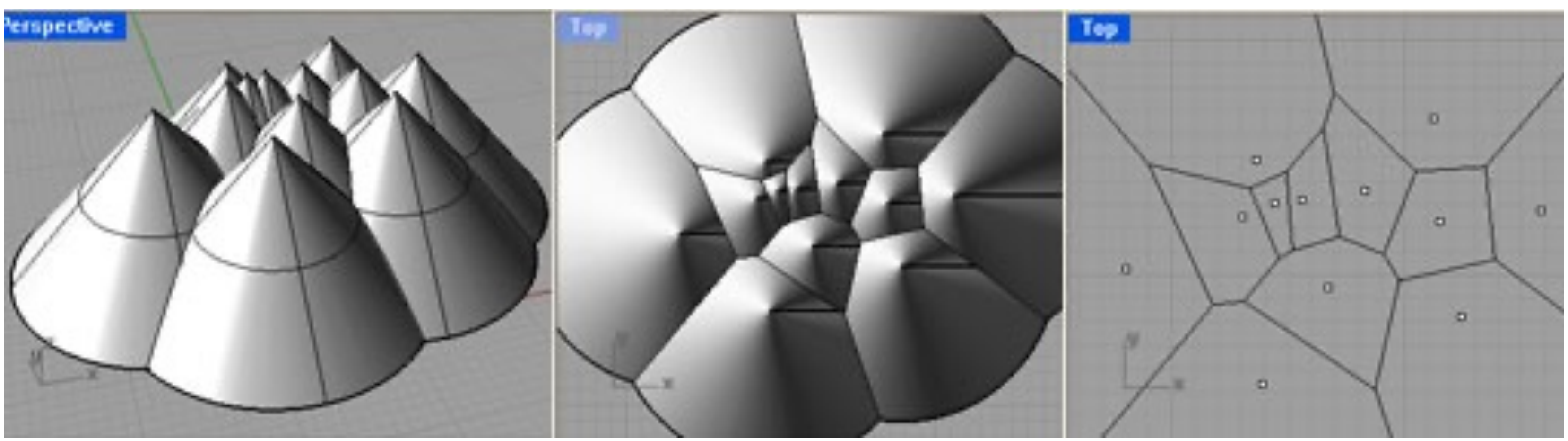
Simulating how trees grow

- The growth of a tree depends on how much “free space” it has around it



Lower envelopes of cones

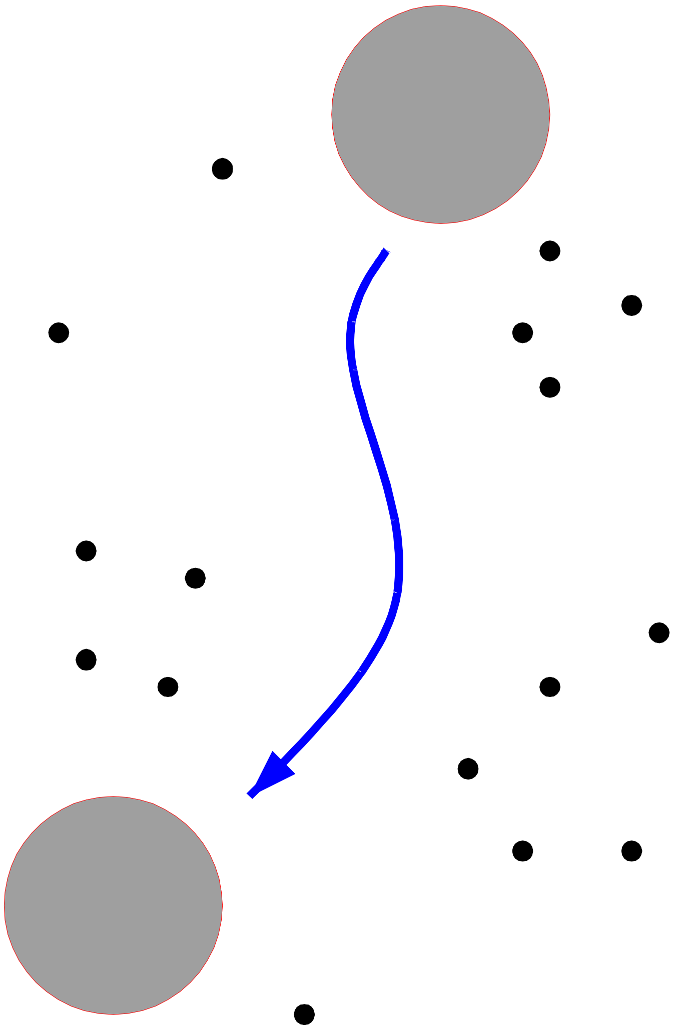
- Alternative definition of Voronoi diagram:
  - 2D projection of lower envelope of distance cones centered at sites



Robot motion planning

- Move robot amidst obstacles

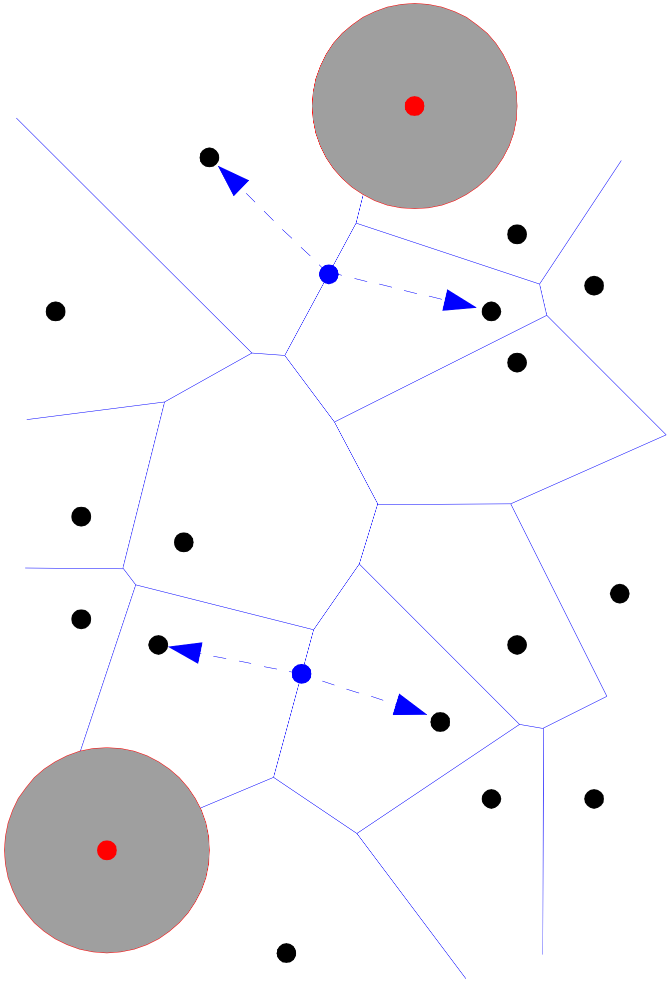
- Can you move a disk (robot) from one location to another avoiding all obstacles?



Most figures in this section are due to Marc van Kreveld

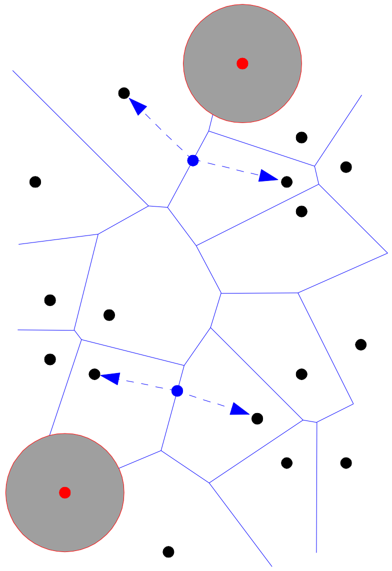
Robot motion planning

- Observation: we can move the disk if and only if we can do so on the edges of the Voronoi diagram
  - edges are (locally) as far as possible from sites



Robot motion planning

- General strategy
  - Compute Voronoi diagram of obstacles
  - Remove edges that get too close to sites
    - i.e. on which robot would not fit
  - Locate starting and end points
  - Move robot center along VD edges



- This technique is called retraction