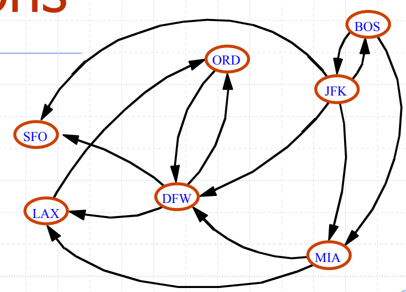
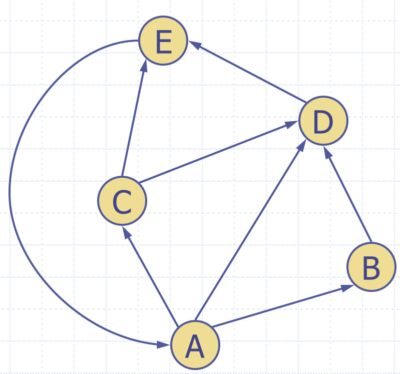
Presentation for use with the textbook, Algorithm Design and Applications, by M. T. Goodrich and R. Tamassia, Wiley, 2015

Directed Graphs

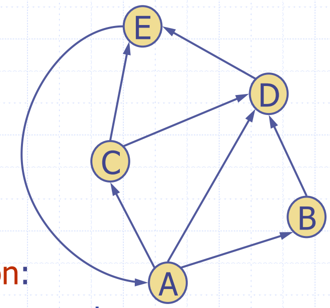


Digraphs

- A digraph is a graph whose edges are all directed
  - Short for "directed graph"
- Applications
  - one-way streets
  - flights
  - task scheduling



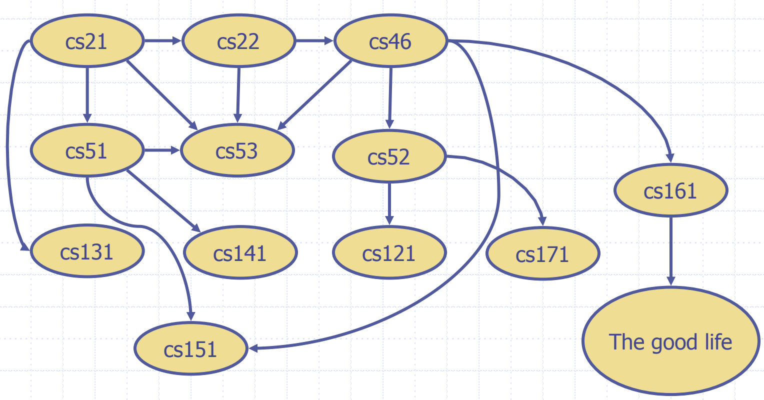
Digraph Properties



- A graph G=(V,E) such that
  - Each edge goes in one direction:
  - Edge (a,b) goes from a to b, but not b to a
- If G is simple, $m \leq n \cdot (n-1)$
- If we keep in-edges and out-edges in separate adjacency lists, we can perform listing of incoming edges and outgoing edges in time proportional to their size

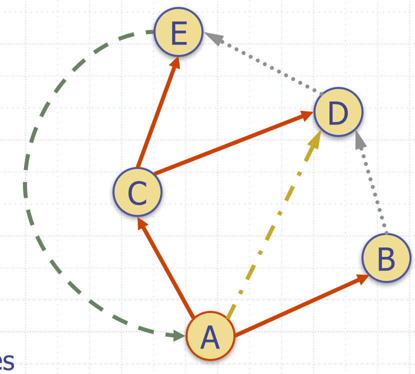
Digraph Application

- Scheduling: edge (a,b) means task a must be completed before b can be started



Directed DFS

- We can specialize the traversal algorithms (DFS and BFS) to digraphs by traversing edges only along their direction
- In the directed DFS algorithm, we have four types of edges
  - discovery edges
  - back edges
  - forward edges
  - cross edges
- A directed DFS starting at a vertex s determines the vertices reachable from s

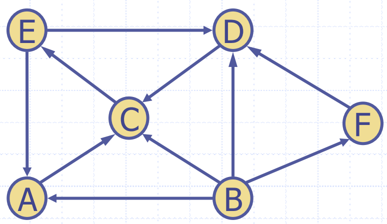


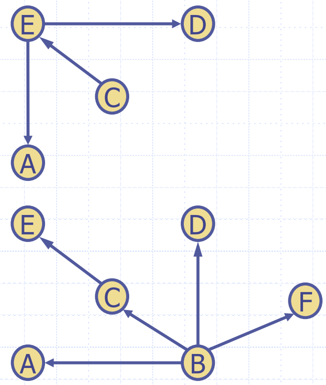
The Directed DFS Algorithm

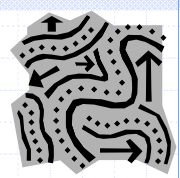
```
Algorithm DirectedDFS(G, v):
  Label v as active        // Every vertex is initially unexplored
  for each outgoing edge, e, that is incident to v in G do
      if e is unexplored then
          Let w be the destination vertex for e
          if w is unexplored and not active then
              Label e as a discovery edge
              DirectedDFS(G, w)
          else if w is active then
              Label e as a back edge
          else
              Label e as a forward/cross edge
  Label v as explored
```

Reachability

- DFS tree rooted at v: vertices reachable from v via directed paths

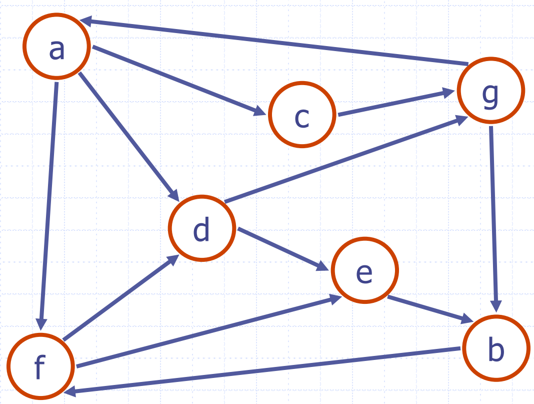




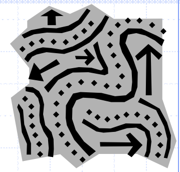


Strong Connectivity

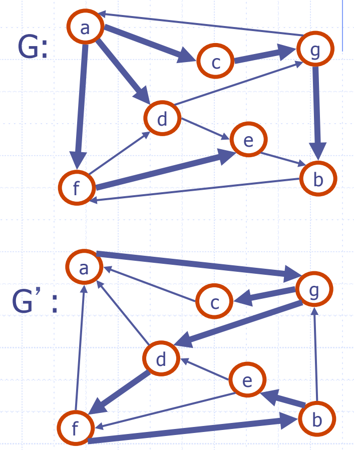
- Each vertex can reach all other vertices



Strong Connectivity Algorithm



- Pick a vertex v in G
- Perform a DFS from v in G
  - If there's a w not visited, print "no"
- Let G' be G with edges reversed
- Perform a DFS from v in G'
  - If there's a w not visited, print "no"
  - Else, print "yes"
- Running time: O(n+m)



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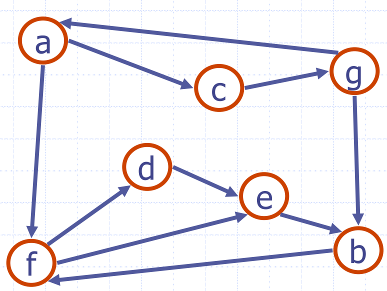
Strongly Connected Components

- Maximal subgraphs such that each vertex can reach all other vertices in the subgraph
- Can also be done in O(n+m) time using DFS, but is more complicated (similar to biconnectivity).



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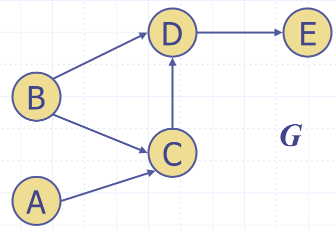
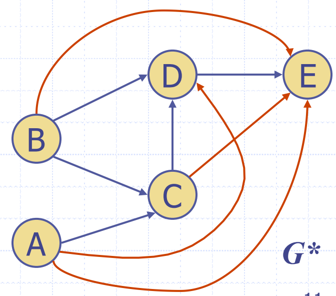
Transitive Closure

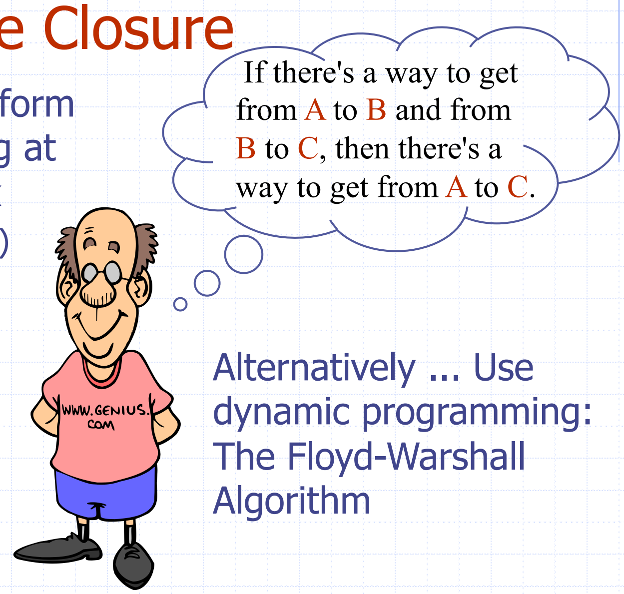
- Given a digraph G, the transitive closure of G is the digraph G^* such that
  - G^* has the same vertices as G
  - if G has a directed path from u to v ($u \neq v$), G^* has a directed edge from u to v
- The transitive closure provides reachability information about a digraph





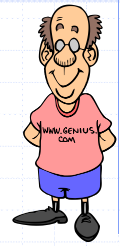
Computing the Transitive Closure

- We can perform DFS starting at each vertex
  - O(n(n+m))

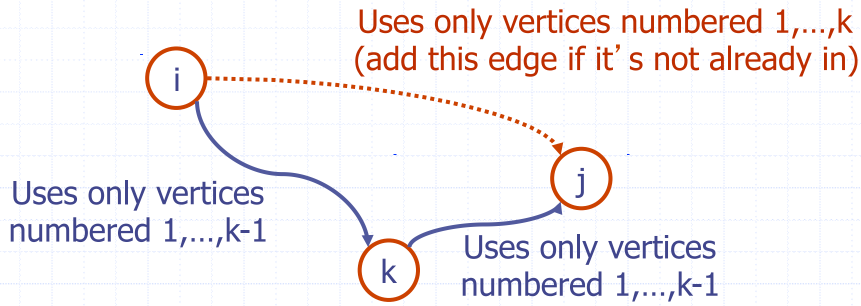


Alternatively ... Use dynamic programming: The Floyd-Warshall Algorithm

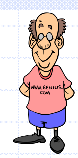
Floyd-Warshall Transitive Closure



- Idea #1: Number the vertices 1, 2, …, n.
- Idea #2: Consider paths that use only vertices numbered 1, 2, …, k, as intermediate vertices:



Floyd-Warshall's Algorithm: High-Level View



- Number vertices $v_1, \ldots, v_n$
- Compute digraphs $G_0, \ldots, G_n$
  - $G_0 = G$
  - G_k has directed edge (v_i, v_j) if G has a directed path from v_i to v_j with intermediate vertices in $\{v_1, \ldots, v_k\}$
- We have that $G_n = G^*$
- In phase k, digraph G_k is computed from G_{k-1}
- Running time: $O(n^3)$, assuming areAdjacent is $O(1)$ (e.g., adjacency matrix)

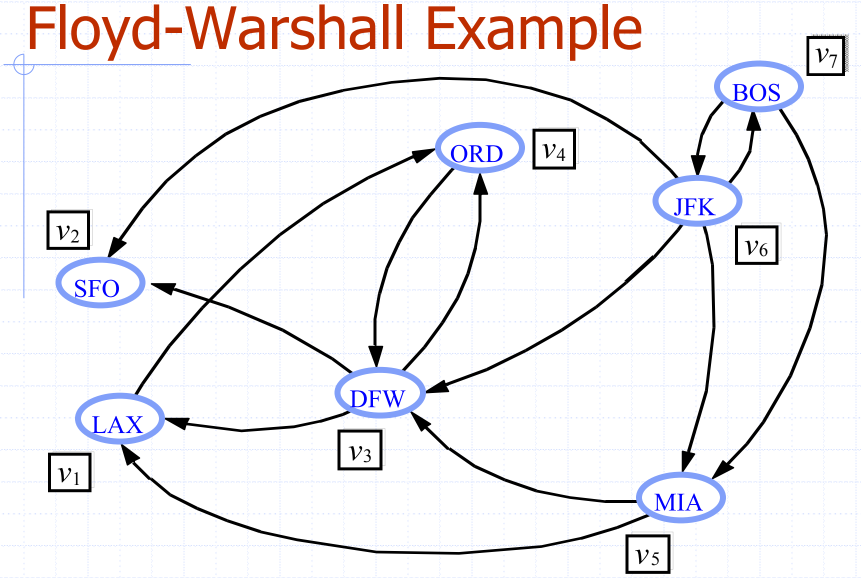
The Floyd-Warshall Algorithm

Algorithm FloydWarshall($\vec{G}$):
  Input: A digraph $\vec{G}$ with n vertices
  Output: The transitive closure $\vec{G}^*$ of $\vec{G}$

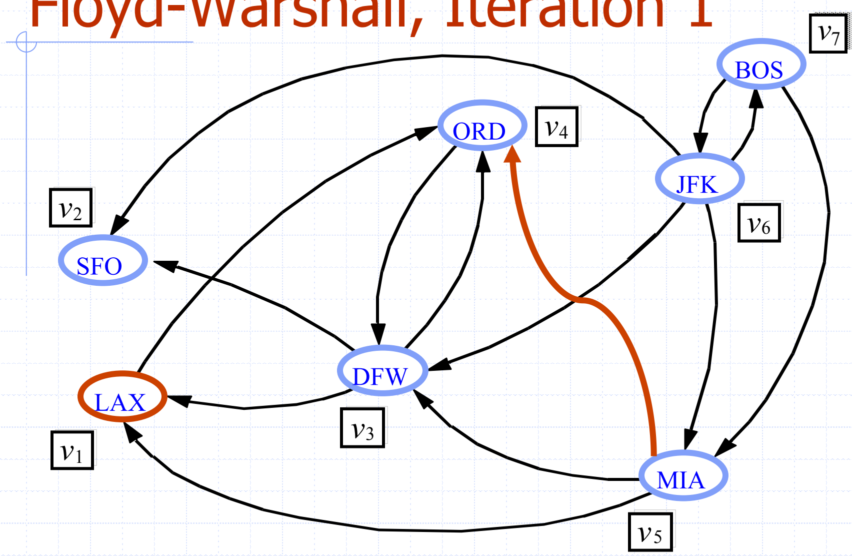
  Let $v_1, v_2, \ldots, v_n$ be an arbitrary numbering of the vertices of $\vec{G}$
  $\vec{G}_0 \leftarrow \vec{G}$
  for $k \leftarrow 1$ **to** n **do**
    $\vec{G}_k \leftarrow \vec{G}_{k-1}$
    for $i \leftarrow 1$ **to** $n, i \neq k$ **do**
      for $j \leftarrow 1$ **to** $n, j \neq i, k$ **do**
        if both edges (v_i, v_k) and (v_k, v_j) are in $\vec{G}_{k-1}$ **then**
          if $\vec{G}_k$ does not contain directed edge (v_i, v_j) **then**
            add directed edge (v_i, v_j) to $\vec{G}_k$
  return $\vec{G}_n$

- The running time is clearly $O(n^3)$.

Floyd-Warshall Example



Floyd-Warshall, Iteration 1



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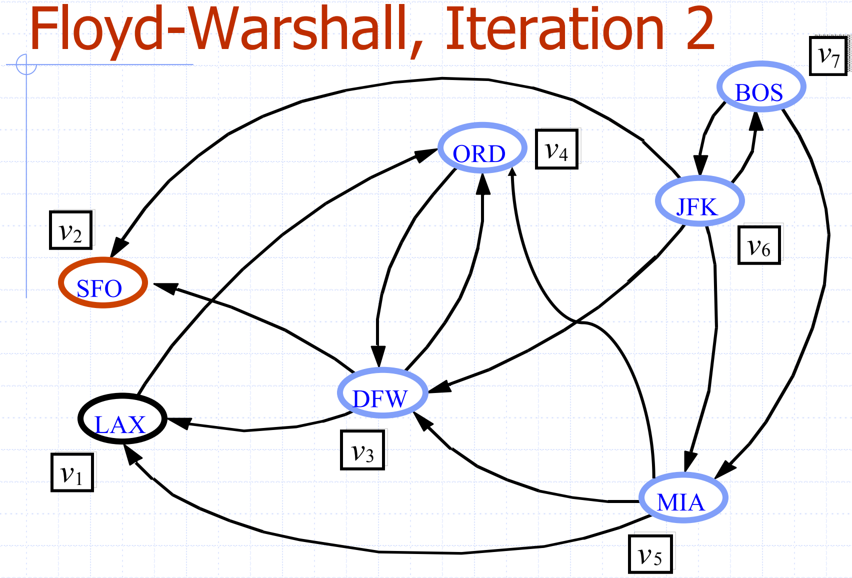
Floyd-Warshall, Iteration 2



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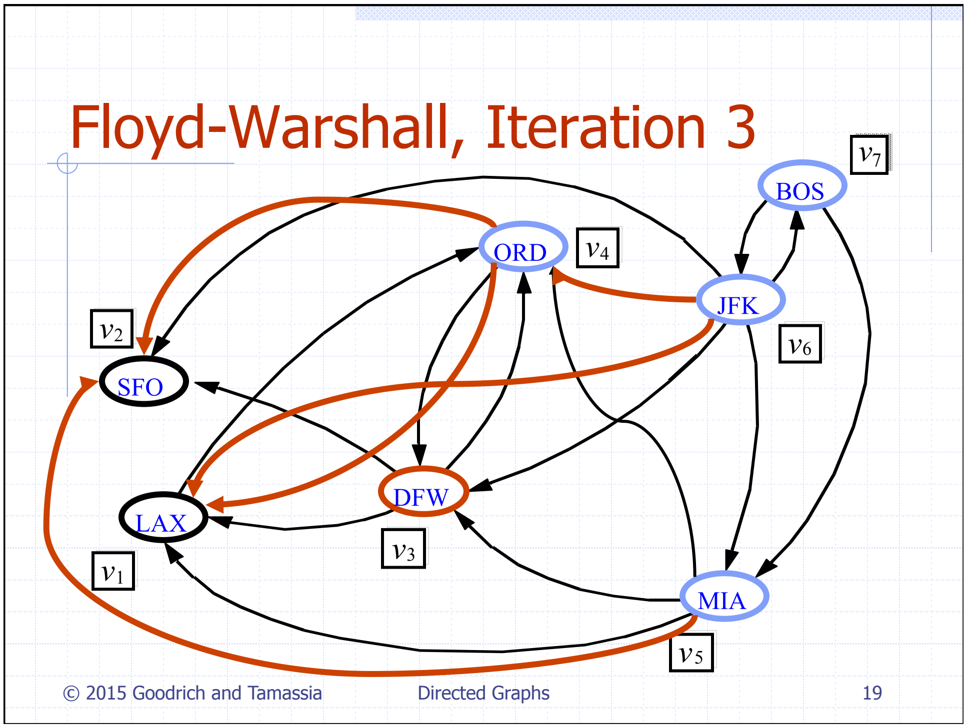
Floyd-Warshall, Iteration 3
BOS
v_7
ORD
v_4
JFK
v_6
v_2
SFO
DFW
LAX
v_3
v_1
MIA
v_5
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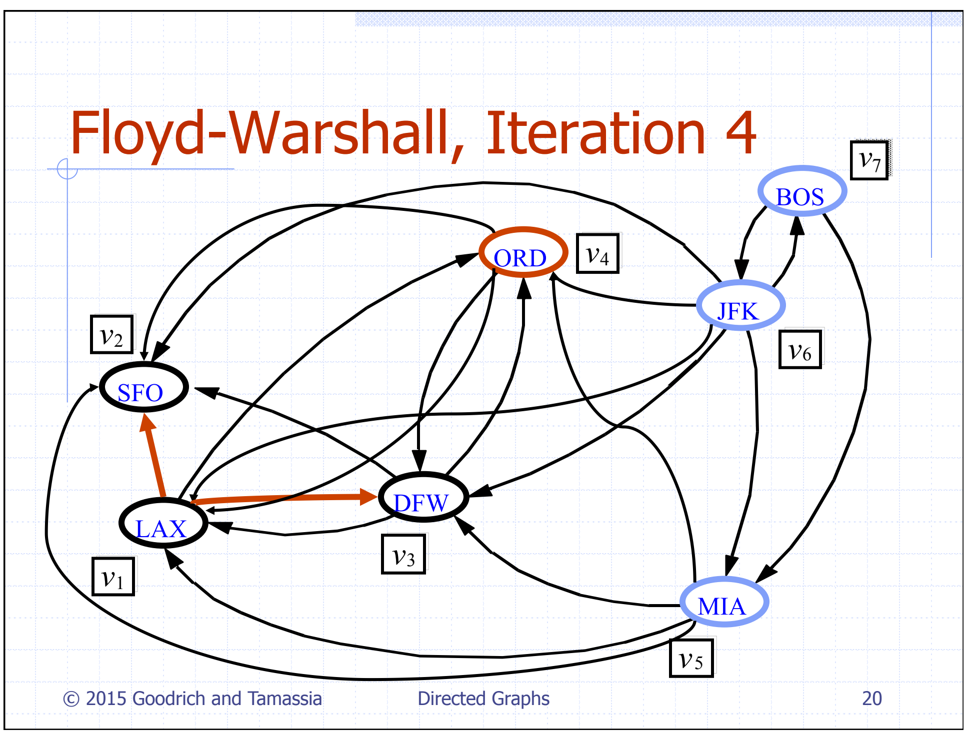
Floyd-Warshall, Iteration 4
BOS
v_7
ORD
v_4
JFK
v_6
v_2
SFO
DFW
LAX
v_3
v_1
MIA
v_5
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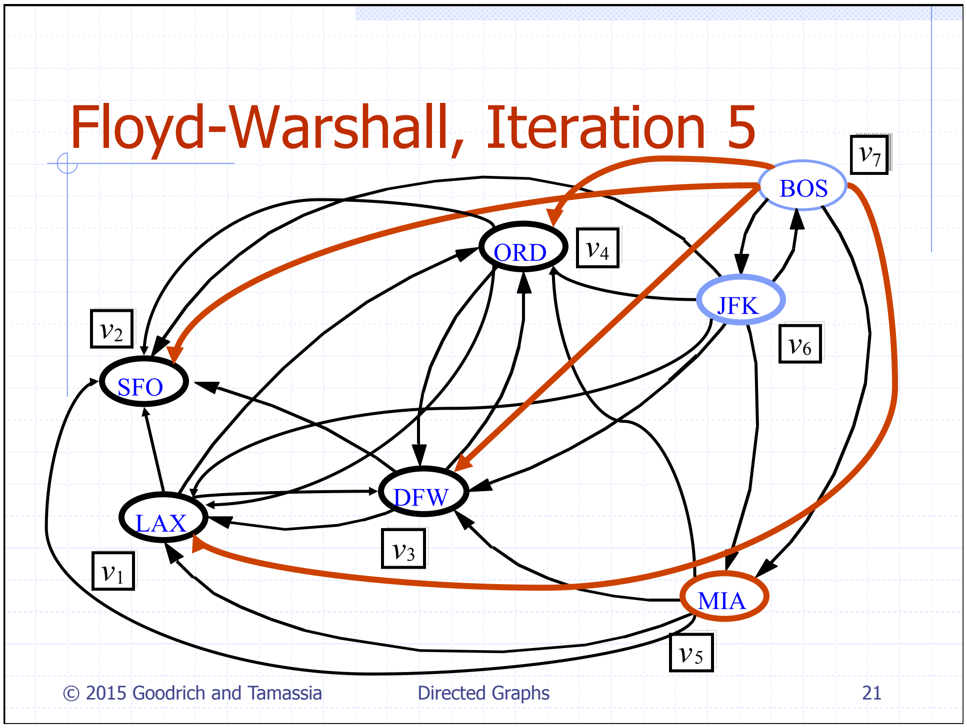
Floyd-Warshall, Iteration 5
BOS
v_7
ORD
v_4
JFK
v_6
v_2
SFO
DFW
LAX
v_3
v_1
MIA
v_5
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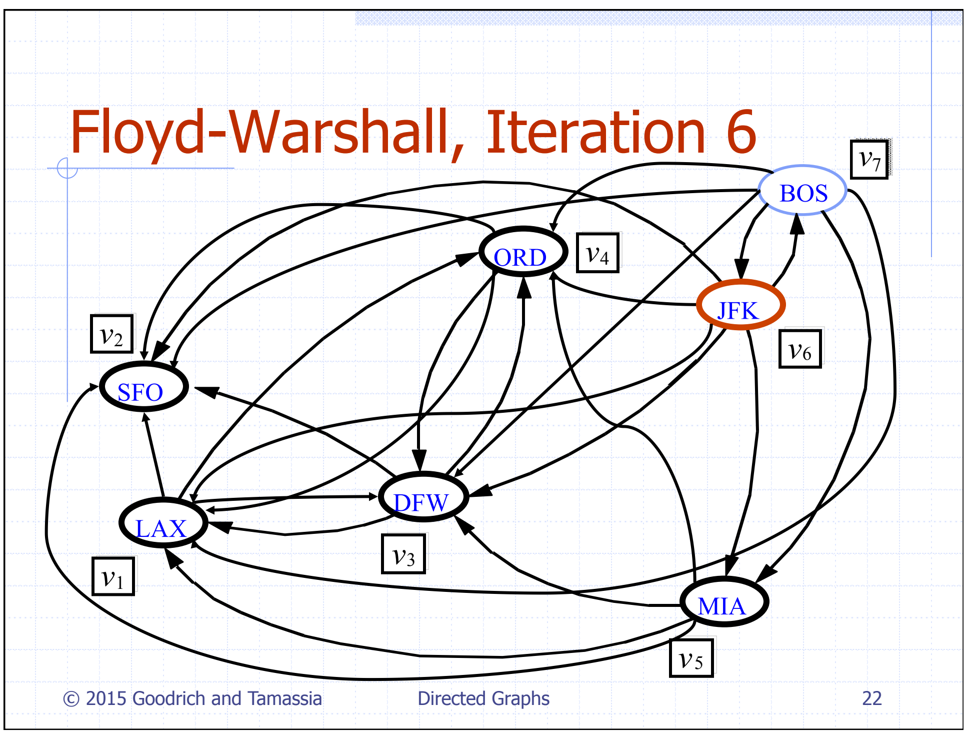
Floyd-Warshall, Iteration 6
BOS
v_7
ORD
v_4
JFK
v_6
v_2
SFO
DFW
LAX
v_3
v_1
MIA
v_5
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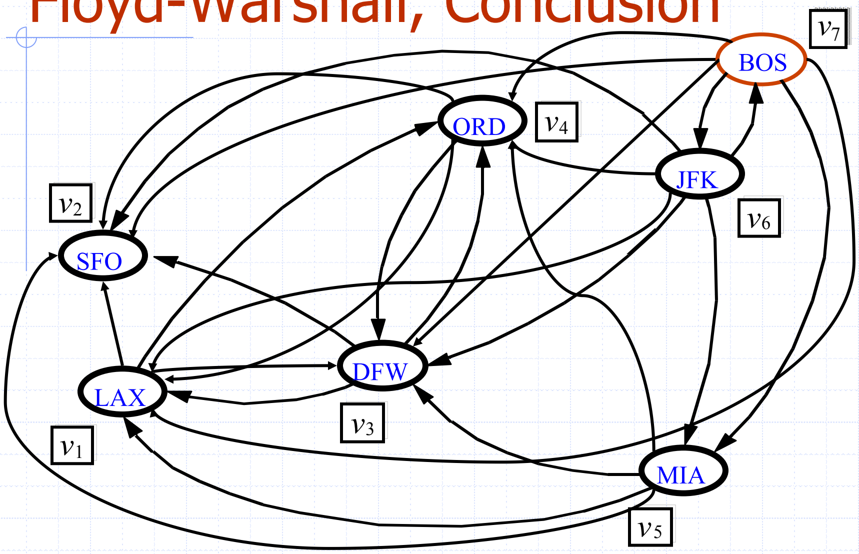
Floyd-Warshall, Conclusion



DAGs and Topological Ordering

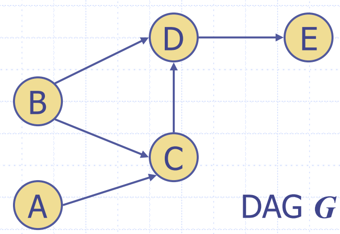
- A directed acyclic graph (DAG) is a digraph that has no directed cycles
- A topological ordering of a digraph is a numbering

  $$v_1, \ldots, v_n$$

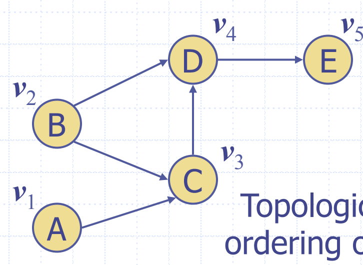
  of the vertices such that for every edge (v_i, v_j), we have $i < j$
- Example: in a task scheduling digraph, a topological ordering a task sequence that satisfies the precedence constraints

Theorem

A digraph admits a topological ordering if and only if it is a DAG



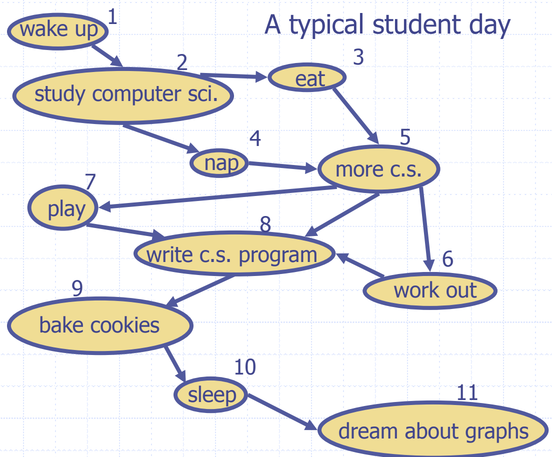
DAG G



Topological ordering of G

Topological Sorting

- Number vertices, so that (u,v) in E implies u < v



Algorithm for Topological Sorting

- Note: This algorithm is different than the one in the book

```
Algorithm TopologicalSort(G)
    H ← G          // Temporary copy of G
    n ← G.numVertices()
    while H is not empty do
        Let v be a vertex with no outgoing edges
        Label v ← n
        n ← n − 1
        Remove v from H
```

- Running time: O(n + m)

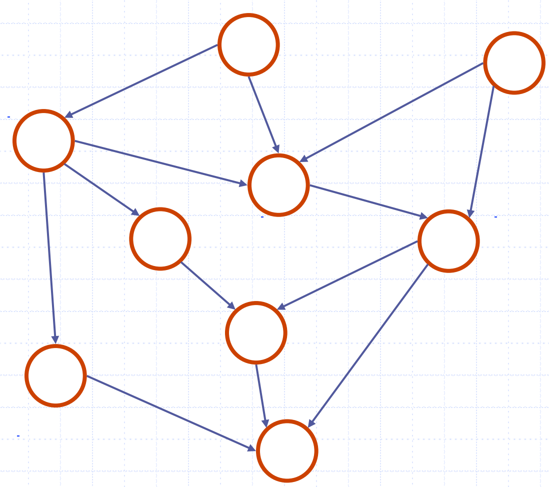
Implementation with DFS

- Simulate the algorithm by using depth-first search
- O(n+m) time.

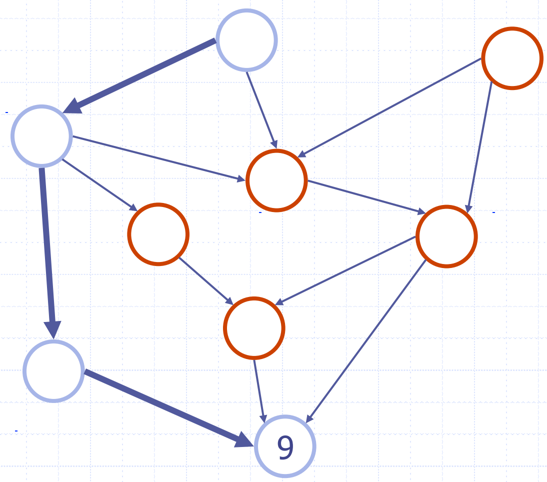
Algorithm *topologicalDFS*(*G*)
  Input dag *G*
  Output topological ordering of *G*
 n ← *G.numVertices*()
 for all *u* ∈ *G.vertices*()
  setLabel(*u, UNEXPLORED*)
 for all *v* ∈ *G.vertices*()
  if *getLabel*(*v*) = *UNEXPLORED*
   topologicalDFS(*G, v*)

Algorithm *topologicalDFS*(*G, v*)
 Input graph *G* and a start vertex *v* of *G*
 Output labeling of the vertices of *G* in the connected component of *v*
 setLabel(*v, VISITED*)
 for all *e* ∈ *G.outEdges*(*v*)
  { outgoing edges }
  w ← *opposite*(*v,e*)
  if *getLabel*(*w*) = *UNEXPLORED*
   { *e* is a discovery edge }
   topologicalDFS(*G, w*)
  else
   { *e* is a forward or cross edge }
 Label *v* with topological number *n*
 n ← *n* - 1

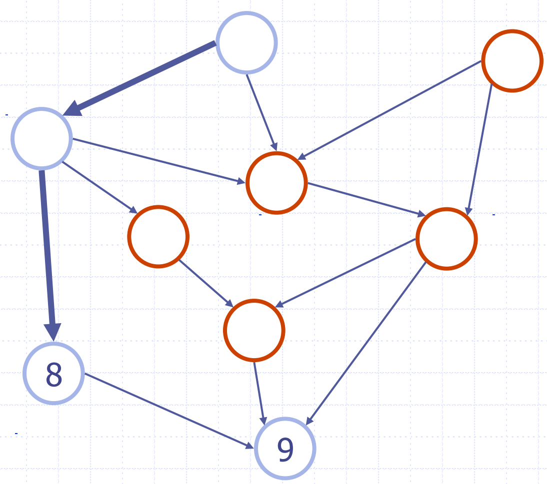
Topological Sorting Example



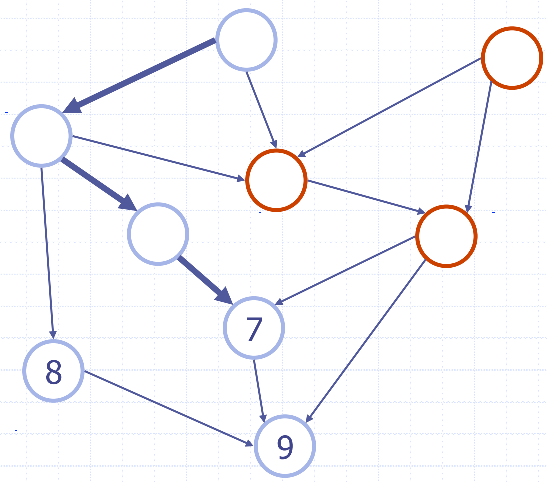
Topological Sorting Example



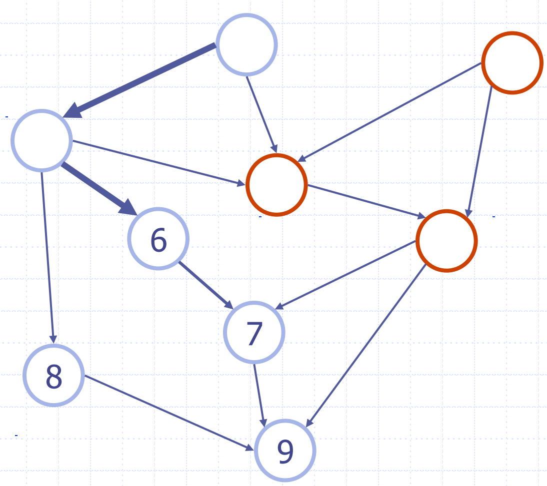
Topological Sorting Example



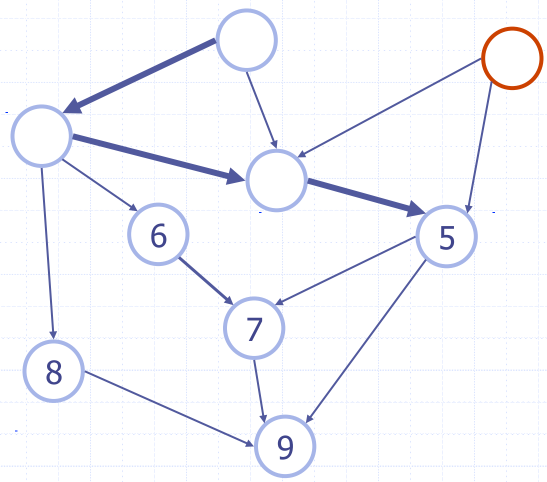
Topological Sorting Example



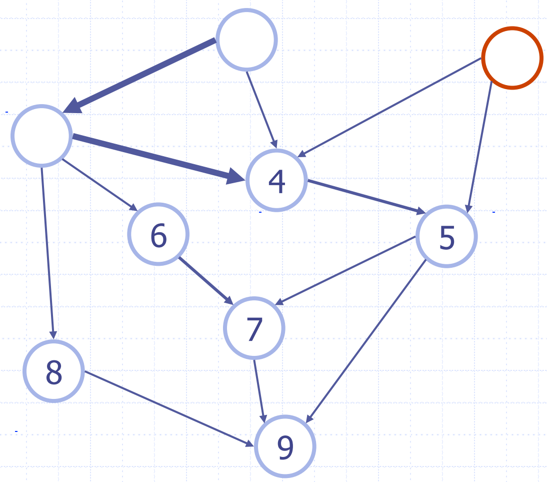
Topological Sorting Example



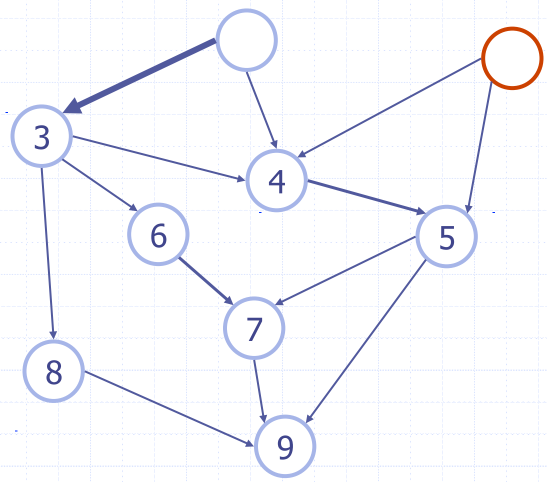
Topological Sorting Example



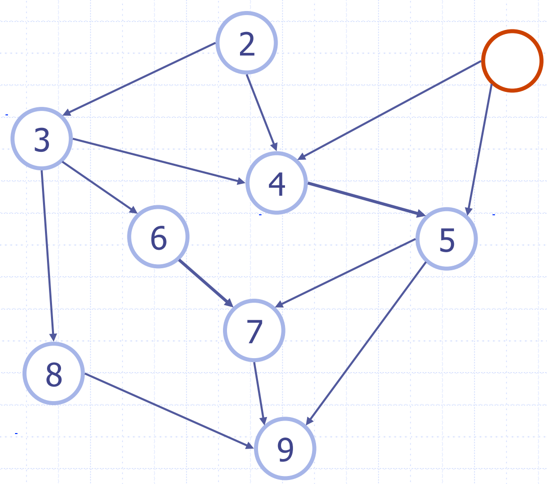
Topological Sorting Example



Topological Sorting Example



Topological Sorting Example



Topological Sorting Example