Edmunds-Karp Algorithm: Choosing Good Augmenting Paths

Use care when selecting augmenting paths.
- Some choices lead to exponential algorithms.
- Clever choices lead to polynomial algorithms.

Goal: choose augmenting paths so that:
- Can find augmenting paths efficiently.
- Few iterations.

Edmonds-Karp (1972): choose augmenting path with
- Max bottleneck capacity. \(\text{fat path}\)
- Sufficiently large capacity. \(\text{capacity-scaling}\)
- Fewest number of arcs. \(\text{shortest path}\)

Shortest Augmenting Path

Intuition: choosing path via breadth first search.
- Easy to implement.
  - may implement by coincidence!
- Finds augmenting path with fewest number of arcs.

\[
\text{ShortestAugmentingPath}(V, E, s, t)\
\]

\[
\begin{array}{l}
\text{FOREACH } e \in E \\
\quad f(e) \leftarrow 0 \\
\quad G_f \leftarrow \text{residual graph} \\
\text{WHILE (there exists augmenting path)} \\
\quad \text{find such a path } P \text{ by BFS} \\
\quad f \leftarrow \text{augment}(f, P) \\
\quad \text{update } G_f \\
\text{RETURN } f
\end{array}
\]
Shortest Augmenting Path: Overview of Analysis

L1. Throughout the algorithm, the length of the shortest path never decreases.
   - Proof ahead.

L2. After at most m shortest path augmentations, the length of the shortest augmenting path strictly increases.
   - Proof ahead.

Theorem. The shortest augmenting path algorithm runs in $O(m^2 n)$ time.
- $O(m+n)$ time to find shortest augmenting path via BFS.
- $O(m)$ augmentations for paths of exactly $k$ arcs.
- If there is an augmenting path, there is a simple one.
  $\Rightarrow 1 \leq k < n$
  $\Rightarrow O(mn)$ augmentations.

Shortest Augmenting Path: Analysis

Level graph of $(V, E, s)$.
- For each vertex $v$, define $\ell(v)$ to be the length (number of arcs) of shortest path from $s$ to $v$.
- $L_G = (V, E_G)$ is subgraph of $G$ that contains only those arcs $(v,w) \in E$ with $\ell(w) = \ell(v) + 1$. 

![Graph G and Level Graph L_G](image-url)
Shortest Augmenting Path: Analysis

Level graph of \((V, E, s)\).
- For each vertex \(v\), define \(\ell(v)\) to be the length (number of arcs) of shortest path from \(s\) to \(v\).
- \(L = (V, F)\) is subgraph of \(G\) that contains only those arcs \((v, w) \in E\) with \(\ell(w) = \ell(v) + 1\).
- Compute in \(O(m+n)\) time using BFS, deleting back and side arcs.
- \(P\) is a shortest \(s\-v\) path in \(G\) if and only if it is an \(s\-v\) path \(L\).

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Shortest Augmenting Path: Analysis

L1. Throughout the algorithm, the length of the shortest path never decreases.
- Let \(f\) and \(f'\) be flow before and after a shortest path augmentation.
- Let \(L\) and \(L'\) be level graphs of \(G_f\) and \(G_{f'}\).
- Only back arcs added to \(G_{f'}\).
  - path with back arc has length greater than previous length
Shortest Augmenting Path: Analysis

L2. After at most m shortest path augmentations, the length of the shortest augmenting path strictly increases.
   - At least one arc (the bottleneck arc) is deleted from L after each augmentation.
   - No new arcs added to L until length of shortest path strictly increases.

Shortest Augmenting Path: Review of Analysis

L1. Throughout the algorithm, the length of the shortest path never decreases.

L2. After at most m shortest path augmentations, the length of the shortest augmenting path strictly increases.

Theorem. The shortest augmenting path algorithm runs in $O(m^2n)$ time.
   - $O(m+n)$ time to find shortest augmenting path via BFS.
   - $O(m)$ augmentations for paths of exactly $k$ arcs.
   - $O(mn)$ augmentations.

Note: $\Theta(mn)$ augmentations necessary on some networks.
## History

<table>
<thead>
<tr>
<th>Year</th>
<th>Discoverer</th>
<th>Method</th>
<th>Big-Oh</th>
</tr>
</thead>
<tbody>
<tr>
<td>1951</td>
<td>Dantzig</td>
<td>Simplex</td>
<td>$mn^2U$</td>
</tr>
<tr>
<td>1955</td>
<td>Ford, Fulkerson</td>
<td>Augmenting path</td>
<td>$mnU$</td>
</tr>
<tr>
<td>1970</td>
<td>Edmonds-Karp</td>
<td>Shortest path</td>
<td>$m^2n$</td>
</tr>
<tr>
<td>1970</td>
<td>Dinitz</td>
<td>Shortest path</td>
<td>$mn^2$</td>
</tr>
<tr>
<td>1972</td>
<td>Edmonds-Karp, Dinitz</td>
<td>Capacity scaling</td>
<td>$m^2 \log U$</td>
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<tr>
<td>1973</td>
<td>Dinitz-Gabow</td>
<td>Capacity scaling</td>
<td>$mn \log U$</td>
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<tr>
<td>1974</td>
<td>Karzanov</td>
<td>Preflow-push</td>
<td>$n^3$</td>
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<td>1983</td>
<td>Sleator-Tarjan</td>
<td>Dynamic trees</td>
<td>$mn \log n$</td>
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<tr>
<td>1986</td>
<td>Goldberg-Tarjan</td>
<td>FIFO preflow-push</td>
<td>$mn \log (n^2 / m)$</td>
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<td>1997</td>
<td>Goldberg-Rao</td>
<td>Length function</td>
<td>$m^{3/2} \log (n^2 / m) \log U$</td>
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<td></td>
<td></td>
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<td>$mn^{2/3} \log (n^2 / m) \log U$</td>
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