Graph coloring

- Examples
- Applications
- Algorithms
Graph coloring

Adjacent nodes must have different colors.

How many colors do we need?
Graph coloring

Neighbors must have different colors

How many colors do we need?
Graph coloring

Neighbors must have different colors

Not valid
Definitions

- **k-Colorable**: can be colored with k colors
- **Chromatic number**: min number of colors needed
Cycle graph
Odd-length cycles require 3 colors
Cycle graph

No way to color with only 2 colors
Bipartite graph
Bipartite graph

Bipartite $\iff$ 2-colorable
Tree

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Tree
Clique
If $G$ contains a Clique of size $k$, then it requires at least $k$ colors
Planar graph
Planar graph

exam-like question: draw the smallest planar graph with chromatic number 4
Planar graph coloring

- Some planar graphs require 4 colors
- Any planar graph is 5-colorable (1890)
- Can every planar graph be colored with 4 colors? (Stated in 1852, solved in 1976)
map coloring

map coloring $\iff$ planar graph coloring
4 color theorem

Every planar graph (and map) can be colored with 4 colors.
Graph coloring

- Examples
- Applications
  - Register allocation
  - Logistics problems
  - Scheduling problems
  - Puzzles
- Algorithms
Applications

$K_9$

$K_9$
Applications
Register allocation

- Limited registers (16 in a x86-64 CPU)
- Goal: avoid storing local variables in main memory
- If a variable $X$ is put in a register, other variables can’t use that register until the last use of $X$
- Need to decide which variable goes to which register

```c
for (i=0; i<n; i++) {
    ...
}
for (j=0; j<n; j++) {
    ...
}
i and j can go in the same register, but i and n can’t
```
Interval graph

- Nodes: intervals
- Edges: overlapping intervals
Register allocation

```
a := input
b := 2*a
c := b*b+a
if a<0 jump(L)
d := b+c
b := d-1
jump(M)

L: e := b-c

b := e+1

M: f := b*2
```

same color = same register

Source: scienceblogs.com/goodmath/2007/06/28/graph-coloring-algorithms-1/
Logistics problems

Grouping things into as few groups as possible

- Nodes: things
- Edges: things that can’t be together
- Each color is a group

Examples: register allocation, chemical storage, ...
Scheduling problems

Doing a set of tasks in the shortest time
  - Nodes: tasks
  - Edges: tasks that can’t be done at the same time
  - Each color is a time slot

Examples: exam scheduling, ...
Graph coloring

- Examples
- Applications
- Algorithms
  - Can this graph be colored with $k$ colors?
  - Find an optimal coloring
  - Or suboptimal coloring more quickly
2-colorability

Is a graph 2-colorable? (is a graph bipartite?)

- Run BFS
- Color nodes by level in the BFS tree
- If there are cross edges between same-parity levelss $\Rightarrow$ not 2-colorable

- BFS Tree edges
- Cross edges between even and odd
- Cross edges between same-parity levels
3-colorability

Is a graph 3-colorable?

- different colorings with $n$ nodes and 3 colors:

$$3^n \quad \text{or} \quad n^3 \quad ?$$

In general: $n$ nodes and $k$ colors $\Rightarrow k^n$ different colorings

Exponential!

- Next lecture: 3-colorability is NP-complete (even for planar graphs)
Greedy coloring

**Algorithm:**

1. Consider the nodes in some order

2. Give each node in this order the first color not used by its neighbors

   - Not optimal: depends on order

```
1 3 4 2
1 3 2 4
⇒
⇒
```

Order | Coloring
---|---

![Diagram](image-url)
Greedy coloring

It can be really bad
k-regular graph

How many colors are needed?
Greedy upper bound

**Algorithm:**
1. Consider the nodes in some order
2. Give each node in this order the first color not used by its neighbors

If the max degree of any node is $r$, the graph is $(r + 1)$-colorable

Proof: let $u$ be any node. When it’s time to color $u$, even if all its neighbors are already colored, they can use at most $r$ colors, so $r + 1$ colors suffice to color $u$. 
k-regular graph

Greedy coloring $\Rightarrow$ $(k+1)$-colorable
Greedy upper bound

If the max degree of any node is $r$, the graph is $(r + 1)$-colorable.

Can still be a bad upper bound.
Greedy for interval graphs

- Suppose the biggest clique has size $k$
- The chromatic number must be at least $k$
- If nodes are sorted by starting point, greedy coloring finds a $k$-coloring
- Therefore, it is optimal
Greedy for interval graphs

If nodes are sorted by starting point, greedy coloring finds a $k$-coloring. **Proof:**

1. Let $I = (I_s, I_e)$ be any interval

2. Any neighbor of $I$ must end after $I_s$

3. Any already-colored neighbor of $I$ must start before $I_s$

4. (2. and 3.) $\Rightarrow$ $I$ and the already-colored neighbors of $I$ intersect at $I_s$

5. If the max clique size is $k$, there are at most $k - 1$ already-colored neighbors of $I$, and $I$ can use color $k$
Overview

Graph coloring
» Examples
  » Bipartite graphs
  » Cliques
  » Planar graphs

» Applications
  » Sudoku
  » Register allocation
  » Logistics problems
  » Scheduling problems

» Algorithms
  » 2-coloring
  » Greedy coloring

Thanks!