On Sampling and Reconstructing Surfaces with Boundaries

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Abstract

Surface reconstruction algorithms can ensure correctness of the reconstructed surface by imposing conditions on sampling and by assuming that the given point set satisfies the prescribed conditions. Traditionally, such sampling conditions impose *minimum required sampling density* for correct reconstruction. Let us assume that no additional information is provided to the reconstruction algorithm other than the input set of points. Under this assumption, in this paper we show that imposing minimum required sampling density is not sufficient to ensure correct reconstruction of *surfaces with boundaries*. In other words, to prove the correctness of the reconstructed surface, either the sampling conditions have to be strengthened or more information has to be provided to the reconstruction algorithm. Further, we prove that the strengthened sampling condition for sampling surfaces with boundaries has a unique property called *non-monotonicity*. We analyze the applicability of this framework to various algorithms like medial axis estimation algorithms, curve reconstruction algorithms, and reconstruction of surfaces from noisy point sets. Finally, we classify surface reconstruction algorithms based on the sampling conditions and the input to their reconstruction stage.

Keywords: Surface Reconstruction, Manifolds with Boundaries, Geometric Sampling

1 Introduction

Sampling is a process of discretizing a continuous surface or a curve into discrete samples. The reconstruction process uses these samples to reconstruct the original continuous surface. The reconstructed surface is *correct* if it is *topologically equivalent* and *geometrically close* to the original surface. To ensure correct reconstruction of the original surface, certain restrictions are imposed on the discretization process. These restrictions are called *sampling conditions*.

For example, in signal processing, one of the well known sampling condition is the *Nyquist criterion* to sample a continuous signal, and this is used in digitization of audio, video, and is applicable to images also. According to the Nyquist theorem [14, 7], the discrete time sequence of a sampled continuous band limited signal $V(t)$ contains enough information to reproduce the function $V(t)$ exactly provided that the sampling rate is at least twice that of the highest frequency contained in the original signal $V(t)$. In other words, *it imposes a minimum required sampling density for correct reconstruction* of the signal from the samples. Conditions on geometric sampling of curves and surfaces with no boundaries follow the same philosophy and impose a minimum required sampling density for correct reconstruction. In this paper, we show that these algorithms cannot reliably reconstruct surfaces with boundaries using such sampling conditions.

The next section lists a few definitions, assumptions, and properties of sampling. Section 3 proves our main theorem about the sampling of surfaces with boundaries, and Section 4 analyzes the design space of reconstruction algorithms based on this theory. Section 4.3 also introduces the property of *non-monotonicity* for sampling conditions and proves that the sampling conditions for surfaces with boundaries exhibit this property. Section 5 applies the concepts developed in this paper for various algorithms including medial axis reconstruction algorithms and curve reconstruction algorithms. Section 6 classifies the reconstruction algorithms for surfaces with boundaries and finally Section 7 concludes this paper.

2 Surface and Sampling

In this section we list a few definitions, properties, and assumptions about the surface and sampling.

As mentioned earlier, the reconstructed surface is said to be *correct* if it is *topologically equivalent* and *geometrically close* (with respect to a given error bound) to the original surface.

A valid surface is one that can be handled by the sampling condition and the reconstruction algorithm. We consider any non-self-intersecting two manifold with or without boundary as a valid surface. In the rest of the paper, we denote any valid surface by M and a small geometric perturbation of M as M'. Further, M and M' have the same topology.

Definition 1 *A* valid sampling *of a surface is one that satisfies the sampling condition and hence a "correct" surface is reconstructed by the reconstruction algorithm using this sampling. Infinite sampling of the whole surface is a trivial valid sampling.*

Definition 2 *An* invalid sampling *of a surface is one that does not satisfy the sampling condition, and if used in the reconstruction process, would generate an incorrect surface. An empty sample point set is an invalid sampling.*

2.1 Boundary Size

For different reconstruction algorithms there will be some maximum distance between sample points on the boundary of the surface for which the algorithm will correctly reconstruct the boundary. We call this the *boundary size*, and its definition is based on the reconstruction algorithm used. For example, in terms of the sampling conditions prescribed by [2, 5], the boundary size is the minimum distance from the boundary points to the medial axis (smallest *feature size* of all the boundary points). In terms of alpha shapes[10], the boundary size is the minimal value of alpha that cannot reconstruct a topologically correct and geometrically close surface.

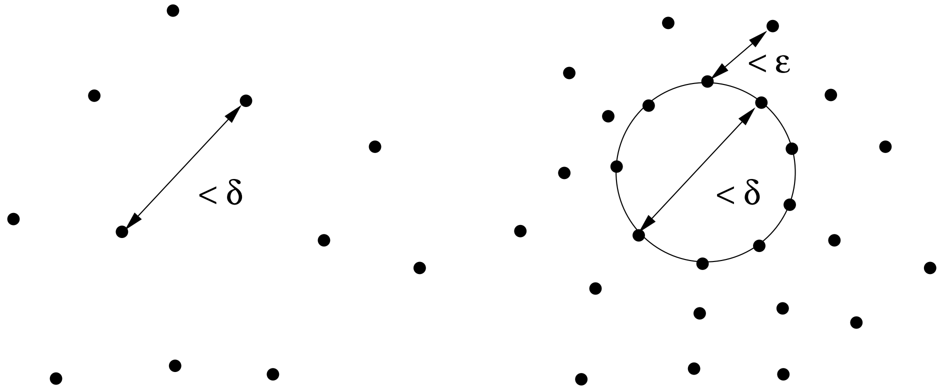

Figure 1: Sampling: Left – Without Boundary, Right – With Boundary.

2.2 Finiteness of sampling

A trivial sampling condition can demand infinite sampling everywhere on the surface. We would like to avoid such conditions in our analysis.

Definition 3 *A sampling condition is said to satisfy the* finiteness property *if there exists a finite set of points that samples M or M', and satisfies the sampling condition.*

For example, the sampling conditions of [2, 5] require infinite sampling when the medial axis of the surface meets the surface, as in sharp corners and edges. But these features can be removed by geometric perturbation of the surface retaining its topology. Hence these sampling conditions satisfy the finiteness property.

Let us consider another hypothetical example where the sampling condition requires infinite sampling along the boundary curves of the surface. Such a sampling condition does not satisfy the finiteness property as even with a geometric perturbation, a boundary curve remains a boundary curve requiring infinite sampling.

2.3 Unrestricted Point Placement

The sampling conditions should not be too restrictive regarding the placement of points. There should be a non-zero probability of successful placement of a sample point on the surface M or M'. For example, the sampling conditions of [2, 5] require placement of sample points on corners. As a corner occupies zero area on M, there is zero probability of placing a sample on a corner of M. But with small geometric perturbation, this corner feature and hence this restrictive requirement can be removed. (While sampling a boundary curve, we cannot avoid the requirement of sample points on the curve. Hence, sampling of boundary curves has to be handled separately as a lower dimensional "surface" where placing a point on this curve has a non-zero probability.)

The above property deals with the *requirement* of sample points at *specific locations*. Another aspect of this problem is the *acceptability* of sample points at *arbitrary locations*. In other words, it should still be possible to find a valid sampling of M even if there exist a few sample points at arbitrary locations on M.

Definition 4 *A sampling condition is said to satisfy the* unrestricted point placement property *if, given an arbitrary finite set of sample points S on M, there exists a finite valid sampling T of M such that $S \subset T$.*

2.4 Assumptions

In this paper we assume the following:

1. The surface that is sampled is a valid surface, that is, a non-self-intersecting manifold with or without boundary.
2. The sampling conditions satisfy the finiteness property and the unrestricted point placement property.
3. The input set of points is unorganized, that is, any permutation of the input set of points results in the same reconstructed model. This assumption is made so that there is no inherent order by which the connectivity can be deduced.
4. The input sample point set S reflects only the positions of the points on the surface. Since S is a set, there is no duplication of sample points. To avoid encoding of information other than the positions of points in the bit level of the input set, we assume that any small perturbation of the point locations does not change the connectivity in the reconstructed surface.

3 Sampling Surfaces with Boundaries

Sampling conditions depend on the reconstruction algorithm used. The few algorithms that specify sampling conditions focus on surfaces with no boundaries. Almost all of the reconstruction algorithms designed till today, with the exceptions of [12, 11], have the "Nyquist philosophy" that the denser the sampling, the closer the approximation of original surface with the reconstructed surface. Hence the sampling conditions directly or indirectly specify the maximum distance between close samples, in other words, the minimum required sampling density. In this section we show that sampling surfaces with no boundaries is different from sampling surfaces with boundaries.

Theorem 1 *Conditions on the minimum required sampling density are not sufficient to design algorithms that reconstruct correct surfaces with boundaries using only the point samples as input.*

Proof: Let us assume that we are given a hypothetical surface reconstruction algorithm A that claims to reliably reconstruct surfaces both with and without boundaries using only the input set of points. Let A be based on a sampling condition C that directly or indirectly specifies just the maximum distance between samples in a given region. Let δ be the maximum distance between the closest samples demanded by C in a specific region on M (Figure 1). In that region of M, if we consider a boundary B of boundary size less than δ (and greater than another constant, say ϵ), then the sampling demanded would not be able to capture the feature B. Hence C is forced to demand a denser sampling in the region around the boundary B. Let the new limit on maximum distance between closest samples be ϵ which is less than δ. Since, by the new sampling condition, the distance between closest samples is less than both ϵ and δ, the sampling satisfies the conditions required to sample M *both with and without the boundary* B. With no additional information about the presence or absence of boundary, the algorithm A has no way to find out whether the given set of sample points are of a surface with or without boundary. Hence conditions on just the minimum required sampling density is not sufficient for reliable reconstruction of M with B when only the sample points are used during reconstruction. $\diamondsuit$

As a result, for reliable reconstruction of surfaces with boundaries, either the sampling condition has to be strengthened or more information, other than the sample point set, has to be provided to the reconstruction algorithm.

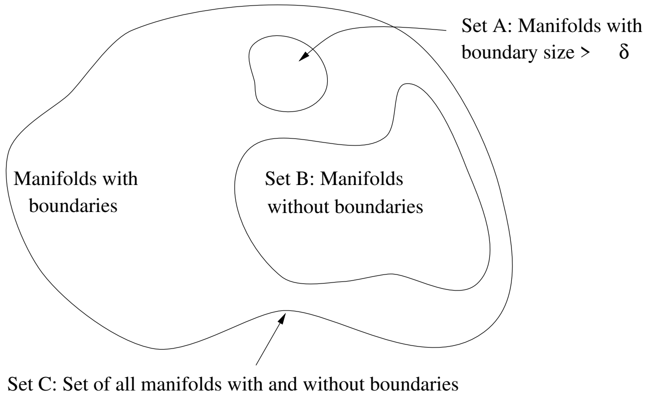

Figure 2: Solution space for different sampling conditions.

4 Analysis of the Algorithm Design Space

Let us use Theorem 1 to analyze the design space of the sampling conditions and algorithms to reconstruct surfaces with or without boundaries. Consider the diagram in Figure 2. In this diagram, manifolds without boundaries are considered to be a subset of manifolds with boundaries with zero boundary size. We need to design algorithms to reconstruct all instances of the set of all manifolds with and without boundaries.

There are two stages in the reconstruction process, the sampling stage and the reconstruction stage. To handle surfaces with boundaries, either the sampling condition should be strengthened in the sampling stage, or more information should be provided, other than just the sample points, to the reconstruction stage. Let us first analyze the option of providing more information to the reconstruction stage.

4.1 More Information to the Reconstruction Stage

Trivial (and maximum) additional information for correct reconstruction is the complete connectivity information, which we do not consider in this paper for obvious reasons. A "medium-sized" additional information can be in the form of tags for all the boundary points. All tagged points can be used as input to a curve reconstruction algorithm that would reconstruct closed boundary curves. The rest of the connectivity can be found using present (closed) surface reconstruction algorithms with minor modification. This approach needs further investigation which this paper will not address.

The minimum additional information that can be provided is one single number, and let that be the smallest boundary size, say δ, of the model. A reconstruction solution can use existing (closed) surface reconstruction algorithms and, as a post processing, remove all edges with length larger than δ. The sampling should be dense enough such that the required edges are ensured to have length less than δ. This approach can reconstruct the instances of surfaces both in sets A and B in Figure 2. Reducing the value of δ allows the reconstruction algorithm to handle set B and a superset of set A, but increases the required sampling density. In the limit, to reconstruct all instances of set C, infinite sampling is required, making it theoretically uninteresting. Nevertheless, this approach is suitable for data sets from uniform, minimally controlled, and automatic sampling processes, and where the reconstruction stage is well controlled. For example, large data sets of rooms and outdoors from laser scanners can be handled using this approach. The surface reconstruction algorithms based on α shapes might fall under this category where the value of α is the additional information used. In summary, with this approach the sampling will not adapt itself to the varying boundary sizes and imposes uniform sampling throughout the model based on the smallest boundary size.

4.2 Strengthening the Sampling Condition

In order to change the sampling to dynamically adapt to boundary-sizes, conditions on sampling should be strengthened. Let us assume that we are not providing any additional information other than the sample points to the reconstruction stage. In this case, we have to encode the presence of boundary using parameters that can be calculated from the positions and number of sample points.

For example, the all-sample-point-pairs closest distance can be used as an indicator of α in the reconstruction algorithms based on α-shapes. But such algorithms will be prone to even the slightest perturbation of a sample point, and any attempt to make this relatively more robust have to use tolerance measures based on heuristics.

Another parameter that can be measured using the positions and the number of points would be the sampling density. We can encode the presence or absence of boundaries using the variation in the sampling density. To accomplish this, here we establish a relationship between the sampling density and the presence of boundary.

To represent boundaries, we have to prevent the addition of certain simplices (triangles) that close holes in the final triangulation. Intuitively, we would not connect two points **p** and **q** that are "far" away. The concept of "farness" is relative, and depends on how close other points are to **p** and **q**. This leads us to one possible solution. The distance to the farthest point that can be connected to a sample **p** should be related to the distance to the closest point of **p**. If the distance to the farthest and the nearest neighbors are indicators of minimum and maximum sampling densities respectively, then this argument shows that *the boundary can be identified using sampling conditions which expresses relative minimum and maximum sampling densities*. An example of such a sampling condition is the one given in [12] in which the distance between a sample **p** and its farthest neighbor in the final triangulation is less than m times the distance between **p** and its closest neighbor, where m is a constant. This constant m, unlike the value of α in the α-shapes, does not depend on the characteristics of the sampled model and hence is not an input to the reconstruction stage.

Such an encoding of information in the sampling density requires the sampling condition to specify more than just the minimum required sampling density.

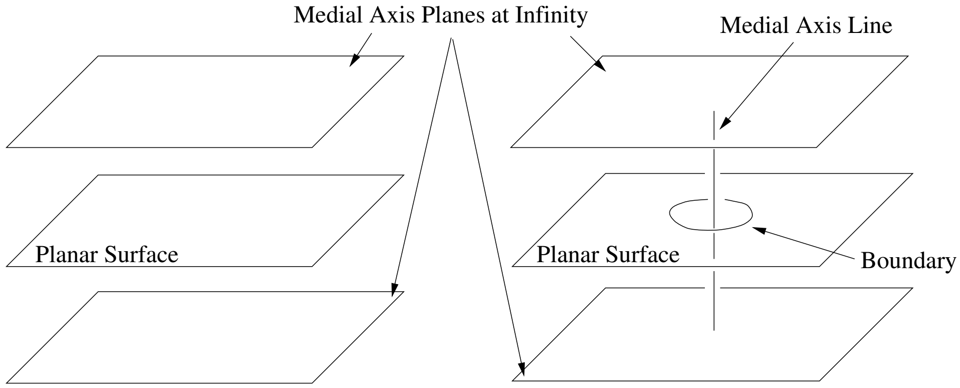

Figure 3: Medial axis of a plane: Left – Without Boundary, Right – With Boundary.

4.3 Non-monotonic Sampling

In this section, we introduce a property called *non-monotonicity* for the sampling conditions and prove the situations under which the sampling conditions exhibit this property.

For surfaces with *no* boundaries, a sampling condition prescribes only the minimum required sampling density. A sampling becomes a valid sampling of a surface M once it satisfies the sampling condition. It would remain a valid sampling with any additional sampling over this minimum requirement.

For surfaces with boundaries, let us consider the approach of strengthening the sampling condition and provide no more information other than the input set of points to the reconstruction stage as detailed in Section 4.2. In the case, addition of a sample point to a valid sample set might make the whole sampling an invalid sampling.

Definition 5 *Validity of sampling is non-monotonic if for monotonic increase in the number of sample points the validity of sampling transitions from a valid sampling to an invalid sampling at least once.*

Since we are considering only a valid surface for sampling, and the sampling satisfies the unrestricted point placement property, the final state of sampling can always be brought to a valid state. Further, note that, by definition, an empty sample set is an invalid sampling.

Definition 6 *A sampling condition is said to be non-monotonic if there exists at least one sequence of sample points for which the validity of the sampling is non-monotonic.*

Theorem 2 *Let the input to the reconstruction algorithm be only a finite set of sample points. Any sampling condition that ensures correct reconstruction of surfaces with boundaries is a non-monotonic sampling condition.*

Proof: Let the surface that is sampled be M, and let S be the finite valid sampling of M that satisfies the sampling condition C.

Since S is finite, there exists an open disk, $U \subset M$ that does not include any element of S. The surface $N = M - U$ is a valid surface with boundary and is topologically different from M. Consider a valid sampling S' of N such that S' satisfies C and $S \subset S'$. There exists such an S' because N is a valid surface and C satisfies the unrestricted point placement property.

Let us assume that the theorem is not true, and hence C is monotonic. This implies that S' is also a valid sampling of M, as $S \subset S'$ is a valid sampling of M. In other words, the reconstructed surface with the sample set S' should be correct with respect to both M and N. This is not possible as M and N are of different topologies. Hence S' should be an invalid sampling of M. This implies that C is non-monotonic since $S \subset S'$ is a valid sampling of M. $\diamondsuit$

5 Analysis of Algorithms

In this section, we analyze algorithms for reconstructing medial axes, curves with end points, and surfaces from noisy data set, and the applicability of the results presented in this paper to these algorithms.

5.1 Medial Axes and Surface with Boundaries

Medial axis representation can be considered as the dual to the surface representation. Medial axes of a surface can be reconstructed correctly if and only if the surface can be reconstructed correctly. Hence Theorem 1 is applicable to medial axis reconstruction algorithms also.

The reconstructed medial axis D from the sample points S of the surface M is correct if the D is topologically same as and geometrically close (within a given error bound) to the medial axis of M.

Corollary 1 *Conditions on the minimum required sampling density are not sufficient to design algorithms that reconstruct correct medial axes of surfaces with boundaries using only the point samples as input.*

Proof: The proof follows from Theorem 1 and the fact that the surface and its medial axis are dual to each other. $\diamondsuit$

For example, the medial axis reconstruction algorithm for surfaces with boundaries using Voronoi vertices used in [3] prescribe only minimum required sampling density as sampling condition. It uses the fact that the Voronoi vertices lie close to the medial axis, to reconstruct the medial axis. This algorithm would fail to reconstruct the medial axis of a plane with a circular hole (Figure 3). However dense the sampling is, there will be no representative Voronoi vertex in the medial axis passing through the hole, and hence the medial axis cannot be reconstructed without any additional information from the sampling conditions or to the reconstruction algorithm.

5.2 Reconstructing Curves with Boundary Points

There are a few algorithms for curve reconstruction which guarantees correct reconstruction. The work that is relevant to us is the reconstruction of curves with boundaries, that is, curves with endpoints. Dey *et al.*[9] developed the *conservative crust* algorithm and justify their reconstruction of curves with endpoints.

The sampling condition of [9] is based on the feature size as in [2]. The closest distance of a point **x** on the curve to the medial axis of the curve is called the *feature size* of **x**. The sampling condition

Class	Strengthen Sampling	More information to Reconstruction	Characteristics/Features
A	No	No	Cannot reconstruct surfaces with boundaries.
B	No	Complete Connectivity or ordering of points	Trivial solutions exist.
		Tagging boundary samples	Needs more exploration.
		Smallest boundary size	Imposes uniform sampling density.
C	Yes	No	Non-monotonic sampling.
D	Yes	Yes	*Not explored yet.* Probably can be used for reconstructing surfaces with boundaries from noisy point set.

Table 1: Classification of reconstruction algorithms for surfaces with boundaries.

of [2] states that for any point **x** on the curve there exists a sample point **p** whose distance from **x** is less than ϵ times the feature size of **x**. In other words, the sampling condition prescribes minimum required sampling density for correct reconstruction of the curve.

Instead of prescribing a sampling condition for correct reconstruction of curves with boundary points, [9] proves that there exists an ϵ for the curve reconstructed by the algorithm. Further, this method reconstructs a family of curves with endpoints (corresponding to various values of ϵ) from the given sample points. Conservative crust algorithm can be made to reconstruct the correct (required) and only the correct curve by providing additional information as input to the reconstruction algorithm. This additional information can be the value of ϵ used in the sampling of the curve with endpoints.

5.3 Reconstructing Surfaces from Noisy Samples

The sample points of surfaces obtained by present sampling devices are probabilistic estimates of the positions of the points, and hence noisy. With noisy samples, interpolatory reconstruction algorithms (where the surface passes through the samples) fail to reconstruct the correct surface, and the refuge is to rely on approximating reconstruction algorithms (where the surface is guided by the samples). Since approximating algorithms assume that the samples are probabilistic estimates, conceptually, the "information content" of each sample is also reduced to a fraction of its value. The Power Crust algorithm proposed by Amenta *et al.*[5, 4] to reconstruct manifolds (with no boundaries) from noisy samples is one of the best combination of interpolatory and approximating algorithms. This algorithm can be seen as dynamically assigning varying "probabilistic information content" to each sample to decide whether the surface should *pass* through or *guided* by the sample.

Boundaries are features of a model, and more information has to be provided to preserve these features. Noise is a "feature" of the sampling process, and more information has to be provided to nullify this "feature". The following question is worth investigating. Since both features require more samples, can "boundaries" be substituted by "noise" so that we can use the same techniques for sampling and reconstruction? In other words, can new algorithms be designed for sampling and reconstruction of *surfaces with no boundaries from noisy samples*, by either strengthening the sampling condition or by providing more information to the reconstruction algorithm? The Power Crust algorithm [5, 4] is an example in this direction. To handle noise, the reconstruction stage of this algorithm requires a user-defined estimate of the noise.

6 Classification of Surface Reconstruction Algorithms

Based on the analysis of the design space of the surface reconstruction algorithms, here we classify these algorithms for its use in reconstructing surfaces with boundaries. The classification is based on the changes in the two stages, the sampling stage and the reconstruction stage, over its basic requirements. In the sampling stage, the basic requirement for a sampling condition is to specify the minimum required sampling density. In the reconstruction stage, the basic input is the set of sample points. Changes in these stages include strengthening of the sampling condition in the sampling stage and providing additional information to the reconstruction stage. The Table 1 enumerates the classification.

All classes of algorithms, except Class A, can be designed to reconstruct surfaces with boundaries. The algorithms of [2, 3, 1, 6, 5, 4] belong to Class A. The algorithms based on α-shapes [13, 15, 8] belong to Class B. The value of α can be used as additional information to reconstruct surfaces with boundaries. The method used for reconstructing laser scan data of large areas also belong to Class B, as the boundary size is used as additional information. If the required reconstruction parameters, like α for α-shapes based algorithms, are estimated from the positions and number of sampling points, then such algorithms can be classified under Class C. These Class C algorithms, if used to reconstruct surfaces with boundaries are bound to exhibit the *non-monotonicity* property. The algorithm presented in [11] is also a Class C algorithm. Algorithms in Class D algorithms are yet to be explored completely, and we believe that they can be used to reconstruct surfaces with boundaries from noisy point data sets. In fact, a few algorithms in Class B can also be used to reconstruct surfaces with boundaries from noisy data sets.

There are a few problems due to noise when surfaces with boundaries are considered. The sampling density is based on the positions of points. As the positions are inaccurate due to noise, the estimates of sampling density will be inaccurate. If the approach of encoding the presence of boundary using the variation in sampling density is used, the sampling and reconstruction should take into account the inaccurate sampling density estimates also. Another problem is when the size of the boundary is comparable to the noise level. Hence we need more information to disambiguate boundary from noise. The field of sampling and reliably reconstructing surfaces with boundaries in the presence of noise is open for future research.

7 Conclusion

In this paper, we analyzed the problem of sampling and reconstructing surfaces with boundaries. We showed that, the sampling condition should prescribe more than the minimum required sampling density to ensure correct reconstruction of surface with boundary from only the sample points. Then we showed that such a sampling condition exhibits the *non-monotonic sampling* property. We discussed the sampling and reconstruction of curves with boundary points, medial axis reconstruction algorithms, and reconstruction of surfaces from noisy sample points. Finally, we classified surface reconstruction algorithms based on their sampling conditions and input to the reconstruction stage of the algorithm.

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