# On Sampling and Reconstructing Surfaces with Boundaries

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## Abstract

Surface reconstruction algorithms can ensure correctness of the reconstructed surface by imposing conditions on sampling and by assuming that the given point set satisfies the prescribed conditions. Traditionally, such sampling conditions impose *minimum required sampling density* for correct reconstruction. Let us assume that no additional information is provided to the reconstruction algorithm other than the input set of points. Under this assumption, we show that imposing minimum required sampling density is not sufficient to ensure correct reconstruction of *surfaces with boundaries*. In other words, to prove the correctness of the reconstructed surface, either the sampling conditions have to be strengthened or more information has to be provided to the reconstruction algorithm. Further, we prove that the strengthened sampling condition for sampling surfaces with boundaries has a unique property called *non-monotonicity*. We analyze the applicability of this framework to various algorithms like medial axis estimation algorithms, curve reconstruction algorithms, and reconstruction of surfaces from noisy point sets.

## 1 Introduction

Sampling is a process of discretizing a continuous surface or a curve into discrete samples. The reconstruction process uses these samples to reconstruct the original continuous surface. The reconstructed surface is *correct* if it is *topologically equivalent* and *geometrically close* to the original surface. To ensure such correct reconstruction, certain restrictions called *sampling coniditions* are imposed on the discretization process.

For example, in signal processing, one of the well known sampling condition is the *Nyquist criterion* to sample a continuous signal. According to the Nyquist theorem [17, 7], the discrete time sequence of a sampled continuous band limited signal $V(t)$ contains enough information to reproduce the function $V(t)$ exactly provided that the sampling rate is at least twice that of the highest frequency contained in the original signal $V(t)$. In other words, *it imposes a minimum required sampling density for correct reconstruction* of the signal from the samples. Conditions on geometric sampling of curves and surfaces with no boundaries follow the same philosophy and impose a minimum required sampling density for correct reconstruction. In this paper, we show that these algorithms cannot reliably reconstruct surfaces with boundaries using such sampling conditions.

**Main Contributions:** Following are the main contributions of this paper.

- We develop the theory of sampling surfaces with boundaries.
- We analyze the properties of the sampling for surfaces with b undaries, especially the non-monotonicity property.
- We analyze the reasons for the failure of the attempts made in the literature to develop theory to sample surfaces with boundaries.
- We analyze various reconstruction algorithms based on the theory developed in this paper.

The next section we review the previous work done in this area. Section 3 lists a few definitions, assumptions, and properties of sampling. Section 4 proves our main theorem about the sampling of surfaces with boundaries. Section 4.1 also introduces the property of *non-monotonicity* for sampling conditions and proves that the sampling conditions for surfaces with boundaries exhibit this property. Section 5 applies the concepts developed in this paper to various algorithms including medial axis reconstruction algorithms and curve reconstruction algorithms. Section 6 summarizes the results and classifies the reconstruction algorithms. Section 7 concludes this paper.

## 2 Previous Work

There has been a substantial amount of work done in the area of surface reconstruction. One of the earlier works on surface reconstruction with theoretical guarantees is from Attali [6], where reconstructing curves and surfaces using Voronoi diagrams was initiated. A different perspective of the problem in terms of the relationship between medial axis and Voronoi vertices was presented by Amenta et al. [1, 5, 4] to reconstruct surfaces with no boundaries. The sampling conditions used in these works of Amenta et al. are based on the distance of the points on the surface from the medial axis (aka *feature size*), which was first proposed for curve reconstruction in [2]. Such sampling conditions with minor modifications were used extensively by Dey et al. [12, 10]. The sampling conditions used in all these work were primarily used to reconstruct surfaces with no boundaries. There has also been an attempt to use these sampling conditions to develop heuristics to identify boundaries as regions of under-sampling [11].

One of the earliest works to reconstruct surfaces with boundaries with fundamentally different sampling conditions from the above was presented in [15]. In this work, the sampling condition was strengthened by specifying not only the minimum required sampling density but also relative maximum allowed sampling density. This work also introduced locally uniform sampling for reconstructing surfaces with boundaries. Later work on dimension detection [10] also used the concepts of locally uniform sampling (but with a different way of achieving it) as a required feature to remove ambiguity in dimension classification. Strengthening of sampling condition as proposed in [15] for reconstructing surfaces with boundaries was proved in [14] to be required under certain conditions. Dey et al. [10] also proposed the same result to identify the dimension of the manifold the sample points belong to, but still failed to claim that their algorithm would work correctly in case of surfaces with boundaries as they mention in the conclusion of their work. (In Section 5.1, we analyze the reason for this failure.) Further, [10] claims that we need a stricter sampling condition for automatic dimension detection and this cannot be avoided if one wants to guarantee correctness. This claim completely ignores the option of providing additional information to the algorithm that would enable it to detect the boundary automatically, without any stricter sampling condition. For example, the value of $\alpha$, and an uniform sampling of the surface would suffice to reconstruct all surfaces without boundaries or surfaces with boundary size greater than $\alpha$. In other words, from an information theoretic point of view, the claims made in [10] are true only under certain conditions.

In this paper, we formally and systematically prove the above intuition behind the sampling conditions to sample surfaces with boundaries, the situations in which such sampling conditions are required, and the properties of such sampling conditions.

# 3 Terminology and Definitions

In this section we list a few definitions, properties, and assumptions about the surface and sampling.

**Definition 1** *The reconstructed surface is said to be* correct *if it is* topologically equivalent *and* geometrically close *(with respect to a given error bound) to the original surface.*

**Valid Surface and Sampling:**

**Definition 2** *We consider any non-self-intersecting compact two manifold with or without boundary as a* valid surface.

In the rest of the paper, we denote any valid surface by $M$ and a small geometric perturbation of $M$ as $M'$. Further, $M$ and $M'$ have the same topology.

**Definition 3** *A* valid sampling condition *is one that* requires *non-empty set of samples to reconstruct a valid surface correctly.*

The above defintion might look obvious, but is important to see why certain elegant sampling conditions do not work in case of surfaces with boundaries. We will discuss about this briefly in Section 5.1.

**Definition 4** *A* valid sampling *of a surface is one that satisfies the given valid sampling condition.*

For example, infinite sampling of the whole surface is a trivial valid sampling, and an empty sample point set is an invalid sampling. We assume that a correct surface can be reconstructed if and only if the sampling satisfies the sampling conditions.

**Boundary Size:** Since, in this paper, we are considering manifolds with boundaries also, we define what we call the *boundary size*. For different reconstruction algorithms there will be some maximum distance between sample points on the boundary of the surface for which the algorithm will correctly reconstruct the boundary. We call this the *boundary size*, and its definition is based on the reconstruction algorithm used. For example, in terms of the sampling conditions prescribed by [3, 5], the boundary size is the minimum distance from the boundary points to the medial axis (smallest *feature size* of all the boundary points). In terms of alpha shapes[13], the boundary size is the minimal value of alpha that cannot reconstruct a topologically correct and geometrically close surface. This definition of boundary size is used in our proof of theorem 1.

**Finiteness of sampling:** A trivial sampling condition can demand infinite sampling everywhere on the surface. We would like to avoid such conditions in our analysis.

**Definition 5** *A sampling condition is said to satisfy the* finiteness property *if there exists a finite set of points that samples $M$ or $M'$, and satisfies the sampling condition.*

For example, the sampling conditions of [3] require infinite sampling when the medial axis of the surface meets the surface, as in sharp corners and edges. But these features can be removed by geometric perturbation $M'$ of the surface retaining its topology. Hence this sampling condition satisfies the finiteness property.

**Unrestricted Point Placement:** The sampling conditions should not be too restrictive regarding the placement of points. There should be a non-zero probability of successful placement of a sample point on the surface $M$ or $M'$. For example, the sampling conditions of [3] require placement of sample points on corners. As a corner occupies zero area on $M$, there is zero probability of placing a sample on a corner of $M$. But with small geometric perturbation, this corner feature and hence this restrictive requirement can be removed. (While sampling a boundary curve, we cannot avoid the requirement of sample points on the curve. Hence, sampling of boundary curves has to be handled separately as a lower dimensional "surface" where placing a point on this curve has a non-zero probability.)

The above property deals with the *requirement* of sample points at *specific locations*. Another aspect of this property is the *acceptability* of sample points at *arbitrary locations*. In other words, it should still be possible to find a valid sampling of $M$ even if there exist a few sample points at arbitrary locations on $M$.

**Definition 6** *A sampling condition is said to satisfy the* unrestricted point placement property *if, given an arbitrary finite set of sample points $S$ on $M$, there exists a finite valid sampling $T$ of $M$ such that $S \subset T$.*

## 3.1 Assumptions

In this paper we assume the following:

1. The surface that is sampled is a valid surface.
2. The sampling conditions satisfy the finiteness property and the unrestricted point placement property.
3. The input set of points is unorganized, that is, any permutation of the input set of points results in the same reconstructed model. This assumption is made so that there is no inherent order by which the connectivity can be deduced.
4. The input sample point set $S$ reflects only the positions of the points on the surface and there is no duplication of sample points. Further, we assume that any small perturbation of the point locations does not change the connectivity in the reconstructed surface. This means that there is no encoding of information other than the positions of points in the input data.

# 4 Sampling Surfaces with Boundaries

Sampling condition for surfaces with boundaries directly or indirectly specify the minimum required sampling density, in other words, maximum distance between close samples. In this section we show that sampling surfaces with boundaries is different from sampling surfaces with no boundaries.

There are two stages to the surface reconstruction process, the sampling stage and the reconstruction stage. Let us assume that the input to the reconstruction stage is only the point samples that satisfy the assumptions 3 and 4. Under this scenario, the following

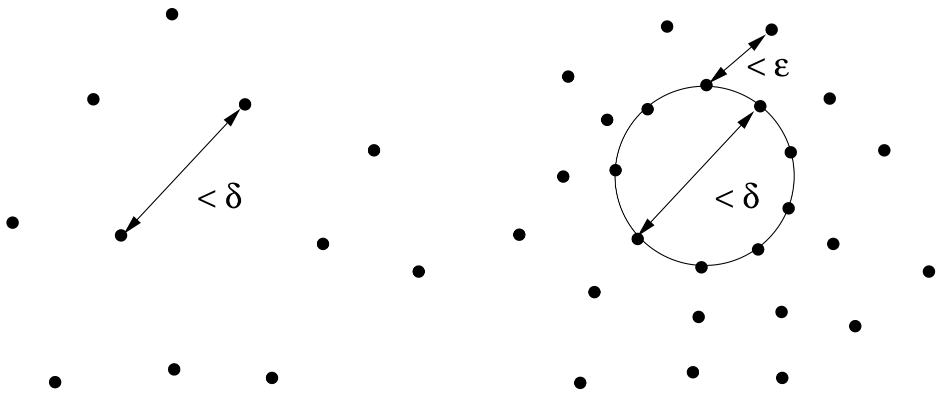


Figure 1: Sampling: Left – Without Boundary, Right – With Boundary.

theorem is true. Even though this theorem has been speculated and used in [15, 10], this has not been formally proved in the literature.

**Theorem 1** *Conditions on the minimum required sampling density are not sufficient to design algorithms that reconstruct correct surfaces with boundaries using only the point samples as input.*

**Proof:** Let us assume that we are given a hypothetical surface reconstruction algorithm $A$ that claims to reliably reconstruct surfaces both with and without boundaries using only the input set of points. Let $A$ be based on a sampling condition $C$ that directly or indirectly specifies just the maximum distance between samples in a given region. Let $\delta$ be the maximum distance between the closest samples demanded by $C$ in a specific region on $M$ (Figure 1). In that region of $M$, if we consider a boundary $B$ of boundary size less than $\delta$ (and greater than another constant, say $\epsilon$), then the sampling demanded would not be able to capture the feature $B$. (Refer to Section 3 for an explanation of the term "boundary size".) Hence $C$ is forced to demand a denser sampling in the region around the boundary $B$. Let the new limit on maximum distance between closest samples be $\epsilon$ which is less than $\delta$. Since, by the new sampling condition, the distance between closest samples is less than both $\epsilon$ and $\delta$, the sampling satisfies the conditions required to sample $M$ *both with and without the boundary* $B$. With no additional information about the presence or absence of boundary, the algorithm $A$ has no way to find out whether the given set of sample points are of a surface with or without boundary. Hence conditions on just the minimum required sampling density is not sufficient for reliable reconstruction of $M$ with $B$ when only the sample points are used during reconstruction. $\diamondsuit$

**Observation 1:** *In summary, for reliable reconstruction of surfaces with boundaries, either (a) more information, other than the sample point set, has to be provided to the reconstruction algorithm or (b) the sampling condition has to be strengthened (or both).*

### 4.1 Non-monotonic Sampling

In this section, we introduce a property called *non-monotonicity* for the sampling conditions and prove the situations under which the sampling conditions exhibit this property.

For surfaces with *no* boundaries, a sampling condition prescribes only the minimum required sampling density. A sampling becomes a valid sampling of a surface $M$ once it satisfies the sampling condition. It would remain a valid sampling with any additional sampling over this minimum requirement.

For surfaces with boundaries, let us consider the approach of strengthening the sampling condition and provide no more information other than the input set of points to the reconstruction stage as observed in Observation 1. In that case, addition of a sample point to a valid sample set might make the whole sampling an invalid sampling.

**Definition 7** *Validity of sampling is non-monotonic if for monotonic increase in the number of sample points the validity of sampling transitions from a valid sampling to an invalid sampling at least once.*

Since we are considering only a valid surface for sampling, and the sampling satisfies the unrestricted point placement property, the final state of sampling can always be brought to a valid state. Further, note that, by definition, an empty sample set is an invalid sampling.

**Definition 8** *A sampling condition is said to be non-monotonic if there exists at least one sequence of sample points for which the validity of the sampling is non-monotonic.*

**Theorem 2** *Let the input to the reconstruction algorithm be only a finite set of sample points. Any sampling condition that ensures correct reconstruction of surfaces with boundaries is a non-monotonic sampling condition.*

**Proof:** Let the surface that is sampled be $M$, and let $S$ be the finite valid sampling of $M$ that satisfies the sampling condition $C$.

Since $S$ is finite, there exists an open disk, $U \subset M$ that does not include any element of $S$. The surface $N = M - U$ is a valid surface with boundary and is topologically different from $M$. Consider a valid sampling $S'$ of $N$ such that $S'$ satisfies $C$ and $S \subset S'$. There exists such an $S'$ because $N$ is a valid surface and $C$ satisfies the unrestricted point placement property.

Let us assume that the theorem is not true, and hence $C$ is monotonic. This implies that $S'$ is also a valid sampling of $M$, as $S \subset S'$ is a valid sampling of $M$. In other words, the reconstructed surface with the sample set $S'$ should be correct with respect to both $M$ and $N$. This is not possible as $M$ and $N$ are of different topologies. Hence $S'$ should be an invalid sampling of $M$. This implies that $C$ is non-monotonic since $S \subset S'$ is a valid sampling of $M$. $\diamondsuit$

## 5 Analysis of Algorithms

In this section, we analyze algorithms for reconstructing medial axes, curves with end points, and surfaces from noisy data set, and the applicability of the results presented in this paper to these algorithms.

### 5.1 Medial Axes Based Sampling Conditions

Reconstruction algorithms mentioned in [3, 5, 4] clearly specify that they cannot handle surfaces with boundaries. The sampling conditions mentioned in these works along with their minor variants mentioned in [10, 12], require no sample to sample a planar region. This is not a problem when considering compact manifolds as the planar polygons have shared vertices and edges that require sample points according to the sampling conditions. Consider a manifold with boundary, a planar convex polygon $P$. These sampling conditions demand no sample point to sample $P$. Clearly, they cannot reconstruct $P$ either. By Defintion 3, these sampling conditions are invalid sampling conditions.

First, we have to make these sampling conditions valid. If the sampling conditions treat boundary curves independent of their surfaces as free standing space curves, then these conditions would become valid. If $M$ is the manifold with boundary, and $N$ is an open set such that Closure($N$)=$M$, then the sampling conditions should treat $M - N$ independent of $M$ while sampling. Such a sampling condition, in case of sampling the polygon $P$, would demand sample points at least on the vertices of $P$ (due to internal medial axis), and hence have a non-empty sample point set. There might be other ways of achieving validity of sampling conditions and requires detailed research.

Second, we have to identify these samples as boundary samples. Once we have sampled the boundary curves, we hypothesize based on Theorem 1 and Observation 1 that the strengthened sampling condition given in [10] that imposes locally uniform sampling and relative maximum sampling density, would suffice to identify the boundary samples.

### 5.2 Other Reconstruction Algorithms

**Reconstructing Medial Axes:** Medial axis representation can be considered as the dual to the surface representation. Medial axes of a surface can be reconstructed correctly if and only if the surface can be reconstructed correctly. Hence Theorem 1 is applicable to medial axis reconstruction algorithms also, in case of surfaces with boundaries.

**Definition 9** *The reconstructed medial axis $D$ from the sample points $S$ of the surface $M$ is correct if $D$ is topologically same as and geometrically close (within a given error bound) to the medial axis of $M$.*

**Corollary 1** *Conditions on the minimum required sampling density are not sufficient to design algorithms that reconstruct correct medial axes of surfaces with boundaries using only the point samples as input.*

**Proof:** The proof follows from Theorem 1 and the fact that the surface and its medial axis are dual to each other. $\diamondsuit$

**Reconstructing Curves with Boundary Points:** There are a few algorithms for curve reconstruction that guarantee correct reconstruction. The work that is relevant to us is the reconstruction of curves with boundaries, that is, curves with endpoints. Dey *et al.*[9] developed the *conservative crust* algorithm and justify their reconstruction of curves with endpoints.

The sampling condition of [9] is based on the $\epsilon$-sampling as in [2]. Instead of prescribing a sampling condition for correct reconstruction of curves with boundary points, [9] proves that there exists an $\epsilon$ for the curve reconstructed by the algorithm. Further, this method reconstructs a family of curves with endpoints (corresponding to various values of $\epsilon$) from the given sample points. Clearly, if the the value of $\epsilon$, with which the curve was sampled, was given as additional information to the reconstruction algorithm, it can reconstruct the correct (required) and only the correct curve.

| Class | Strengthen Sampling | More information to Reconstruction |
|---|---|---|
| $A$ | No | No |
| $B$ | No | Yes |
| $C$ | Yes | No |
| $D$ | Yes | Yes |

Table 1: Classification of reconstruction algorithms for surfaces with boundaries.

**Reconstructing Surfaces from Noisy Samples:** Boundaries are features of a model, and more information has to be provided to preserve these features. Noise is a "feature" of the sampling process, and more information has to be provided to nullify this "feature". Due to this similarity of "noise" with the "boundary" we can conclude the following:

**Corollary 2** *Conditions on the minimum required sampling density are not sufficient to design algorithms that reconstruct correct surfaces with or without boundaries using only the noisy point samples as input.*

Following problems are worth investigating:

1. Provably correct reconstruction of manifolds (without boundaries) from noisy samples by providing more information to the reconstruction algorithm and with sampling conditions on only minimum required sampling density.

2. Provably correct reconstruction of manifolds (without boundaries) with no more information other than the noisy samples, but by strengthening the sampling conditions.

3. Provably correct reconstruction of manifolds with boundaries from noisy samples.

The problem 1 is addressed by the Power Crust algorithm [5, 4] to reconstruct manifolds (without boundaries) from noisy samples, in which the algorithm requires additional information, an user-defined estimate of the noise. Problems 2 and 3 still remain open.

## 6 Summary

In this section, we classify these algorithms for its use in reconstructing surfaces with boundaries. The classification is based on the changes over its basic requirements in the two stages: the sampling stage and the reconstruction stage. In the sampling stage, the basic requirement for a sampling condition is to specify the minimum required sampling density. In the reconstruction stage, the basic input is the set of sample points. Changes in these stages include strengthening of the sampling condition in the sampling stage and providing additional information to the reconstruction stage. The Table 1 enumerates the classification.

All classes of algorithms, except Class $A$, can be designed to reconstruct surfaces with boundaries. The algorithms of [2, 3, 1, 6, 5, 4] belong to Class $A$. The algorithms based on $\alpha$-shapes [16, 18, 8] belong to Class $B$. The value of $\alpha$ can be used as additional information to reconstruct surfaces with boundaries. The method used for reconstructing laser scan data of large areas also belong to Class $B$, as the boundary size is used as additional information. If the required reconstruction parameters, like $\alpha$ for $\alpha$-shapes based algorithms, are estimated from the positions and number of sampling points, then such algorithms can be classified under Class $C$. These Class $C$ algorithms, if used to reconstruct surfaces with boundaries are bound to exhibit the *non-monotonicity* property. The algorithm presented in [14] is also a Class $C$ algorithm. Algorithms in Class $D$ algorithms are yet to be explored completely, and we believe that they can be used to reconstruct surfaces with boundaries from noisy point data sets. In fact, a few algorithms in Class $B$ can also be used to reconstruct surfaces with boundaries from noisy data sets.

## 7 Conclusion

In this paper, we analyzed the problem of sampling and reconstructing surfaces with boundaries. We showed that, the sampling condition should prescribe more than the minimum required sampling density to ensure correct reconstruction of surface with boundary from only the sample points. Then we showed that such a sampling condition exhibits the *non-monotonic sampling* property. We analyzed the reason for the failure of medial axis based algorithms to reconstruct surfaces with boundary. We also discussed the sampling and reconstruction of curves with boundary points, medial axis reconstruction algorithms, and reconstruction of surfaces from noisy sample points. Finally, we classified surface reconstruction algorithms based on their sampling conditions and input to the reconstruction stage of the algorithm. We believe that the results presented in this paper would help researchers to formally and systematically design algorithms for reconstructing surfaces with or without boundaries, and to analyze the applicability of their algorithms to various data sets.

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