

Example of a direct proof:

Theorem: If  $0 \leq x \leq 2$ , then  $-x^3 + 4x + 1 > 0$ .

Proof: Assume that  $x$  is in the range from 0 to 2.

That is, assume  $0 \leq x \leq 2$ . For  $x$  in this range the following quantities are all at least 0:

$$x \geq 0$$

$$2 - x \geq 0 \quad (\text{This follows from the fact that } x \leq 2).$$

$$2 + x \geq 0$$

Multiplying all three quantities together yields an expression which is also at least 0:

$$x(2-x)(2+x) \geq 0$$

Multiplying the expression out, we get that:

$$4x - x^3 \geq 0$$

Adding 1 to both sides gives.

$$4x - x^3 + 1 \geq 1 > 0$$

Therefore  $-x^3 + 4x + 1 > 0$  \(\square\)

### Proof by contrapositive

Theorem: If  $3n+8$  is an odd integer then  $n$  is an odd integer.

Proof: We will assume that  $n$  is even and will prove that  $3n+8$  is even. Recall that if  $n$  is even, it is an integer multiple of 2 and can be written as  $n = 2c$ , where  $c$  is an integer. Plug the expression

for  $n$  into  $3n+8$ :

$$3n+8 = 3(2c)+8 = 2(3c+4).$$

Note that  $3c+4$  is an integer since  $c$  is an integer.

The expression  $3n+8$  is expressed as 2 times an integer.  
Therefore  $3n+8$  must be even.  $\square$

Theorem: If  $a$  is an integer and  $a^2$  is even, then  $a^2$  is a multiple of 4.

Proof: We prove this theorem in two parts:

- If  $a$  is an integer and  $a^2$  is even, then  $a$  is even.
- If  $a$  is an even integer then  $a^2$  is a multiple of 4.

Together A) + B) imply the theorem, so we now prove each part.

Proof of A): A) is proven by contrapositive. We assume that  $a$  is an odd integer and we will show that  $a^2$  is odd.

If  $a$  is odd it can be written as  $a=2c+1$  where  $c$  is an integer. Now we plug the expression for  $a$  into  $a^2$ :

$$a^2 = (2c+1)^2 = 2(2c^2+2c)+1.$$

Note that since  $c$  is an integer,  $2c^2+2c$  is also an integer. Since  $a^2$  is  $2c'+1$  where  $c'$  is an integer, then  $a^2$  is odd.

Proof of B): B) is proven directly.

If  $a$  is even then  $a=2c$  for some integer  $c$ .

Therefore  $a^2 = (2c)^2 = 4c^2$ . Since  $c^2$  is an integer  $a^2$  is an integer multiple of 4.  $\square$

Theorem:  $\sqrt{2}$  is irrational.

Proof: The proof is by contradiction. First assume that  $\sqrt{2}$  is rational. If  $\sqrt{2}$  is rational then  $\sqrt{2}$  can be expressed as  $\sqrt{2} = a/b$  where  $a$  and  $b$  are integers. Every fraction  $a/b$  has a reduced form, so we can assume that  $a$  and  $b$  have no common divisors greater than 1.

Take the inequality  $\sqrt{2} = a/b$  and square both sides to get  $2 = a^2/b^2$ . Multiply both sides by  $b^2$  to get  $2b^2 = a^2$ . Since  $a^2$  can be expressed as 2 times an integer (because  $a^2 = 2b^2$ ), we know that  $a^2$  is even.

Part A) of the previous proof can be applied to get that  $a$  is even.

If  $a$  is even then it can be expressed as  $a = 2c$ , for some integer  $c$ . Plug this value for  $a$  into the equation  $2b^2 = a^2$ .

$$2b^2 = (2c)^2 = 4c^2$$

Dividing both sides by 2 gives  $b^2 = 2c^2$ . This implies that  $b^2$  is even. Again, we can apply Part A) of the previous proof to conclude that  $b$  is even.

We have now proven that  $a$  and  $b$  are both even. This means that  $a$  and  $b$  both have 2 as a common divisor which contradicts our assumption in the beginning of the proof. Therefore  $\sqrt{2}$  is irrational.  $\square$