Note Title 11/12/2013

Find eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

Find all scalars λ such that the equation

$$(A - \lambda I)\vec{x} = \vec{0}$$

has a non-trivial solution. $\iff$ $A - \lambda I$ is not invertible

$\iff \det(A - \lambda I) = 0.$

$$(A - \lambda I) = \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= (2-\lambda)(-6-\lambda) - 9 \\ &= \lambda^2 + 4\lambda - 21 \\ &= (\lambda + 7)(\lambda - 3) \end{aligned}$$

$\lambda = -7, 3.$

Summary of determinants: $A \rightsquigarrow U$ (echelon form obtained by row exchange and row repl).

$$\det(A) = \det(U)$$
$$= \underbrace{(u_{11} u_{22} \cdots u_{nn})}_{\text{this is non-zero iff } A \text{ is invertible.}} (-1)^r \rightarrow \#\text{ row swaps.}$$

Invertible Matrix Theorem (cont).

Let A be an $n \times n$ matrix. A is invertible iff

s. The number 0 is not an eigenvalue of A.

t. The det of A is not 0.

$\rightarrow$ Corresponds to geometric view from before. If cols of A are lin dep, they span only 2D vol of parallelipiped is 0.

<u>Properties of Dets</u> A & B $n \times n$ matrices.

a) A is invertible iff $\det A \neq 0$.

b) $\det AB = \det A \cdot \det B$.

c) $\det A^T = \det A$.

d) If A is triangular then $\det A =$ product of diagonal entries

e. $A \rightsquigarrow B$ by 1 row op.

row repl : $\det(A) = \det B$

swap : $\det(A) = -\det B$.

mult by k : $\det B = k \det A$.

The determinant of the matrix $(A - \lambda I)$ is 0 exactly when λ is an eigenvalue of A.

The algebraic expression $\det(A - \lambda) = 0$ is called the characteristic equation.

A scalar λ is an eigenvalue of an $n \times n$ matrix A iff λ satisfies the characteristic eqn:

$$\det(A - \lambda) = 0.$$

Make an upper tri matrix + express char eqn. → multiplicity.

$\Rightarrow$ $\det(A - \lambda I)$ eqn for $n \times n$ matrix A is a polynomial of degree n. called the characteristic polynomial.

↳ multiplicity of an eigenvalue.

the char polynomial always has n roots — counting mult although some may be complex.

For large matrices, eigenvalues are computed by computer.

Similarity:

↳ useful in approximating eigenvalues.

Let A & B be two $n \times n$ matrices. A is similar to B if there is an invertible matrix P such that

$$\begin{cases} B = P^{-1}AP \\ A = Q^{-1}BQ. \end{cases} \quad \text{or if } Q = P^{-1}$$

$\hookrightarrow$ A & B are similar to each other.

If A & B are similar then they have the same characteristic equation $\Rightarrow$ same eigenvalues. (though different eigenvectors).

$$B = P^{-1}AP \qquad B - \lambda I = P^{-1}AP - \lambda P^{-1}P$$
$$= P^{-1}(AP - \lambda P)$$
$$= P^{-1}(A - \lambda I)P$$

$$\det(B - \lambda I) = \det(P^{-1})\det(A - \lambda I)\det(P)$$
$$= \det(A - \lambda I)\underbrace{\det(P^{-1})\det(P)}_{\det(P^{-1}P) = \det(I) = 1.}$$

$\boxtimes$

* Similarity is not the same as row equivalence.
* Two matrices can have the same eigenvalues & not be similar.