

Want to express $n \times n$ matrix A as $A = PDP^{-1}$

where P is invertible & D is diagonal (i.e. all 0's off the diagonal).

This is useful for taking large powers of A : $A^k = P D^k P^{-1}$

$$\text{If } D = \begin{bmatrix} \lambda_1 & & \\ 0 & \ddots & \\ & & \lambda_n \end{bmatrix} \text{ then } D^k = \begin{bmatrix} (\lambda_1)^k & & \\ 0 & \ddots & \\ & & (\lambda_n)^k \end{bmatrix}$$

which is easy to compute.

NOT all square matrices are diagonalizable. Here's how to determine if A is diagonalizable.

Find characteristic polynomial $\det(A - \lambda I)$

This is a degree n polynomial $p(\lambda)$ → characteristic polynomial

Solve for $p(\lambda) = 0$. (Either by factoring by hand or by computer). → characteristic equation.

We will ignore the situation that $p(\lambda)$ has imaginary roots.
Any solution to $\det(A - \lambda I) = 0$ is an eigenvalue.

(Case 1) Characteristic polynomial has n distinct roots:

$\lambda_1, \lambda_2, \dots, \lambda_n \Rightarrow A$ is diagonalizable!

Why? Since λ_i is a solution to the characteristic equation,

$(A - \lambda_i I) \vec{x} = \vec{0}$ has at least one non-trivial soln. \vec{v}_i

\vec{v}_i is the eigenvector corresponding to eigenvalue λ_i .

Then $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ are linearly independent and the matrix $P = [\vec{v}_1 \dots \vec{v}_n]$ is invertible.

$$\text{Let } D = \begin{bmatrix} \lambda_1 & & 0 \\ \lambda_2 & \ddots & \\ 0 & \ddots & \lambda_n \end{bmatrix} \text{ Then } A = PDP^{-1}$$

{ must be the same order as the \vec{v}_i .

so λ_i is eigenvalue for \vec{v}_i .

2) Characteristic polynomial has $r < n$ roots but
multiplicities add to n : $\det(A - \lambda I) = (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_r)^{m_r}$

m_i is the multiplicity of root λ_i .

For each $i = 1 \dots r$

Find the set of solutions to $(A - \lambda_i I) \vec{x} = \vec{0}$

The methods we learned to solve homogeneous systems like
the one above expresses the solution set as a linear combination
of linearly indep vectors: $x_i \begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_r \end{bmatrix} + x_j \begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_r \end{bmatrix} \dots x_k \begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_r \end{bmatrix}$

Case 2a) If the number of linearly indep vectors (i.e. the dimension
of the solution space to $(A - \lambda_i I) \vec{x} = \vec{0}$) is less than
the multiplicity of λ_i in the characteristic equation
for some $\lambda_i \rightarrow$ then A is not diagonalizable.

Case 2b) If for all λ_i , the multiplicity of λ_i matches the
dimension of the solution space to $(A - \lambda_i I) \vec{x} = \vec{0}$ then
 A is diagonalizable.

$$D = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix} \quad \begin{array}{l} \text{→ } \# \text{ copies of} \\ \lambda_i = \text{multiplicity} \end{array}$$

$$P = \begin{bmatrix} \vec{v}_1 & \cdots & \underbrace{\vec{v}_{i_1} \vec{v}_{i_2} \cdots \vec{v}_{i_n}}_{\text{lin indep}} & \cdots & \vec{v}_r \end{bmatrix}$$

Solutions to $(A - \lambda_i I) \vec{x} = \vec{0}$



Some examples.

Case 1) n distinct eigenvalues.

$$A = \begin{bmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{bmatrix}$$

$$\det(A - \lambda I) = \lambda(\lambda+1)(\lambda+2) = 0$$

$$\lambda = 0, -1, -2$$

Eigenvalue $\lambda_1 = 0$
 Solve $(A - 0 \cdot I)\vec{x} = 0$ $A\vec{x} = 0$. $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
 $A\vec{v}_1 = \vec{0}$

Eigenvalue $\lambda_2 = -1$
 Solve $(A - -I)\vec{x} = 0$ $\begin{bmatrix} 3 & -2 & -2 \\ 3 & -2 & -2 \\ 2 & -2 & -1 \end{bmatrix}\vec{x} = \vec{0}$. $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$
 $A\vec{v}_2 = -\vec{v}_2$

Eigenvalue $\lambda_3 = -2$
 Solve $(A - -2I)\vec{x} = 0$ $\begin{bmatrix} 4 & -2 & -2 \\ 3 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}\vec{x} = \vec{0}$. $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
 $A\vec{v}_3 = -2\vec{v}_3$
 $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$
 $\uparrow \quad \uparrow \quad \uparrow$
 $\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3$

Case 2g: $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = -(\lambda - 1)(\lambda + 2)^2$$

$$\lambda = 1 \quad \underbrace{\lambda = -2}_{\text{multiplicity 2}}$$

Eigenvalue $\lambda_1 = 1$

Solve $(A - I)\vec{x} = \vec{0}$ $\begin{bmatrix} 1 & 4 & 3 \\ -4 & -7 & -3 \\ 3 & 3 & 1 \end{bmatrix}\vec{x} = \vec{0} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$
 $A\vec{v}_1 = \vec{v}_1$

Eigenvalue $\lambda_2 = -2$

Solve $(A + 2I)\vec{x} = \vec{0}$ $\begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix}\vec{x} = \vec{0} \Rightarrow \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$
 $A\vec{v}_2 = -2\vec{v}_2$

only one vector
 λ_2 has multiplicity 2

$\Rightarrow A$ is not
 diagonalizable.

$$\text{Case 2b)} \quad A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \quad \det(A - \lambda I) = -(\lambda - 1)(\lambda + 2)^2$$

$\lambda = 1$ $\underbrace{\lambda = -2}$
mult 2.

eigenvalue

$$\lambda_1 = 1$$

$$\text{Solve } (A - I)\vec{x} = 0$$

$$A\vec{v}_1 = \vec{v}_1$$

$$\begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \vec{x} = \vec{0}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{eigenvalue } \lambda_2 = -2$$

$$\text{Solve } (A + 2I)\vec{x} = \vec{0}$$

$$A\vec{x} = -2\vec{x}$$

for any $c\vec{v}_2 + c'\vec{v}_3$

$$\begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \vec{x} = \vec{0}$$

$$c \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c' \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

2 vectors

= multiplying by $\lambda_2 = -2$.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A = PDP^{-1}$$