Note Title 10/4/2013

$\vec{a}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \vec{a}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

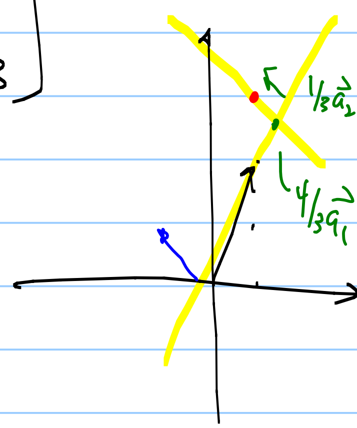
Is there a linear comb of $\vec{a}_1$ & $\vec{a}_2$ that $= \vec{b}$??

$x_1 \vec{a}_1 + x_2 \vec{a}_2 = \vec{b}$ find x_1 & x_2 that satisfies

$$\begin{aligned} x_1 - x_2 &= 1 \\ 2x_1 + x_2 &= 3 \end{aligned}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 4/3 \\ 0 & 1 & 1/3 \end{bmatrix}$$

$$\frac{4}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$



Can write a matrix: with columns $\vec{a}_1 \; \vec{a}_2 \; \cdots \; \vec{a}_n \; \vec{b}$

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n & \vec{b} \end{bmatrix}$$

Vector equation $x_1 \vec{a}_1 + \cdots x_n \vec{a}_n = \vec{b}$ $\quad \vec{a}_i \in \mathbb{R}^m$

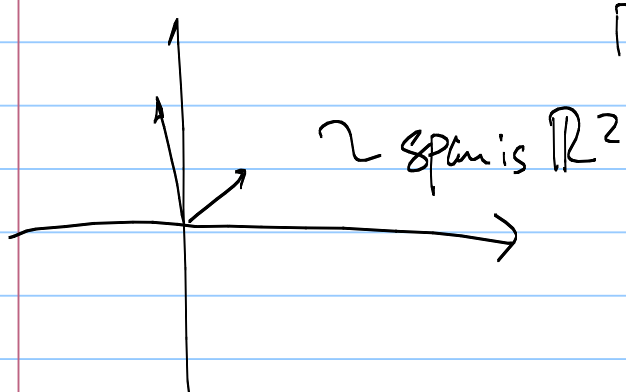
has the same set of solutions as the linear system whose augmented matrix is $[\vec{a}_1 \cdots \vec{a}_n \; \vec{b}]$

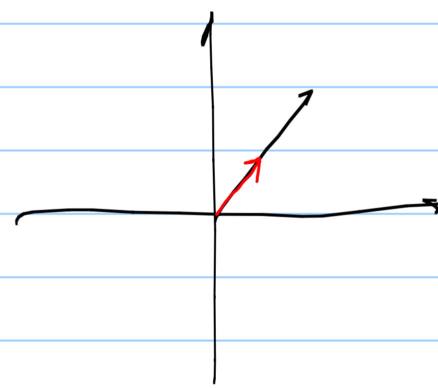
$\vec{b}$ can be expressed as a linear combination of $\vec{a}_1 \cdots \vec{a}_n$ if & only if the linear system corresponding to $[\vec{a}_1 \cdots \vec{a}_n \ \vec{b}]$ is consistent.
iff $\vec{b}$ in Span of $\vec{a}_1 \cdots \vec{a}_n$.

If $\vec{V}_1 \cdots \vec{V}_p \in \mathbb{R}^n$ the set of all possible linear combinations of $\vec{V}_1 \cdots \vec{V}_p$ is called Span $\{\vec{V}_1 \cdots \vec{V}_p\}$

The set that can be written as $c_1\vec{V}_1 + c_2\vec{V}_2 + \cdots + c_p\vec{V}_p$.

$\mathbb{R}^2$





2 vectors in $\mathbb{R}^3$ $\vec{a}_1 \neq \vec{0}$ $a_2 \neq \vec{0}$
$\vec{a}_1$ + $\vec{a}_2$

$\vec{a}_1 \neq c\,\vec{a}_2$.

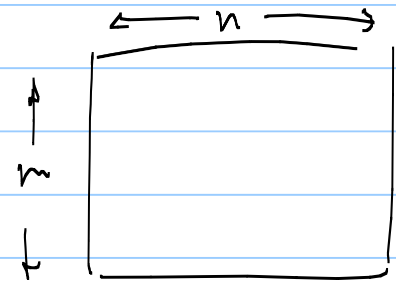
Span $\{a_1, a_2\}$. $\rightsquigarrow$ a plane.

System of linear equations as a matrix equation.

A is an $m \times n$ matrix

columns in $\mathbb{R}^m$

$\vec{a}_1 \cdots \vec{a}_n$



If $\vec{x} \in \mathbb{R}^n$

$A \cdot \vec{x}$ is $= x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_n \vec{a}_n$.

$$\overbrace{\begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}}^{n}_{m} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} n = x_1 \vec{a}_1 + \cdots + x_n \vec{a}_n.$$

$$\begin{bmatrix} 2 & 0 & 1 \\ -3 & -1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ -3 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 13 \\ -17 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 1 \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + (-2) \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ -5 \\ 2 \end{bmatrix}$$

$$\begin{aligned} x_1 - 2x_2 - 3x_3 &= 0 \\ 2x_2 + x_3 &= -8 \\ -x_1 + x_2 + 2x_3 &= 3 \end{aligned} \equiv x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -8 \\ 3 \end{bmatrix}$$

$$\equiv$$

$$\begin{bmatrix} 1 & -2 & -3 \\ 0 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -8 \\ 3 \end{bmatrix}$$

Thm A is $m \times n$ matrix w/ cols $\vec{a}_1 \cdots \vec{a}_n \in \mathbb{R}^m$, $b \in \mathbb{R}^m$

- Solutions to $A\vec{x} = \vec{b}$
- $\equiv$ $x_1 \vec{a}_1 + \cdots + x_n \vec{a}_n = \vec{b}$
- $\equiv$ linear system w/ aug matrix $[\vec{a}_1 \cdots \vec{a}_n \; \vec{b}]$.

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 5 \\ 2 & 7 & 1 \end{bmatrix}$$

Is $A\vec{x} = \vec{b}$ consistent for all $\vec{b}$??

No only for $b_3 - 2b_1 + b_2 = 0$

$$-2r_1 \quad \begin{bmatrix} 1 & 4 & 3 & b_1 \\ 0 & 1 & 5 & b_2 \\ 2 & 7 & 1 & b_3 \end{bmatrix} \rightarrow \begin{matrix} 1 & 4 & 3 & b_1 \\ 0 & 1 & 5 & b_2 \\ 0 & -1 & -5 & b_3 - 2b_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 4 & 3 & b_1 \\ 0 & 1 & 5 & b_2 \\ 0 & 0 & 0 & b_3 - 2b_1 + b_2 \end{bmatrix}$$

$$A: \begin{bmatrix} 1 & 3 & 2 \\ -1 & 2 & 1 \\ -1 & -8 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 2 & b_1 \\ -1 & 2 & 1 & b_2 \\ -1 & -8 & 3 & b_3 \end{bmatrix}$$

$A\vec{x} = \vec{b}$

Consistent for any $\vec{b}$?

Yes!

$$\begin{bmatrix} 1 & 3 & 2 & b_1 \\ 0 & 5 & 3 & b_1 + b_2 \\ 0 & -5 & 5 & b_1 + b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 2 & b_1 \\ 0 & 5 & 3 & b_1 + b_2 \\ 0 & 0 & 8 & 2b_1 + b_2 + b_3 \end{bmatrix}$$

Theorem: A be an $m \times n$ matrix

The following statements are all true or all false.

(1) For any $\vec{b} \in \mathbb{R}^m$ $A\vec{x} = \vec{b}$ has a solution.

(2) For any $\vec{b} \in \mathbb{R}^m$ $\vec{b}$ can expressed as a linear comb of the columns of A.

(3) Span of columns of A $= \mathbb{R}^m$

(4) A has a pivot in every row.

$$\begin{bmatrix} \; A \; \end{bmatrix} \rightsquigarrow \cancel{\left[\begin{array}{cccccc|c} \ddots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & b \end{array}\right]}$$

$$A\vec{x} \qquad \begin{bmatrix} 3 & -1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2 \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 3 + (-1)(-1) \\ 2 \cdot 2 + (-1) \cdot 0 \\ 2 \cdot 1 + (-1) \cdot 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 1 & 1 \\ 2 & -1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \cdot 1 + 0 \cdot 2 + 1(-1) + 1 \cdot 0 \\ \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 1 & 1 \\ 2 & -1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \cdot 1 + 0 \cdot 2 + 1(-1) + 1 \cdot 0 \\ 2 \cdot 1 + (-1) \cdot 2 + 3(-1) - 2 \cdot 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

A is an $m \times n$ matrix $\vec{u}, \vec{v} \in \mathbb{R}^n$

c scalar.

$$A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$$

$$A(c\vec{u}) = c(A\vec{u})$$